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Physics-PHY101-Lecture #01

INTRODUCTION TO PHYSICS & THIS COURSE

1.1. Introduction

Welcome to Physics. We will embark on a long journey that will consist of 45 lectures. Nevertheless, I hope will find it interesting and enjoyable. As I continued to study and research in this subject my interest in this subject grew. But before we come to physics, I want to remind all students that physics is the branch of science and must be aware of the fact that our modern world is based on concepts from science, modern machines/equipment rely on science and inventions such as telephones, satellites, etc. are based on science. But in reality, science is a way of thinking that only accepts the rule of reasoning. In which the judgment of truth and falsehood is based on the sets of results from an experiment. So as mentioned earlier physics is the branch of science that is often termed as the “queen” of science and is rightly said to be the greatest science.

The history of science is as old as the story of mankind. When did it start? It is probably tens of thousands of years old. Every civilization has contributed to it. It may have started from the time of Babylon, but after that, the Greeks created great perfection in it, and then Chinese, Hindu and Islamic civilizations advanced it. Physics in its present form is not that much old. It started about three and a half hundred years ago, and it was a time when a great scientific revolution took place in Europe, which is called the Scientific Revolution, and this was the time when great scientists like Newton revolutionized it which is the reason why our present-day world is so different from previous eras.

Physics is related to every worldly thing (actually everything in the whole universe) with its main purpose being to fully understand the whole physical world/universe and it means that everything, no matter how big or small it is. For example, if we look at our solar system (as shown in Figure 1.1), the sun is at the center and planets revolve around it. Human beings have thought a lot over many fundamental questions like, how heat is produced in the sun and then how it reaches the Earth. Why and how does the Earth rotate around the sun? Contrary to this, in general, if anything moves it moves in a straight line but the earth continues to move around the sun in a circular path.

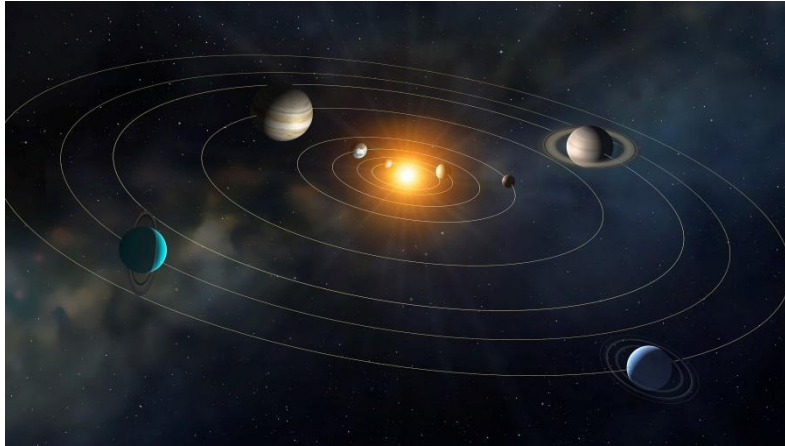


Figure 1.1. Schematic of Solar System. Sun at the center and planets revolving around the Sun.

Atom was considered to be the smallest thing in the world making up all the matter. The atom was considered to be a basic building block but now we know that it can be further broken down as it has a center called the nucleus and electrons revolve around the nucleus as shown in Figure 1.2(a). Upon closer inspection of the nucleus Protons and neutrons can be observed. Interestingly protons and neutrons can be further subdivided into particles called quarks as shown in Figure 1.2(b).

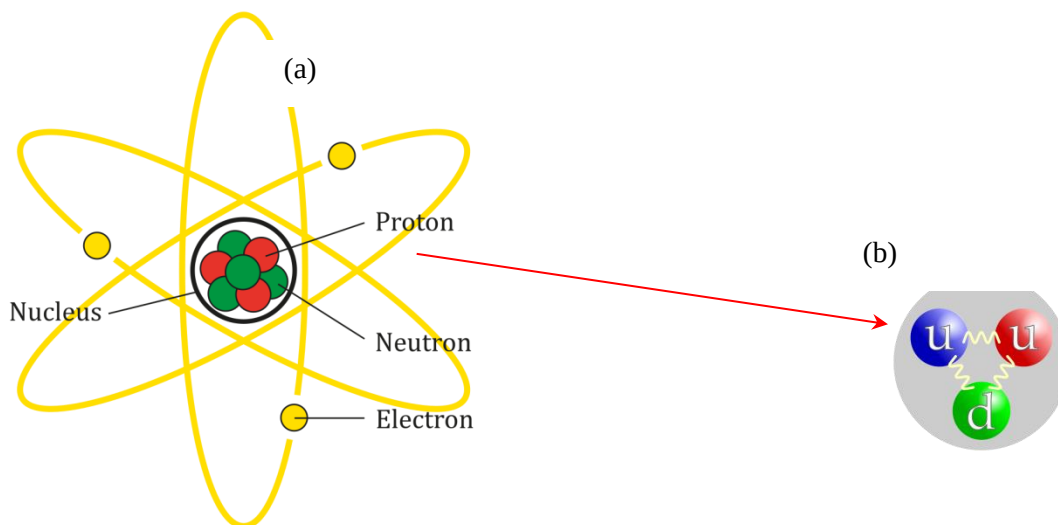


Figure 1.2. (a) Atomic Structure showing nucleus at the center having Proton and Neutron. Electrons revolve around the nucleus. (b) Protons can be subdivided into quarks.

So, on the one hand physics is related to the sun, the solar system, galaxies, and the universe itself, and on the other hand, it is also related to atoms and subatomic particles, and lastly all kinds of entities that come in between these two extremes. These entities can be classified as forms of matter which are as follows

1. Solids
2. Liquids
3. Gases

The first form of matter is called solids in which atoms can't move very far from each other for example table salt. The sodium atom in the salt cannot move away from the chlorine atoms. Atoms are fixed to each other and immobile as shown in Figure 1.3.

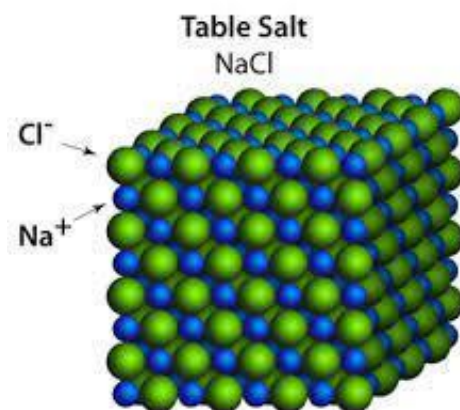


Figure 1.3. Structure of common table Salt (Sodium Chloride, NaCl)

Matter can be in a liquid state and when the substance is in the liquid state, the atoms or molecules that are there can move quite far from each other but still there is some kind of attraction among them confining them. For example, if we put water in a vessel, then this water takes the shape of the vessel and this is a characteristic of the matter when it is in a liquid state. Lastly, there is another situation in which matter can be in a gas state. Well, if the matter is in a gaseous state, then its atoms and molecules can move around freely anywhere and far away from each other. Gases like oxygen, hydrogen, and nitrogen make up our atmosphere and without oxygen, we cannot even breathe.

The purpose of this course is to introduce students to the vast subject of physics. Although even experts in the field only know a fraction of it, we will do our best to learn and gain knowledge related to physics in all 45 lectures. The main goal of this course is to teach problem-solving to

students. Science is not just about solving existing problems, but also about understanding and solving new and challenging problems.

To achieve this goal, students must listen to all video lectures, read all materials (such as handouts and referred books) given for this course, and complete the assignments (samples) provided after going through the materials. It's essential to pay attention to the assignments since they will help students to apply the knowledge they have gained.

Studying physics has multiple purposes, one of which is to gain knowledge for future courses. In every field of engineering, ranging from bridge construction to designing electric components, fundamental physics knowledge is required. However, it is crucial to develop the habit of critical thinking and evaluation based on intellect.

Science relies on reason, logic, and experience to determine the truth or falsehood of a claim. Physics is often considered a difficult subject because it involves mathematics. Calculus is required, but we will develop as much calculus as we need. Knowledge of algebra, trigonometry, linear equations, or quadratic equations is necessary prerequisite knowledge.

1.2. Main Areas of Physics:

Following are the main areas of physics

1. Classical Mechanics
2. Electricity and Magnetism
3. Thermal Physics
4. Quantum Mechanics

First of all, classical mechanics is the field of physics, which is attributed to Isaac Newton. It is related to objects, the movement of objects, momentum, force, and energy, these are all concepts that come under classical mechanics. After this, we will talk about electricity and magnetism. Now, electricity and magnetism are not so different from each other and this is a discovery of James Clerk Maxwell from about 200 years ago. And now all modern technologies in the world (telephone, radio, television etc.) are based on these discoveries. So, we will study electricity and magnetism after classical mechanics. When we have this much background, then we will come to thermal physics, i.e. the physics of heat. Concepts like temperature and entropy will be discussed.

In these three areas, classical mechanics, electricity and magnetism and thermal physics, we will spend a lot of time on them. But there is another field, which is more related to the physics of atoms. We will not be able to spend much time on this because to understand it and to study it correctly, you need special mathematics. So, my advice to those people who want to study physics further is that you should study mathematics separately.

1.3.Dimension:

Just like a house is built with bricks, in every field of physics, everything is built with three types of bricks. These are called dimensions. Dimension is a concept which is about 200 years old and it was proposed by a French scientist named Joseph Fourier. Now, let us focus on the three most important fundamental dimensions¹.

1. Time T
2. Length L
3. Mass M

First of all, there is a dimension of time. What is the reality of time? Now, philosophers have been discussing this for centuries, but scientists and physicists are not very interested in these discussions. In physics, we want to know how to measure a thing or a dimension. There is a method to measure time which is called a clock. Now, clocks can be of different types. For example, my pulse is also a clock although not an accurate one. When I stand here, my pulse moves at one speed but when I run, my pulse increases. But there is a better clock, a pendulum. Now, the pendulum moves to and fro. And every time it moves to and fro, the time duration is the same. The earth rotates around its axis and when it rotates, it comes back exactly to the same position after 24 hours. Similarly, when the earth rotates around the sun, we say that one year has passed. The best clock is called the atomic clock which is based on the regular vibration of an atom. With this, we can measure time with better accuracy than one part in a billion. To measure length, normally we need a ruler but its accuracy is not very good. In modern ways, we measure with the wavelength of atoms. So, accuracy is much better as compared to the ruler. The third fundamental dimension is mass, which tells us how much matter is present in a body. For example, if I have an apple and I add another apple to it, the mass will double (assuming both apples are identical). The more mass

¹ [Fundamental and Derived Dimensions](#)

there is, the more difficult it will be to move. For example, if I put equal force on a rickshaw and a taxi, the rickshaw will move faster and the taxi will move slower. So, mass tells us how much resistance to motion there is.

Now, let us look at dimensional quantities, which are made up of mass, length and time. For example, the dimension of the area is $L \times L$ or L^2 . For a volume, it will be L^3 . For example, for a box, its length multiplied by width multiplied by height will give its volume having the dimension of L^3 . We can make another thing from this, which is called density. The dimensions of density will be mass per unit volume or M / L^3 . In the same way, we find out the dimensions of frequency. What is frequency? For example, there is a pendulum that completes 2 cycles in a second. So, its frequency is 2. Or, if we consider alternating current (AC) of electricity, which changes its polarity 50 times every second. So, its frequency is 50. Frequency has a dimension of $1/T$ or T^{-1} . The dimensions of speed are L/T or LT^{-1} . But there are some quantities which have no dimensions such as angle. The angle between two adjacent fingers is dimensionless. Degrees are units not the dimension of the angle². In short, there are seven fundamental dimensions as given in the table. And everything in physics, every other quantity can be constructed in terms of these. Fundamental set dimension is given below in form of table.

Fundamental Quantity	Dimension	SI Unit
Length	[L]	Meter (m)
Time	[T]	Second (s)
Mass	[M]	Kilogram (kg)
Temperature	[θ]	Kelvin (K)
Electric Current	[I]	Ampere (A)
Amount of Substance	[N]	Mole (mol)
Luminous Intensity	[J]	Candela (cd)

² Dimensions refer to the fundamental physical properties that describe a quantity. They are abstract and universal, not tied to any specific measurement system. In physics, there are several fundamental dimensions, such as length (L), mass (M), and time (T). Units, on the other hand, are specific measurements that quantify the magnitude of a physical quantity. Units are used to express how much of a particular dimension is present in a given quantity. Dimensions represent the abstract properties of a quantity, while units are specific measurements that combine a numerical value with dimensions to quantify that property. The combination of dimensions and units provides a complete description of a physical quantity in a specific measurement system.

1.4. Units

As per the previous example, there is an angle between my two adjacent fingers, and measure this angle in terms of degrees or radians. So now we will need a system of units. There are two different unit systems. The first one is called the MKS system. MKS means meter, kilogram, second system. where the length is measured in meters. The mass is measured in kilograms. And the time is measured in seconds. Then there is another system called the CGS system. CGS means centimeter, gram, second system. Here, the length is measured in centimeters. The mass is measured in grams. And the time is measured in seconds, like MKS.

Approximate lengths in meters (m) of a few entities are given below in the table

Distance	Length (m)
Tall Person	2×10^0
Cricket Ground	3×10^2
Radius of Earth	6.4×10^6
Earth to Sun	1.5×10^{11}
Radius of Universe	1×10^{26}
Thickness of Paper	1×10^{-4}
Diameter of Hydrogen atom	1×10^{-10}
Diameter of Proton	1×10^{-15}

For very large lengths, scientific notation is required and a power of 10 is used. So, for example, the size of the earth is 6.4×10^6 meters and the distance of the sun from the Earth is about 1.5×10^{11} meters. Similarly, the universe is very vast, but today we know that it is not unlimited, it is limited and its radius is about 1×10^{26} meters. Now, on the other hand, there are many small things in the world. For example, the thickness of a piece of paper is about 10^{-4} meters and atom,

for example, a hydrogen atom, its radius is about 10^{-10} meters. The proton in it is 100,000 times smaller than that and its size is 10^{-15} meters. So, these are different scales of length.

Similarly, there are different scales of time and mass. The approximate time for particular events is given below.

Event	Time (s)
Light travels from Earth to the moon	1.3×10^0
One hour	3.6×10^3
One year	3.2×10^7
Age of universe	5×10^{17}
Open/close eyelid	1×10^0
One cycle of radio wave	1×10^{-8}

Light moves at a very fast speed, but light also needs time to go from Earth to the moon and this is about 1.3 seconds, i.e. 1.3×10^0 s. In an hour, there are 3600 seconds, i.e. 3.6×10^3 s. In a year, there are 3.2×10^7 s. Today, we also know that the universe was formed about 15 billion years ago and if written in seconds, then it becomes about 5×10^{17} s. On the other hand, there are such events in which the duration is very short. For example, if you blink your eyes, you can do this twice in a second, i.e. the frequency is 2 per second. For a radio wave (electromagnetic wave) in which the descent and ascent happens 10^8 in a second, i.e. its time duration is 1×10^{-8} seconds.

There are also different scales of mass as listed in the table below.

Object	Mass (kg)
Student	7×10^1
Car	1×10^3

Ship	1×10^6
Earth	6×10^{24}
Sun	2×10^{30}
Milky Way (Galaxy)	4×10^{41}
Dust Particle	1×10^{-9}
Oxygen Atom	3×10^{-25}
Electron	9×10^{-31}

For example, the weight of an ordinary person is about 70 kilograms, the weight of a car is about 1000 kilograms, while the weight of a ship is 1000 times more, i.e. 1×10^6 kg. It is worth noting that powers of 10 are used to write very large as well as very small quantities. Earth's weight is 6×10^{24} kg and the sun are about 1 million times heavier; its weight is 2×10^{30} kg and if we calculate the weight of our galaxy, then that is about 4×10^{41} kg. Dust particle barely weighs 1×10^{-9} kg. An atom of oxygen is much lighter than this as its weight is about 3×10^{-25} kg and an electron is 1000 times lighter than that, and its weight is 9×10^{-31} kg.

1.5. Conversion of Units

Going from a one-unit system (like MKS or CGS) to another system is called conversion of units. There are some useful conversion factors i.e. 1 inch is equal to 2.54 centimeters and 1 meter is equal to 3.28 feet. Sometimes it is required to convert other quantities as well. For example, converting miles per hour (mi/hr) into meters per second (m/s). 1 mile per hour (mi/hr) can be easily converted to meter per second (m/s) using only two following conversions and cross multiplications:

$$1 \text{ mi} = 5280 \text{ ft} \rightarrow 1 = \frac{5280 \text{ ft}}{1 \text{ mi}}$$

$$1 \text{ m} = 3.28 \text{ ft} \rightarrow 1 = \frac{1 \text{ m}}{3.28 \text{ ft}}$$

$$1 \text{ hr} = 3600 \text{ s} \rightarrow 1 = \frac{1 \text{ hr}}{3600 \text{ s}}$$

So, multiplying all the above unities by $1 \frac{mi}{hr}$

$$1 \frac{mi}{hr} \cdot 1 \cdot 1 \cdot 1 = \left(1 \frac{mi}{hr}\right) \cdot \left(\frac{5280 ft}{1 mi}\right) \cdot \left(\frac{1 m}{3.28 ft}\right) \cdot \left(\frac{1 hr}{3600 s}\right)$$

By cancelling units, we get

$$1 \frac{mi}{hr} = \frac{5280}{3.28 \times 3600} \frac{m}{s} = 0.477 \frac{m}{s}$$

This was also an example of dimensional analysis. Even though units were of miles per hour on the left side and meters per second on the right side, both were length over time [L/T]. So, the dimensions were equal in both cases. So, it becomes very important that the dimensions on the left side should be on the right side in every equation. If this is not the case, then it means that the equation cannot be correct.

For example, let's assume the following equation

$$d = vt^2$$

Where d is distance, v is velocity and t is time. Dimension on the left side of the equation is [L] and on the right side, it is $[LT^{-1}] [T^2] = [LT]$ which is not equal to the left side so this equation is incorrect. Explain whether the following equation(s) could possibly be right or wrong based on Dimensional analysis.

$$v^2 = u^2 + at$$

$$v^2 = u^2 + 3a^2t^2$$

Here v and u are velocities, a is acceleration, and t is time.

1.6. Rules of Dimensions

Dimensions (M, L, and T) can be considered as algebraic quantities. Likewise, symbols and dimensions can be added and subtracted but rules differ from conventional algebraic rules. Multiplication and division are applicable to dimensions. For example, in the case of length divided by length [L/L] dimension will cancel and the result will be a dimensionless quantity. Dimension for speed can be simplified to $[LT^{-1}]$ based on division. So [M], [L] and [T] can be

divided and multiplied but cannot be added or subtracted from each other. For example, $[L] + [T]$ is physically not realizable. Even $[T^2]$ cannot be added to $[T]$. So, addition and subtraction of the same dimension is only possible.

1.7. Accuracy

Whenever we measure a quantity its accuracy cannot be 100%. For example, if a certain length is measured using a conventional ruler to be 2.95 centimeters (cm), it cannot be measured as 2.952467 cm. This is because of the fact that measuring equipment always has inaccuracy associated with it so significant figures become relevant. It is not advised to do calculations using certain equipment that misrepresents their accuracy.

For example, if you are asked to measure the weight of 3 apples and then calculate the average weight. Its average weight cannot be measured at 550.234 grams. This is not possible because the equipment being used to measure weight is not 100% accurate. For this reason, we need to be aware of the fact that what level of accuracy is required for the measurement of an algebraic or an arithmetic quantity. Only an appropriate and reliable number of digits should be used for the measurement process.

If length is being measured by a ruler, then there can be at most 1% accuracy, not more than that. Some general rules should be considered for calculations. For example, when two numbers are added together if one number has an accuracy of 95.9 and the other has an accuracy of 39.32, then you add them together and it becomes 135.22. But it would be more appropriate to call its accuracy 135. So, there is no point in adding the decimal figures to it. It would not be of any benefit. When subtracting when multiplying and when dividing, for example, multiplying, one number has an accuracy of one figure of decimal i.e. 105.8 and the other has an accuracy of 31.4. If we multiply both, then we have six figures. But we should round off this. This means that after the decimal figure, the 0.12 does not have any meaning. We should leave it. So, if we divide 105.8 by 31.4, then we will get many decimal figures. But it would be appropriate to cut it off at 3.37 because the two numbers are known only to one decimal figure of accuracy. Therefore, 5 decimal figures after 3 do not give much information.

1.8. Orders of Magnitude:

Order of magnitudes means that we want to approximately calculate a quantity. In physics, it is often not possible to do exact calculations. Either we do not have enough information or the problem is very difficult or whatever. So, it is extremely important that we can estimate. Now, we would like to discuss the order of magnitude. Order of magnitude means that when we write in powers of 10, then if it is written near the nearest power of 10. Let us give you an example. A man's weight is approximately 100 kilograms. Now, I agree that 100 kilograms is of very few people. Usually, it is half of that. But writing it as 10 to the power of 2 kilograms can be useful in some situations. Similarly, if we take the weight of a child, then it can be approximately 10 kilograms. And if you take a cricket ball, then its weight is approximately 1 kilogram. Now, here we have only talked about the closest powers of 10. So, we are not saying that this is accurate, but as mentioned earlier, whenever an estimate is required, we can use this method.

Let's have a look at an example. How many seconds are there in a person's life? Let's calculate it. If a person is 80 years old, then you can calculate it from 365 days per year. Next, there are 24 hours in a day. And there are 60 minutes in an hour. And there are 60 seconds in a minute. So, calculate by multiplying these conversion factors (i.e. $80 \times 365 \times 24 \times 60 \times 60$). As a result, we will get 2.5×10^9 seconds. On the other hand, if we estimate it, then assume a person's life is 100 years for the purpose of approximation. There are 365 days in a year but we assume 100 days per year which is the closest power of 10 and so on as shown in the table below. Upon multiplication of all conversion factors, we get 10^9 seconds. On one hand, we did a very accurate calculation to get 80 years which is equivalent to 2.5×10^9 seconds. And on the other hand, if we make these coarse estimates, then there is not much difference. In both cases, we get 10^9 seconds only. Although in one calculation we get 2.5 and in the other we get unity. But this difference is very small. Approximate reasoning sometimes gives us very useful results.

Conversion Factor	Closest Power of 10	Closest Power of 10 (scientific notation)
80 yr	100 yr	10^2 yr
365 days/yr	100 days/yr	10^2 days/yr
24 hr/day	10 hr/day	10^1 hr/day
60 min/hr	100 min/hr	10^2 min/hr
60 s/min	100 s/min	10^2 s/min

There once a news that armed men robbed 10 crore rupees (100 million rupees) from shopkeepers and ran away on a motorcycle. Now the question arises, can such a large amount be put on a motorcycle or not? To find out, let us do a small experiment. A balancing scale (as shown in the video lecture) is used to measure the weight of 10 currency notes each having a value of 1000 rupees. As seen in the video lecture, the weight of 10,000 rupees (10 notes $\times$ 1000 rupees/note) is 20 grams. This means that the weight of 100,000 rupees is 200 grams, which is equal to 0.2 kilograms. And the weight of 1 crore rupees is 20 kilograms. The weight of 10 crores is 200 kilograms. Now 200 kilograms is equal to the weight of 3 people. And it is not possible that such a large amount can be put on a motorcycle other than two people riding it. Therefore, there is something doubtful about the whole incident. This was a very ordinary example of scientific methodology. However, we will see in the upcoming lectures that this methodology is used in every field of physics, and it is called the scientific method.

Observations are made based on which an initial thought is created. After that, the hypothesis is born in the mind. For now, we do not know whether this hypothesis is right or wrong. Now, to test it, its predictable outcome has to be compared with more observations. It has to be experimentally verified. When a law is tested again and again and it is seen again and again that it is successful, means the hypothesis combined with the observations and they connect, then it gets the status of scientific theory. We will give you an example of Newton's theory of gravitation and Newton's theory of motion. Now, we will see in the next lectures, those lectures which are related to classical mechanics and which are associated with Newton's theory, what phenomena are achieved by those. But as we know, Einstein who came after Newton proved that Newton's theory cannot be completely correct. So, does this mean that Newton's theory is wrong? No, this does not mean at all. Now, if you apply Newton's theory to any particle, as long as that particle is not moving close to the speed of light, that is, it is not moving at this extreme speed, then Newton's theory can be used very easily and effectively. For example, when we send a spacecraft to a planet far away from the Earth, then we use Newton's laws. We do not need to use Einstein's laws. The theory of relativity created by Einstein is used only in those cases where particles' speed is close to the speed of light. For example, in research of particle physics particle accelerators are used and electrons move at a speed close to the speed of light.

Finally, I will say something which may seem a little strange to you. In this lecture, I emphasized that physics is related to the material world and that its purpose is to find out the rules and laws that are the basis of our universe. So, we are talking about materialistic things. But on the other hand, the concepts of physics are abstract and only exist in our minds. These concepts are the creation of our minds. Now, we would like to clarify this a little. In the upcoming lectures, you will hear again and again that this is a free particle, a free body, that this is a point, that this is a straight line. But there is no such thing as a free body. We say that there is nothing free on Earth because gravity pulls that body towards the Earth. So, take it a little farther from Earth, take it to the outer atmosphere. How far should we take it? Take it 1 crore miles away from the earth. Take it 100 crore miles away. But that will not be enough because there will always be a minute force that pulls that body towards the earth.

And if we take it too far, then there are galaxies and other planets, therefore, a completely free body does not exist. It exists only in one place and that is in our brains. And similarly, if we talk about a dot. We assume that a dot is a very tiny thing and we can make it by putting a pencil on paper. But if we look at it under a microscope, the higher the magnification bigger the dot we will observe. Now a pencil's tip cannot be made so fine because a pencil's tip itself is made up of atoms and atoms themselves are not dots. Therefore, there is no dot in the world and there is no straight line either. The point is that the concepts of physics are in one's mind. Therefore, these concepts are created and evolve, which is why physics is a progressive discipline. In which there is progress all the time. And this is the most interesting thing about physics. So, I hope that in the next lectures, you will feel that this is an interesting topic. That there is progress in it and that it invokes thinking. To learn this subject, you will have to work a little hard. As long as you do not solve problems by yourself. You have to believe that you can solve every problem by yourself. And the more problems you solve, your physics skills will increase accordingly. Secondly, do not assume that you will be able to solve all your problems with a single course be it this course or any other course. There is a vast ocean of knowledge in this world and we can pick up a drop of it at one time. It is a lifelong effort to understand physics and science. And we hope that you will also be a part of this effort. The journey of physics will continue in the next lecture. We will meet again at that time. With your permission, goodbye.

Physics-PHY101-Lecture #02

Kinematics

"**Kinematic**" is derived from the Greek word "kinesis," which means to move or motion. Our world is full of motion. For example, cars move, birds can fly, and fish can move underwater. It is necessary to understand motion and its causes.

The main topics of this lecture are:

- Displacement
- Velocity
- Acceleration

Whenever we discuss the motion of a body, it is essential to locate its position. To achieve this, we need an origin point. For example, a point x has a value of $x = 0$ when stationary, but as it starts moving, its values begin to increase.

If the position of the vector x depends on " t ," then we can express its position as $x(t)$, where x is referred to as a function of " t ." (Think of a function as a factory where you input raw material and receive products, like inputting a number into a function and obtaining a result as output.)

In kinematics, we require a specific function denoted as $x(t)$. This function allows us to input a value of t and obtain the position of a moving particle.

Now, let's delve into the concept of "Displacement."

Displacement:

As we know, the position at time t is denoted as $x(t)$. Here, " t " can be any number, such as 9, 7, 6, etc. Similarly, when a particle moves at different times " t ", it may be denoted with different notations, such as $x(t_1)$ and $x(t_2)$, and its function may be defined as $x(t)$. Now, when we subtract these two quantities, we will obtain a value Δx . The displacement Δx in time interval $\Delta t = t_2 - t_1$ is:

$$\Delta x = x(t_2) - x(t_1)$$

Sometimes we don't want to write equation as $x(t_1)$ and $x(t_2)$ so we do write the above equation as:

$$\Delta x = x_2 - x_1$$

This is what we call displacement. Now consider that a particle is moving in one dimension. Imagine a graph of particle's position along the x -axis as a function of time.

From the Figure. 2.1 we can see that when the particle is at position x_1 the time is t_1 and when at x_2 the time is t_2 .

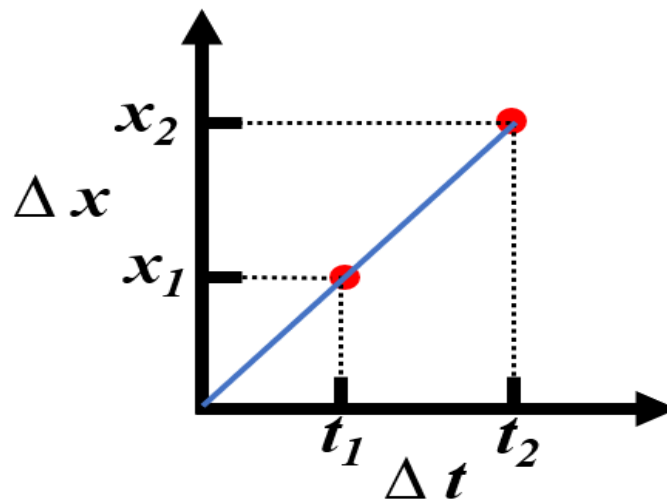


Figure 2.1. Displacement vs. time graph of an object.

If an object as shown in figure 1. is at 10 m from origin at t_1 and reach at 30 m at time t_2 , then, we can define displacement Δx as:

$$\begin{aligned}\Delta x &= x_2 - x_1 \\ &= 30 - 10 \\ \Delta x &= 20 \text{ m}\end{aligned}$$

But if the object at t_1 is at 30 m and reach at 10 m at t_2 , then the magnitude will remain same but negative sign appears due to net displacement in negative direction.

$$\begin{aligned}\Delta x &= x_2 - x_1 \\ &= 10 - 30 \\ \Delta x &= -20 \text{ m}\end{aligned}$$

This is what we call displacement, which can be both positive and negative.

Speed and Velocity:

Speed and velocity measure how position changes with time. There are two major concepts to consider. First, let's look at average speed:

$$\text{Average speed} = \frac{\text{Total distance}}{\text{Total time}}$$

If we divide the particle's total distance travelled by the total time taken, we can determine the average speed of that particle. Average speed can vary, being either maximum or minimum. Meanwhile, average velocity can be defined as:

$$\text{Average velocity} = \frac{\text{Total displacement}}{\text{Total time}}$$

$$= \frac{x_2 - x_1}{t_2 - t_1}$$

$$v = \frac{\Delta x}{\Delta t}$$

Note that distance is always positive, while displacement can be positive or negative. It's important to clarify that this is average velocity because we are dividing displacement by time, not distance by time. This is also known as slope or gradient. For example, consider a car moving on a slope – the steeper the slope, the greater the gradient.

Now, consider the Figure. 2.2 (a) where the distance covered by two points creates a slope from which average velocity is calculated.

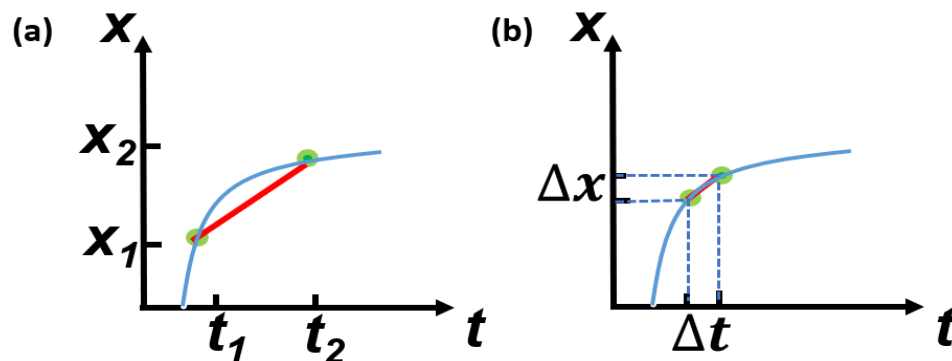


Figure 2.2. Displacement vs. time graph of an object presenting (a) constant velocity, (b) Instantaneous velocity.

In contrast, **instantaneous velocity** is determined by taking two times very close to each other, with the interval between them approaching zero. In Figure 2.2 (b). if we take two values so close to each other that they approach zero, we can call it instantaneous velocity. Now, let's discuss what is meant by "being too close." Being too close implies approaching a distance of zero. (It's important to note that approaching zero doesn't mean you are making the distance zero.)

Acceleration

Acceleration measures how the velocity changes with time. As we have defined average velocity, we can also define average acceleration as:

$$\text{Average acceleration} = \frac{v_2 - v_1}{t_2 - t_1} = \frac{\Delta v}{\Delta t}$$

We can say that average acceleration is change in velocity divided by time taken to undergo that change. It is also known as slope of graph of velocity against time.

$$a = \lim_{\Delta t \rightarrow 0} \frac{\Delta v}{\Delta t}$$

Now to understand definition better we will again construct a graph as shown in Figure. 3:

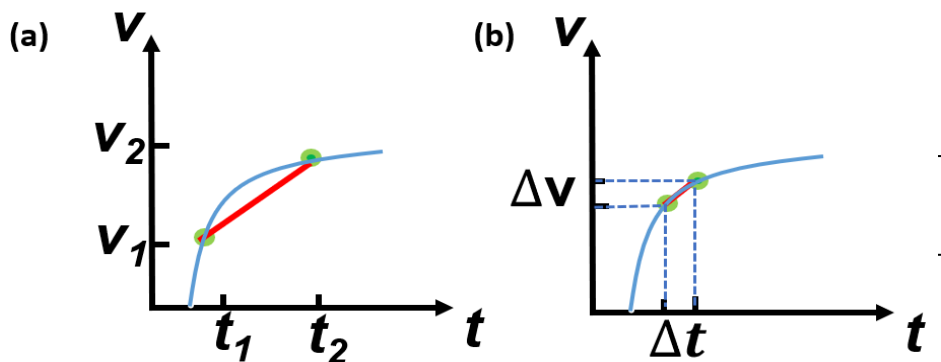


Figure 2.3. Velocity vs. time graph of an object presenting (a) constant acceleration, (b) Instantaneous acceleration.

When t_1 and t_2 come closer, v_1 and v_2 also come closer and we draw a tangent line at that point which is known as slope of tangent acceleration as shown in Figure 2.3 (b). Remember that we are approaching t to zero, not making it zero.

The acceleration can be positive and negative. Moreover, it is not necessary for acceleration to be in the direction of velocity. They can have different direction as shown in Figure 2.4, in which velocity of train is decreasing as time passes which presents deacceleration. By convention we take positive x-axis in right direction and negative x-axis in left direction. If the velocity of an object is increasing in positive x-axis direction than its **acceleration is positive**. If the velocity of the object is increasing in negative x-axis, then its **acceleration will be negative**, but it never be deacceleration or retardation. The SI unit of acceleration is m s^{-2} .

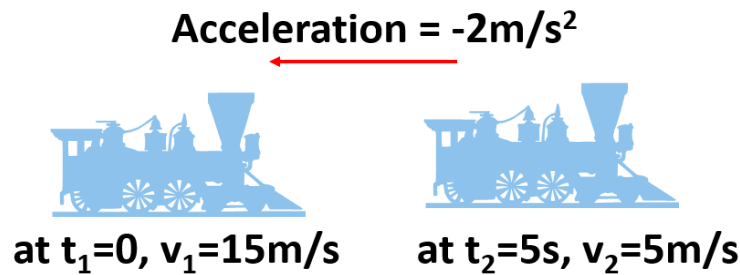


Figure 2.4. The negative acceleration of a train.

Constant Acceleration

When an object's acceleration stays constant over time, it's referred to as having constant acceleration. Stated differently, the object's velocity changes at a constant rate.

Mathematically:

Let us talk about constant acceleration in more detail. For this purpose, let's do simple calculations.

For convenience, take:

$$t_1 = 0, t_2 = t$$

Then:

$$x_1 = x_o \text{ and } x_2 = x$$

$$v_1 = v_o \text{ and } v_2 = v$$

At time t_1 , the velocity of particle is v_1 which is v_o , and at time t_2 its velocity becomes $v_2 = v$.

If acceleration is constant, then we can write the average acceleration as:

$$a_{vg} = \frac{v_2 - v_1}{t_2 - t_1}$$

And as acceleration is constant, so, v is equals to,

$$v = v_o + at \dots\dots\dots (2)$$

$$\therefore a = v/t, \text{ so } v = at$$

The average velocity as “it is the average of v and v_o divided by 2”.

$$v_{av} = \frac{1}{2}(v_o + v)$$

If we want to write in previous notation, then:

$$v_{av} = \frac{x_2 - x_1}{t_2 - t_1}$$

If we substitute values from equation given above, then:

$$\frac{x - x_o}{t - 0} = \frac{v + v_o}{2}$$

$$\frac{x - x_o}{t} = \frac{v_o + at + v_o}{2}$$

$$\frac{x - x_o}{t} = v_o + \frac{1}{2}at$$

$$x = x_o + v_o t + \frac{1}{2}at^2$$

As we have discussed earlier that x is function of t . So, by putting $t = 0$ in above equation, we will have value of x as x_o , which is the initial point from, we started.

From all that procedure we have got two main equations:

$$x = x_o + v_o t + \frac{1}{2}at^2 \dots\dots (A)$$

$$v = v_o + at \dots\dots\dots (B)$$

Now let's plot again them on a graph for better understanding:

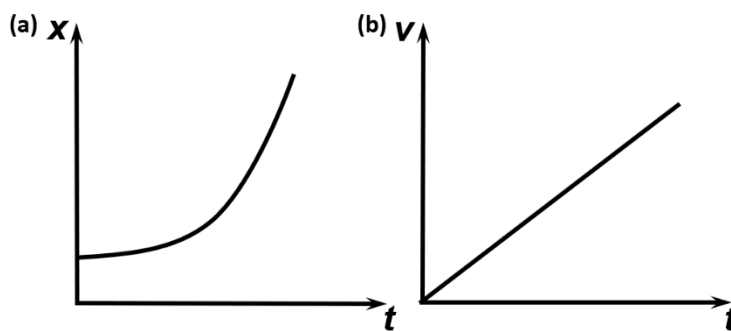


Figure 2.5. (a) Displacement vs. time, (b) velocity vs. time graph of an object with constant acceleration.

From the graphs in Figure. 2.5 it can be observed that as the time increases particle start moving away from the origin and its velocity increases linearly, while its displacement increases quadratically.

From equation (A) and (B),

$$t = \left[\frac{v - v_o}{a} \right]$$

$$x = x_o + v_o \left[\frac{v - v_o}{a} \right] + \frac{1}{2} a \left[\frac{v - v_o}{a} \right]^2$$

$$x - x_o = \frac{v_o v - v_o^2}{a} + \frac{v^2 + v_o^2 - 2v v_o}{2a}$$

$$x - x_o = \frac{2v_o v - 2v_o^2 + v^2 + v_o^2 - 2v v_o}{2a}$$

$$2a(x - x_o) = v^2 - v_o^2$$

$$v^2 = v_o^2 + 2a(x - x_o)$$

Where v is function of x . This equation tells us when the value of x changes, the value of v also changes.

Example:

For better understanding, let's consider an example with cars. In the context of cars equipped with a speedometer ranging from 0 to 160 km/h, the initial state of the car is at rest, denoted as $v = 0$. When the car is started, its speed, let's consider it as $= 70$ km/h, increases to this value and moves at this speed for some time. Afterward, it changes its velocity to 60 km/h, indicating negative acceleration. If we bring the car to a stop, its velocity will gradually decrease and eventually become zero, signifying that the car is at rest.

Introduction to Vectors

Up until now, the discussion has focused on motion in one dimension. Now, the exploration will extend to motion in two and three dimensions. Vectors exhibit two primary properties:

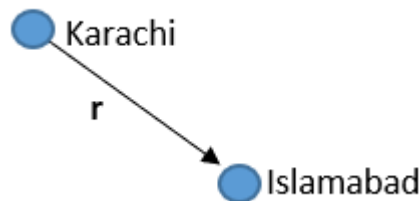
- Magnitude
- Direction

Consider the position vector r in two dimensions. For example, envision drawing a vector on a map of Pakistan, where one end (origin) is in Karachi, and the other end is in Islamabad. To achieve this, two essential considerations are needed:

- Set the origin at Karachi.

- Choose coordinates for distance (in kilometres) and direction (North, West, East, South)

Two sets of numbers or coordinates are necessary. Subsequently, two arrows will be drawn, representing vectors originating from Islamabad and Karachi.



Consider a car moving on a non-uniform surface. We will notice that its velocity is changing its direction. Although it is moving with constant speed, but its direction keeps changing. The components of vector in two dimensions can be expressed as,

$$r_x = x = r \cos \theta$$

$$r_y = y = r \sin \theta$$

The position vector is written as,

$$\mathbf{r} = (r_x, r_y) = (x, y)$$

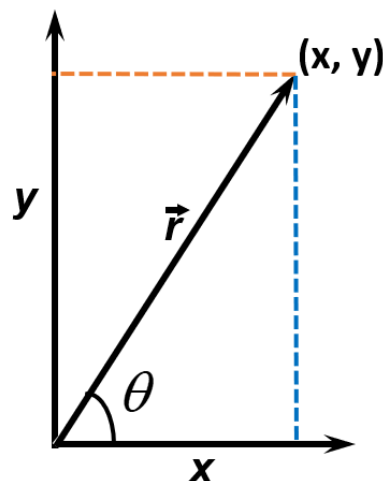


Figure 2.6. The point (x, y) is in two-dimensional plane with origin (0,0). So, we write this vector as **r** or in two coordinates (x, y). We call these coordinates as components.

For $x = r\cos(\theta)$:

From the first trigonometric identity, we have:

$$\cos(\theta) = x/r$$

Multiplying both sides by r , we get:

$$r\cos(\theta) = x$$

Therefore, $x = r\cos(\theta)$

For $y = r\sin(\theta)$:

Similarly, from the second trigonometric identity, we have:

$$\sin(\theta) = y/r$$

Multiplying both sides by r , we get:

$$r\sin(\theta) = y$$

Therefore, $y = r\sin(\theta)$.

Mathematically we write them as x and y as:

$$r_x = x = r\cos\theta$$

$$r_y = y = r\sin\theta$$

We are already familiar from these basic trigonometric formulas.

$$x^2 + y^2 = r^2(\sin^2 \theta + \cos^2 \theta)$$

$$x^2 + y^2 = r^2(1) = r^2$$

Which proves that if we square two rectangular components and then add them, we get a constant number.

Although the length of the vector doesn't depend on the direction of vectors. We can write it mathematically as:

Where $\mathbf{r} = |\mathbf{r}|$

$$\theta = \tan^{-1} \frac{y}{x}$$

Magnitude of r can be found by Pythagorean theorem.

$$|\mathbf{r}| = r = \sqrt{x^2 + y^2}$$

Where r is independent of value of angle.

Vectors can be of different types. We have discussed position vector till now. For instance, we have also discussed velocity vector with example of moving car. Acceleration is a vector itself although it made up from velocity but still it is independent of it.

Vector addition:

We can add vectors in one dimension. For example, consider a vector in one direction with a length of 10 m and another in the opposite direction with a length of 20 m.

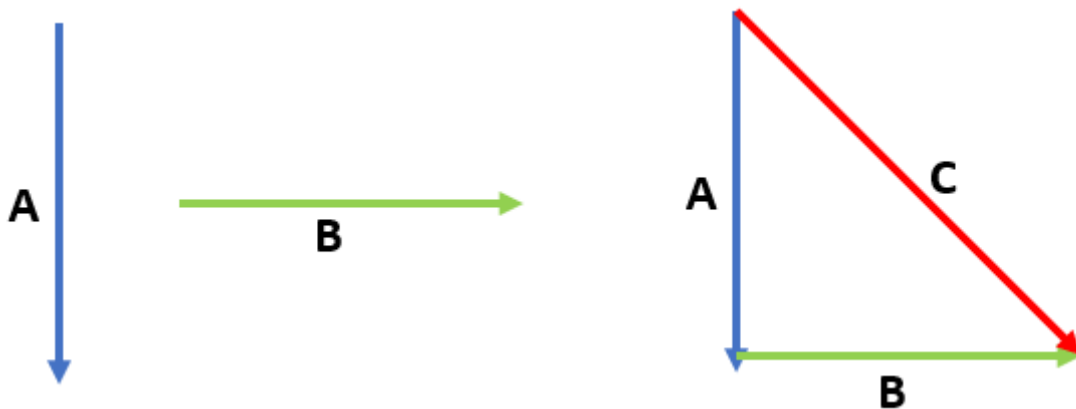
$$20 \text{ m} - 10 \text{ m} = 10 \text{ m}$$

So, adding vectors in one dimension is an easy task. Now, let's discuss two dimensions.

We want to add two vectors **A** and **B** and resultant as **C**. As,

$$\mathbf{C} = \mathbf{A} + \mathbf{B}$$

Graphically if we add these two vectors, they form a triangle with a resultant vector **C**.



We can arrange the vectors any way we want if we maintain their length and direction.

Parallelogram method for vector addition:

We can also add two vectors by parallelogram method, Continuing from previous example,

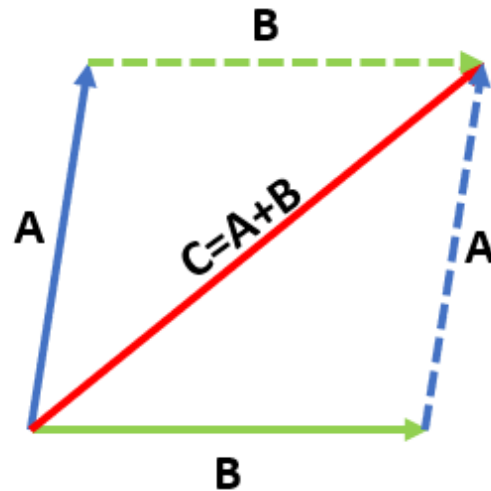


Figure 2.7. We translate the vectors A and B and form the sides of a parallelogram. The diagonal between them represents their resultant vector, denoted as C . This method is known as the parallelogram method.

In the parallelogram method for vector addition, the vectors are translated, (i.e., moved) to a common origin and the parallelogram constructed as follows:

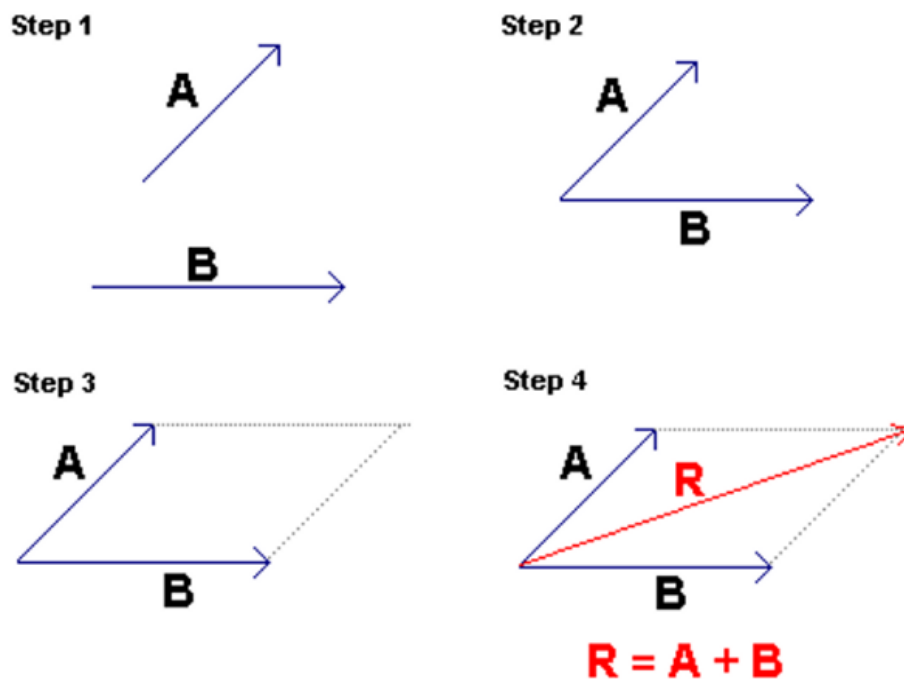


Figure 2.8. The resultant R is the diagonal of the parallelogram drawn from the common origin.

Component Method:

The components of a vector are those vectors which, when added together, give the original vector. The sum of the components of two vectors is equal to the sum of these two vectors. When we add two vectors their components do get added.

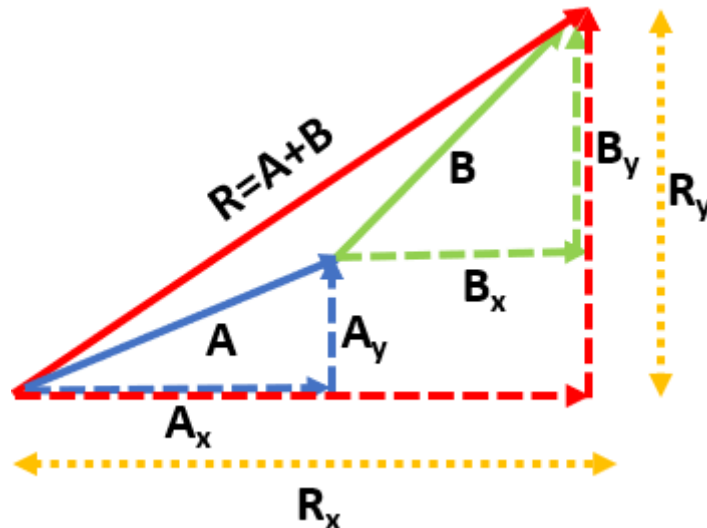


Figure 2.9. Figure presents the concept of vector addition by component method.

Here vector **A** and **B** have two components along x-axis and along y-axis we get a resultant.

vector **R** by adding these vectors components in x and y direction.

Summarizing it as,

- Add components.

$$R_x = A_x + B_x$$

$$R_y = A_y + B_y$$

Bold letters present the vector nature.

- Then calculate the magnitude from following formula,

$$R = \sqrt{R_x^2 + R_y^2}$$

- Calculate the angle by using formula.

$$\theta = \tan^{-1} \frac{R_y}{R_x}$$

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Kinematics-II

Student Learning Outcomes

After listening the lecture, students will be able to,

- Understanding Position Functions
- Analyzing Constants and Dimensions
- Exploring Derivatives and Velocity
- Motion in Two Dimensions
- Vector Operations and Applications

Here are some questions for the students,

1. Is it possible for a car to have acceleration while at rest?
2. Are velocity and acceleration always in the same direction? If a car is accelerating, does it acceleration with constant acceleration?

To comprehend these questions, one must understand the term 'motion.' In this lecture, we will delve into the concept of motion and explore some functions that elucidate this term.

Function

The function represents a machine where an input is inserted, producing an output. The emphasis is specifically on the position function, denoted as x , which is a function of time, t .

$$x(t) = c_0 + c_1t + c_2t^2 + c_3t^3 + \dots$$

Where c_0, c_1, c_2, c_3 are constants and remain fixed. 'Constant' refers to values that do not change over time. It is essential to focus initially on the dimensions on both the left and right side of the equation, ensuring their equality.

Upon observing the dimension on the left side, it is identified as length, even if t is not explicitly stated. This implies that the function describes the movement of a particle in a straight line, indicating the distance covered over time, t . So, this is a function that says the particle is moving in a straight line and the distance it covers in time t . This is what we called $x(t)$.

The dimensions must match,

$$\text{Dim}[c_0] = L$$

$$\text{Dim}[c_1] = L/T$$

$$\text{Dim}[c_2] = L/T^2$$

$$\text{Dim}[c_3] = L/T^3$$

And obviously, when the value of t is zero, then x is C_0 . So, the dimension of C_0 is equal to L . Now, focus on the dimension of C_1 . The dimension of C_1 is equal to L/T . Similarly, the dimension of C_2 is L/T^2 , and the dimension of C_3 is L/T^3 . In short, if the values of C_0 , C_1 , C_2 , etc., are unknown, then at any point, the value of $X(t)$ can be calculated. If dimensions of constants, and function $x(t)$ are known, the dimensions of “ t ” can easily be determined.

Derivatives

Derivative is the invention of Newton. In differential calculus we take the differences and then dividing them by the difference the ratio of both becomes finite. Let's try to understand the concept of derivatives, which represents the change. This change might be in position, velocity, acceleration etc. The change in position “ x ” w.r.t time is written in form of derivative is,

$$\begin{aligned}\frac{dx}{dt} &= \lim_{\Delta t \rightarrow 0} \frac{\Delta x}{\Delta t} \\ &= \lim_{\Delta t \rightarrow 0} \frac{\Delta x}{\Delta t} = \frac{x(t + \Delta t) - x(t)}{\Delta t}\end{aligned}$$

Never make the mistake of cancelling the “ d ” in numerator and denominator; this is not possible and doesn't hold any meaning.

The meaning of Δt is clear in this definition. On one value of “ t ”, you take a value of x , and then on the second value of “ t ”, you take another value of x . Then, take the difference, which we call Δx , and divide by Δt . Here we emphasize that Δt should be very small. You might ask how small? 0.1 is not enough? no. Then 0.01 is not enough either. Even if you take 0.0001, that is still not enough. But if we make it small enough so that Δt approaches 0.”

We will discuss a little detail about it now,

$$\begin{aligned}x(t) &= t \\ \Delta x &= x(t + \Delta t) - x(t) \\ &= (t + \Delta t) - t = \Delta t \\ \frac{dx}{dt} &= \lim_{\Delta t \rightarrow 0} \frac{\Delta x}{\Delta t} = 1\end{aligned}$$

From this it is clear that the function whose value is 1 is called a linear function and the derivative of linear function is constant.

Let's calculate the derivative of “ t^2 ”,

$$\begin{aligned}
 x(t) &= t^2 \\
 \Delta x &= x(t + \Delta t) - x(t) \\
 \Delta x &= (t + \Delta t)^2 - t^2 \\
 &= t^2 + (\Delta t)^2 + 2t\Delta t - t^2 \\
 \frac{\Delta x}{\Delta t} &= \Delta t + 2t \\
 \Rightarrow \lim_{\Delta t \rightarrow 0} \frac{\Delta x}{\Delta t} &= \frac{dx}{dt} = 2t
 \end{aligned}$$

This case is different from the previous case. Now dx/dt the derivative of x of with respect to “ t ” is not constant, but it depends on function. And we know that dx/dt is known as speed or velocity. As in this case dx/dt is not constant but proportional to t .

Similarly,

$$\begin{aligned}
 x(t) &= t^3 \\
 \Delta x &= (t + \Delta t)^3 - t^3 \\
 &= t^3 + 3t^2\Delta t + 3t\Delta t^2 + \Delta t^3 - t^3 \\
 \frac{\Delta x}{\Delta t} &= (\Delta t)^2 + 3t^2 + 3t\Delta t \\
 \Rightarrow \lim_{\Delta t \rightarrow 0} \frac{\Delta x}{\Delta t} &= \frac{dx}{dt} = 3t^2
 \end{aligned}$$

Now generalize it for $t^4, t^5 \dots$ etc.

consider, the function with power “ n ”, where n =integer

$$\begin{aligned}
 x(t) &= t^n \\
 \text{then:} \\
 \frac{dx}{dt} &= \lim_{\Delta t \rightarrow 0} \frac{\Delta x}{\Delta t} = nt^{n-1}
 \end{aligned}$$

Then value of dx/dt is nt^{n-1} . Look for previous cases when $n = 1$ value of dx/dt is 1. And for $n = 2$ value of dx/dt is $2t$. And for $n = 0$ the value of dx/dt should be 0. The derivative of a constant is always zero.

Geometrical interpretation of derivative

When value of t is increased to Δt , the value of x increases as well as Δx . As if dx/dt is like a gradient and in gradients, when you go a bit horizontal, you also go a bit vertical, resulting in a slope.

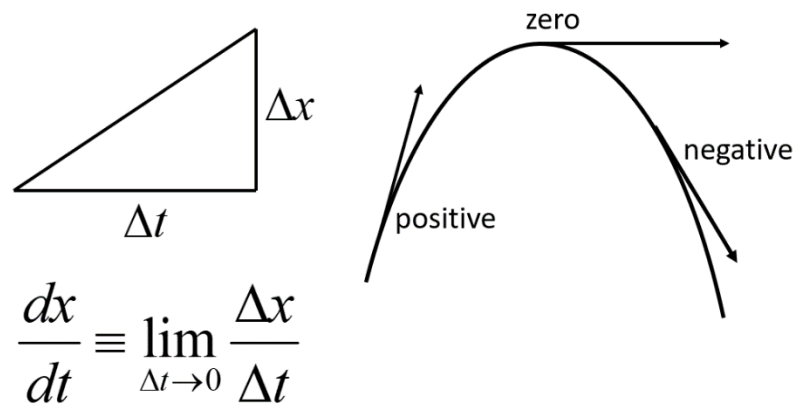


Figure 3.1. present the concept of derivatives.

In Figure 3.1. Three arrows represent the change w.r.t time. The arrow pointing upward indicates a positive gradient, meaning its derivative (dx/dt) is positive or the rate of change with respect to time (t) is positive. As we progress to the right, the value of t increases, and the arrow's direction changes. It transitions from an upward orientation to a horizontal one and eventually moves downward. Consequently, the sign of the derivative changes from positive to zero and then from zero to negative. This shift in sign signifies the geometrical significance of the derivative.

The second eq. of motion is,

$$x = x_0 + v_0 t + \frac{1}{2} a t^2$$

Now let's see how the derivative formulas we have drawn apply to this.

Now differentiate x with respect to t

$$\begin{aligned} x &= x_0 + v_0 t + \frac{1}{2} a t^2 \\ \frac{dx}{dt} &= 0 + v_0 + \frac{1}{2} a (2t) \\ \Rightarrow v &= \frac{dx}{dt} = v_0 + at \\ \frac{dv}{dt} &= 0 + a = a \end{aligned}$$

x_0, v_0 is the initial displacement and velocity, respectively.

Now answer the question asked by a student about how the car will accelerate when it is parked. A car at rest can be accelerating very fast

$$v = at$$

$$\frac{dv}{dt} = a \neq 0$$

The car is not moving at $t = 0$, then it starts to move with some acceleration. So, from here it is clear that speed is a separate thing and acceleration is separate thing. It is possible that the object is moving in a positive direction with a constant velocity, while its acceleration is directed in the negative direction. Both speed and acceleration are always considered relative to a specific origin.

Now take the example of a stone. Stone always falls downward. A stone can be at rest yet accelerating.

$$v = -gt$$

$$\frac{dv}{dt} = -g \neq 0$$

The value of "g" does not remain constant at 9.8 m/s^2 . If an object goes upward, it will decrease, and as they go further up, it will decrease even more. If an object goes out into space, the value of "g" will be zero. However, it's important to note that the units of "g" are in meters per second squared. Since we live on Earth, it would be beneficial to remember this value as 9.8 m/s^2 .

A useful notation:

$$\frac{dv}{dt} = \frac{d}{dt} \left(\frac{dx}{dt} \right)$$

$$= \frac{d^2x}{dt^2}$$

When we take second derivative of x then we write it as: $\frac{d^2x}{dt^2}$

Remember that speed is $\frac{dx}{dt}$ and acceleration is $\frac{d^2x}{dt^2}$.

Motion in 2-dimension

Let's discuss some characteristics of vectors. Each vector has a specific direction and a certain length. Now, there are special vectors known as unit vectors. Their characteristic is that their

length is equal to one, and they only indicate directions. For example, one unit vector may point in a certain direction, while another may be perpendicular to it. A unit vector is a vector with a magnitude of 1 (no units), and it is obtained by dividing a vector by its length or magnitude.

$$\hat{A} = \bar{A}/A$$

Example of unit vector are $\hat{i}$ and $\hat{j}$ in 2 dimensions. The vector $\hat{i}$ and $\hat{j}$ are perpendicular to each other, and their magnitude is 1. The resultant vector is written as,

$$\vec{A} = A_x \hat{i} + A_y \hat{j}$$

Resolution of vectors into its components

The vector component along the x direction is called x-component and the component along the y direction is called y-component.

$$\vec{A} = A_x \hat{i} + A_y \hat{j}$$

In this way, we can resolve a vector into its components as shown in Figure 3.2. Resolution of vectors means breaking down a vector into its components. We can do this in 3D too, except that it requires three vectors, $\hat{i}$, $\hat{j}$ and $\hat{k}$.

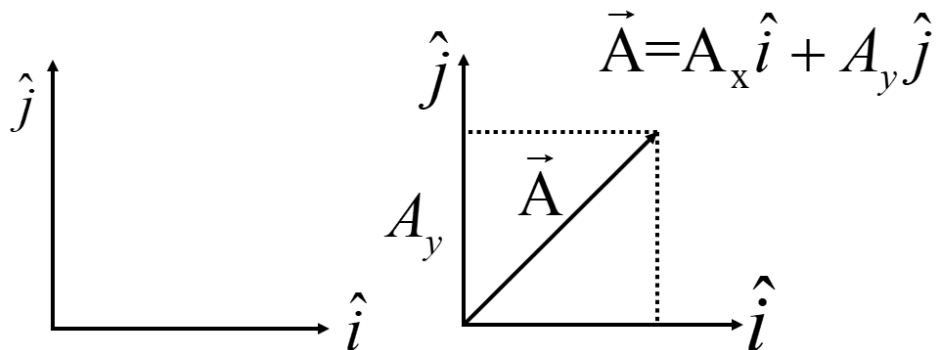


Figure 3.2. Illustrate the concept of addition of vectors.

Velocity in 2 dimensions

We have already discussed velocity in 1 dimension in detail. Velocity in 2 dimensions come from differentiating a displacement vector in 2 dimensions.

$$\vec{r} = x(t)\hat{i} + y(t)\hat{j}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j}$$

$$\vec{v} = v_x \hat{i} + v_y \hat{j}$$

Acceleration in 2 dimensions

Acceleration is the rate of change of velocity. When the velocity in 2-dimension is differentiated with respect to time, the acceleration is

$$\begin{aligned}\vec{a} &= \frac{d\vec{v}}{dt} \\ &= \frac{dv_x}{dt}\hat{i} + \frac{dv_y}{dt}\hat{j} \\ &= a_x\hat{i} + a_y\hat{j}\end{aligned}$$

Addition of vectors

Two vectors **A** and **B** can be added by head to tail rule. Mathematically,

$$\begin{aligned}\vec{A} &= A_x\hat{i} + A_y\hat{j} \\ \vec{B} &= B_x\hat{i} + B_y\hat{j} \\ \vec{R} &= \vec{A} + \vec{B} \\ &= (A_x\hat{i} + A_y\hat{j}) + (B_x\hat{i} + B_y\hat{j}) \\ &= (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j} \\ &= R_x\hat{i} + R_y\hat{j}\end{aligned}$$

Example

$$\begin{aligned}\vec{A} &= 6\hat{i} + 5\hat{j} \\ \vec{B} &= 8\hat{i} + 7\hat{j}\end{aligned}$$

What is the magnitude of $2\vec{A} - \vec{B}$?

Letting

$$\begin{aligned}\vec{R} &= 2\vec{A} - \vec{B} \\ &= 2(6\hat{i} + 5\hat{j}) - (8\hat{i} + 7\hat{j}) \\ &= (12 - 8)\hat{i} + (10 - 7)\hat{j} \\ \vec{R} &= 4\hat{i} + 3\hat{j}\end{aligned}$$

The magnitude is,

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{4^2 + 3^2} = 5$$

Consider two vectors $\vec{A}$ and $\vec{B}$ making an angle θ with each other as shown in Figure 3.3,

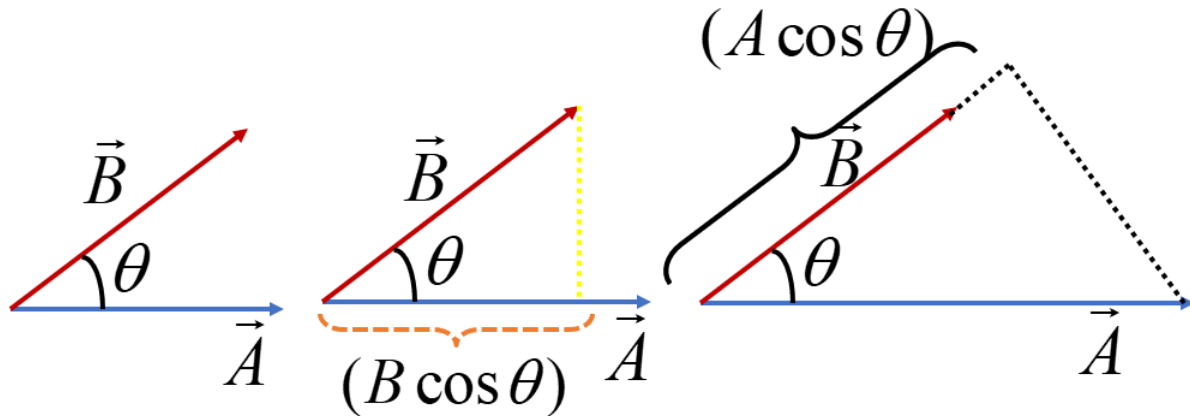


Figure 3.3. Illustrates the schematic of dot product of two vectors.

We can also write it as:

$$\begin{aligned}\vec{A} \cdot \vec{B} &= (A)(B \cos \theta) \\ &= (\text{length of } A) \times (\text{projection of } B \text{ on } A)\end{aligned}$$

Scalar product of unit vectors

The dot product of vector with itself is always 1, and dot product of two mutually perpendicular vectors is always zero.

$$\begin{aligned}\hat{i} \cdot \hat{i} &= \hat{j} \cdot \hat{j} = (1)(1) \cos(0) = 1 \\ \hat{i} \cdot \hat{j} &= (1)(1) \cos(90^\circ) = 0\end{aligned}$$

We can take dot product easily between any two vectors.

$$\begin{aligned}\vec{A} &= A_x \hat{i} + A_y \hat{j} \\ \vec{B} &= B_x \hat{i} + B_y \hat{j} \\ \vec{A} \cdot \vec{B} &= (A_x \hat{i} + A_y \hat{j}) \cdot (B_x \hat{i} + B_y \hat{j}) \\ &= A_x B_x \hat{i} \cdot \hat{i} + A_x B_y \hat{i} \cdot \hat{j} + A_y B_x \hat{j} \cdot \hat{i} + A_y B_y \hat{j} \cdot \hat{j} \\ &= A_x B_x + A_y B_y\end{aligned}$$

Now you can do all these things in the same way in 3 dimensions. There would be only one more unit vector needed which is $\hat{k}$.

Generalization in 3 dimensions

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

$$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$

$$\vec{A} \cdot \vec{B} = A_xB_x + A_yB_y + A_zB_z$$

Application:

Projectile Motion:

Consider the example of a ball being thrown. When the ball is thrown, it follows a trajectory until it reaches its destination. The ball's velocity is composed of two distinct components: one in the y-direction and another in the x-direction. Notably, the x and y components of velocity are independent of each other. Acceleration is a factor affecting the ball's motion. Upon release, the ball descends in the negative y-direction, experiencing acceleration in the y-direction. However, there is no acceleration acting in the x-direction. For the sake of clarity, let's denote the acceleration of the ball in the y-direction as " a_y ."

- Acceleration along y-axis is $a_y = -g$
- Velocity along x is constant
- Acceleration along x-axis is $a_x = 0$

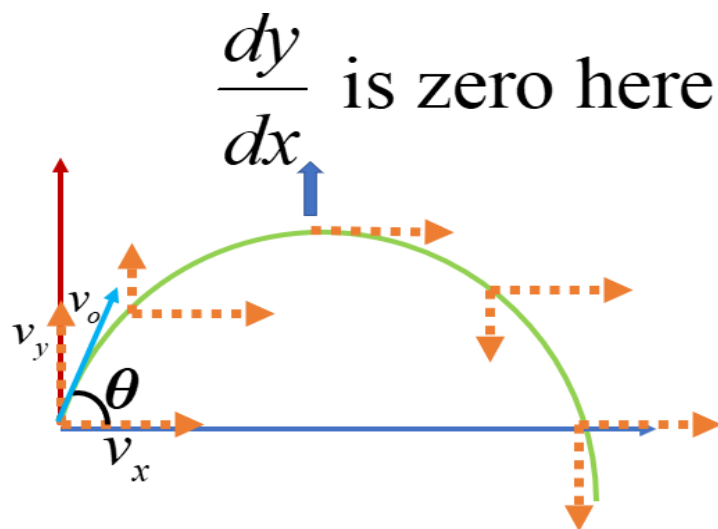


Figure 3.4. Shows the projectile motion of an object at various instants.

$$V_{0x} = V_0 \cos \theta$$

$$V_{0y} = V_0 \sin \theta$$

As presented in Figure 3.4 the velocity V of the ball is represented by two components: horizontal V_x and vertical V_y . As the ball traverses its trajectory, its velocity undergoes changes. Notably, the value of V_x remains constant because the acceleration a_x in the horizontal x direction is equal to zero, indicating no acceleration along the x -direction. In contrast, the value of V_y changes as the ball moves. At its highest point in the trajectory, V_y becomes zero. As discussed earlier, the gradient dy/dx is also zero at this point. As x increases beyond this point, dy/dx continues to be zero. Subsequently, the ball descends, causing its vertical component V_y to become negative. V_y eventually reaches zero again at a lower point in the trajectory, and further descent results in negative values for V_y . A more detailed examination using mathematical formulas will provide further insight into this phenomenon.

Along x-axis

As the variable “ x ” undergoes change, it is important to note that when “ x ” is constant, it equals the product of velocity (v) and time (t). The initial velocity “ v ” is the same as the velocity with which the ball was initially thrown. When the ball is thrown, the component along the horizontal direction remains constant when time is equal to zero. At this point, the value of “ x ” is zero, and this choice of coordinates is convenient, as it aligns with the starting point of the ball. While alternative coordinate choices are valid, setting “ x ” to zero at “ t ” not equal to zero proves to be a practical choice.

Examining this concept in mathematical terms, the relationship is described through the following formulas.

X direction

$$V_x = V_{0x}$$

$$x = x_0 + V_{0x}t$$

$$a_x = 0$$

Y-direction

$$a_y = -g$$

$$V_y = V_{0y} - gt$$

$$y = y_0 + V_{0y}t - \frac{1}{2}gt^2$$

How much its velocity changes depend on the value of “t”. In the equation, $V_y = V_{0y} - gt$, if “t” is equal to zero, then the values of V_y will be equal to the initial vertical velocity, but with the increase of “t”, they are decreasing and finally when they reach their maximum, where V_y will be zero.

Consider the following scenario: an inquiry into the maximum height a ball can attain when thrown lightly versus when thrown with greater force. The analysis suggests that a faster throw results in a higher ascent. To quantify this, calculations are performed with the objective of determining the optimal height. Subsequently, attention is turned to calculating the maximum distance the ball can cover in the horizontal (x) direction—the maximum range. To address these questions, algebraic methods are employed to derive the necessary equations and relationships that govern the ball's trajectory.

$$V_{0x} = V_0 \cos \theta$$

$$V_{0y} = V_0 \sin \theta$$

$$V_y = V_{0y} - gt$$

$$v_y = 0$$

$$v_0 \sin \theta - gt = 0$$

$$t = \frac{v_0 \sin \theta}{g}$$

$$y = (v_0 \sin \theta)t - \frac{1}{2}gt^2$$

$$H = (v_0 \sin \theta) \left(\frac{v_0 \sin \theta}{g} \right) - \frac{1}{2}g \left(\frac{v_0 \sin \theta}{g} \right)^2$$

$$H = \frac{(v_0 \sin \theta)^2}{2g}$$

In the quest to achieve the maximum height for a thrown ball, it is essential to set the angle 90, as the sine function attains its maximum value at 90 degrees. Thus, to propel the ball to its highest point, aligning it straight upward is optimal. On the other hand, if the objective is to maximize the horizontal distance covered by the ball, the strategy involves achieving the maximum range “R”. Given that the speed of the ball remains constant in the horizontal direction, the distance covered “x” in time “t” is crucial. At two distinct points in time, “y”

attains the value of zero: firstly, at the release point (initial point) and secondly, at the landing point (final point). Consequently, there are two unique values of “x” where “y” equals zero. The first is evident at “x = 0”, and the second, denoted as x = R, represents the distance from the starting to the ending point. Notably, the choice of θ is pivotal for achieving the maximum range. To optimize the horizontal distance covered, it is recommended to throw the ball at an angle of $\theta = 45^\circ$.

$$\begin{aligned}
 x &= (v_0 \cos \theta)t \\
 t &= \frac{x}{(v_0 \cos \theta)} \\
 y &= (v_0 \sin \theta)t - \frac{1}{2}gt^2 \\
 &= (v_0 \sin \theta) \frac{x}{(v_0 \cos \theta)} - \frac{1}{2}g \left(\frac{x}{(v_0 \cos \theta)} \right)^2 \\
 &= x \tan \theta - x^2 \left(\frac{g \sec^2 \theta}{2v_0^2} \right) \\
 y &= x \left[\tan \theta - x \left(\frac{g}{2v_0^2 \cos^2 \theta} \right) \right] = 0
 \end{aligned}$$

This equation has two solutions for x,

$$x = 0, \quad \text{and} \quad x = R \text{ (range)}$$

$$\left[\tan \theta - R \left(\frac{g}{2v_0^2 \cos^2 \theta} \right) \right] = 0$$

$$R \left(\frac{g}{2v_0^2 \cos^2 \theta} \right) = \tan \theta$$

$$R = \frac{2v_0^2 \cos^2 \theta}{g} \cdot \tan \theta$$

$$R = \frac{2v_0^2 \cos^2 \theta}{g} \cdot \frac{\sin \theta}{\cos \theta}$$

$$R = \frac{2v_0^2 \sin \theta \cos \theta}{g}$$

$$R = \frac{v_0^2 \sin 2\theta}{g}$$

Since $-1 \leq \sin 2\theta \leq 1$

therefore $(\sin 2\theta)_{\max} = 1$

$$\Rightarrow R_{\max} = \frac{v_0^2}{g} (\sin 2\theta)_{\max} = \frac{v_0^2}{g}$$

Physics-PHY101-Lecture#04

FORCE AND NEWTON'S LAWS

Till now in this course of physics, we have looked at kinematics which includes basic concepts such as displacement, velocity, and acceleration. We talked about one dimension that if the body can move only on one dimension i.e., on a straight line then how do we define velocity and acceleration? We talked about derivatives, then we generalized it into two and three dimensions. Then some interesting problems were also solved. But this question remains to be raised why do bodies/objects move like this, where do they get this acceleration from? We will look at these issues in today's lecture.

The theme of today's lecture is dynamics. By dynamics, we mean the force which acts on the body and gives it acceleration. In this context, we will talk about Newton's three laws of Motion. So, the real question arises where does acceleration come from? For this, we will need a new concept which we will call Force. Dynamics means it is the study of forces and the resulting motion. So how does a body get motion and what is the effect of force on it will be discussed in this lecture? Before Newton's time, it was believed that the natural state of everything is to come to a standstill, which means that everything wants to come to a standstill or stop, so for example: if we roll a ball then the ball stops after some time or if you throw any other thing then it moves forward for a while and then stops as if it is a natural state that is the of rest and if something moves then there must be some force acting on it, hence this idea was common before Newton's time that some force behind moving body.

For this reason, it was believed that the sun, the moon and all the other planets are moving then there must be some force behind them. For example: if mars rotates in its orbit like this then there must be something pushing it. This was an old idea but after Newton a great revolution took place. Newton said that the natural state of everything is that it wants to continue its movement. The modern view is that objects tend to remain in their initial state, that is they want to remain in the same state in which they were and unless some force acts on it will maintain their condition. Isaac Newton was probably the world's greatest scientist and thinker. He wrote a book called Principia Mathematica about 350 years ago in which he proposed three laws of motion.

Newton's First Law of Motion:

It states that everything either remains at rest or moves with constant speed unless some force acts on it. This means that the body moving will keep moving on its own without the application of external force and can be considered as a free body.

Frame of reference:

If we measure the movement of a body and someone else measures the motion of the same body then there can be differences between the two, so here we have to talk to you about reference frames as shown in figure 4.1. These are two frames of reference, one we can call S , and the other can be called S' . Imagine that you are at rest and standing on land and exist in frame S . There is another person who exists in frame S' and it is moving away from us at velocity V . Let's assume a point P having distance x from our frame of reference. Now the person who is moving will say that I measured this distance as x' .

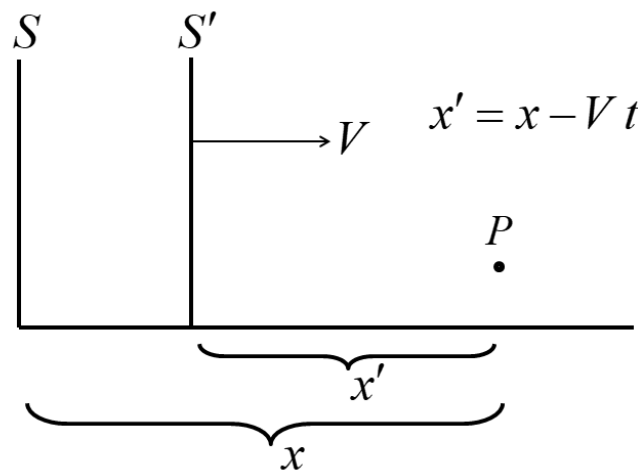


Figure 4.1: Frames of references S and S' .

The relation between these two distances is given by

$$x' = x - Vt$$

and at $t = 0$

$$x' = x$$

So initially ($t = 0$) both distances are measured to be the same, but as the second observer moved forward, there was a difference between x and x' which is called a frame of reference. According to this, two different observers will have different sets of coordinates for the same point and whatever measurements we make will depend on these coordinates. Now the question arises of whose measurements are more accurate.

Inertial and non- Inertial Frame of Reference:

Newton's first law applies in every frame which is moving with constant velocity and such frames are called **inertial frames**. Newton's law is valid in all inertial frames and all non-accelerating frames are called inertial frames.

For example: Imagine that you are sitting in the car and there is a hydrogen or helium balloon in that car, and it is suspended inside the car. Now when the car accelerates, we will observe that the balloon starts moving backwards. It seems like some force is acting on the balloon. But in reality there is no force acting on it, but we feel it as if it's moving backwards. And that is because you are not in the same inertial frame. The frame which accelerates is a **non-inertial frame** and apparently, a force acts in it sometimes called a fictitious force. It appears to be force but in reality, it is only observable because we have chosen the wrong frame of reference. So, Newton's first law is true only in inertial frames.

Difference between inertial and non-inertial frames of reference:

Inertial frame	• The frame of reference which is moving with uniform velocity and does not accelerate ($a=0$). • Obeys Newton's law of motion. • Example, a train moving with uniform velocity is an inertial frame of reference.
Non-inertial frame	• The frame of reference which is accelerating ($a \neq 0$) is called non-inertial frame of reference. • Does not obey Newton's law of motion. • Example, a freely falling elevator is taken as non-inertial frame of reference.

Law of inertia:

Now the question arises when you want to change the state of motion of a body, you feel some resistance. How much does a body resist when you try to change its state of motion? It depends on its mass. The heavier the body, the more it will resist change in its state of motion. There are many examples of this.

For example: If you push a light body (like a shopping cart), it moves easily. If you push a heavy body (like a car or bus) it is possible that you may not even be able to move it. We define this resistance as “**inertia**”. So, inertia is the resistance to change in motion in other words resistance to acceleration and mass quantifies it. The greater the mass, inertia will be more accordingly. Now the question arises whether mass means size? to which the answer is no. Sometimes smaller bodies offer more resistance. It is related to density. To which the answer is also no. Is this related to weight? Is mass the same as weight? This is also not the answer. Mass and weight are not equal.

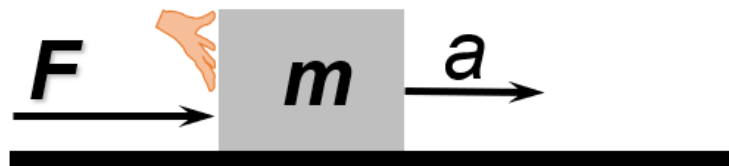


Figure 4.2: Force F acting on a body having Mass m and causing acceleration a .

Now look at Figure 4.2. Force F is acting on a mass whose value is m . Now the more force applied to it the acceleration ‘ a ’ will be greater accordingly. So mathematically it is written as

$$a \propto F$$

More force leads to more acceleration. But on the other hand, if we consider a body having more mass then the acceleration will be less.

$$a \propto \frac{1}{m}$$

Hence

$$a \propto \frac{F}{m}$$

$$F = kma \text{ (k=1)}$$

Increasing the force increases the acceleration but increasing mass reduces acceleration.

Newton's second law of motion:

Newton's second law of motion states that the product of mass and acceleration is equal to the total external force acting upon mass. Now there can be many forces (shown by subscripts) applied simultaneously in different directions. The total net force will produce acceleration.

$$F = ma \rightarrow a = \frac{F}{m}$$

Where

$$F = F_1 + F_2 + F_3 + \dots$$

By this, we can also define mass as:

$$m = \frac{F}{a}$$

So, how to rightly write this equation? There is an equation and there are three variables. Here you can measure the acceleration separately because acceleration is the rate of change of velocity and velocity is the rate of change of position. As position can be measured as a function of time so, velocity and consequently time-dependent acceleration can be measured. There is a need to measure force and how do we do this? We can give you many simple examples of this.

Example: We can consider a spring balance. In spring balance, we can place some mass and there is a spring. The more mass we place, the more spring will extend. Similarly, if we pull something with a spring balance, there is a scale on the spring balance showing the magnitude pulling force. If we pull a rubber band, this rubber band exerts a force on our hand and the more it is pulled, the greater the force. So, a rubber band can measure force.

Introduction to force:

Force is the vector and hence it has a magnitude and also a direction. The direction of force is also the direction of acceleration. There are many types of forces.

Let us consider **contact forces**. When two bodies come in contact with each other, one body exerts a force on another body that is a force acts on it, i.e., when we push a box, a force acts on the box, air exerts pressure on a moving car which is called **air resistance**. Consider a rope with a weight tied at one end that results in **tension** which is also a type of force.

Just like other physical quantities force also has dimensions given by,

$$[Force] = [Mass] \times [Acceleration] = [M] \times \left[\frac{L}{T^2} \right] = \left[\frac{ML}{T^2} \right] = [MLT^{-2}]$$

Units for force in the MKS system is Newton. Newton is defined as, if we apply 1 Newton force on 1 kilogram of mass then its acceleration becomes 1 m/s^2 .

$$1N = 1kg \cdot 1 \frac{m}{s^2}$$

Force F means external forces only. Consider a body made up of atoms (as all matter is made up of atoms), every atom attracts or repels the other atom. These interactions are internal forces so they cancel out each other. When we say that the acceleration of a body is F/m , then that F refers to external force only.

External force actually is a total force so if various forces are acting such as F_1 , F_2 and F_3 till F_N then we have to add all these, given by:

$$F = F_1 + F_2 + F_3 + \dots + F_N = \sum_{i=1}^N F_i$$

here we see a new symbol which is called the Greek symbol “Sigma Σ ” or symbol of summation. So, this summation means we add F_1 , and F_2 till F_N and this makes the total force.

As force is a vector quantity it can be added by two methods. One method is already discussed (the Parallelogram method) earlier. This can also be done and the other way is through vectors addition by components. The following sample problem (figure 4.3) can help us better understand the addition of two forces by components method. We are given two forces (F_1 whose

value is 4 N along the negative y-axis) and F_2 whose value is 5 N at an angle of 36.9° with the x-axis (angle not mentioned on PPT slide).

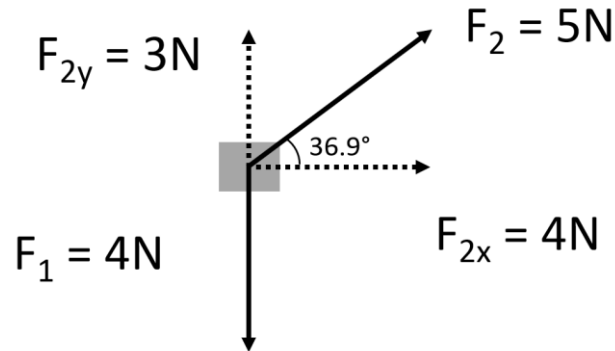


Figure 4.3: Sample problem.

Components of F_1 :

$F_{1x} = 0$ (as no component along x-axis)

$F_{1y} = -4N$ (as it is along negative y-axis)

Components of F_2 :

$$F_{2x} = F_2 \cos(36.9) \approx 5 * 0.8 = 4N$$

$$F_{2y} = F_2 \sin(36.9) \approx 5 * 0.6 = 3N$$

The sum of components along the x-axis,

$$\sum F_x = F_{1x} + F_{2x} = 0 + 4 = 4N$$

$$\sum F_y = F_{1y} + F_{2y} = -4 + 3 = -1N$$

So, the **magnitude of resultant force F** can be calculated by,

$$|F| = \sqrt{\sum F_x^2 + \sum F_y^2} = \sqrt{(4)^2 + (-1)^2} = \sqrt{17}N = 4.12N$$

Exercise for student: Determine the direction of resultant force F .

Mass and Weight:

Now we will discuss the difference between weight W and mass m . Weight actually is a force, the force due to gravity. If I have a kilogram of matter it will have a specific weight on earth's surface, but if I take it to the moon's surface, its weight will be different. The mass will be the same but its weight is different on both (earth and moon). Let's look at it in the formula, the weight is the force of gravity and if you apply the following equation.

$$F = ma$$

Where F becomes the weight W , mass m remains the same, and acceleration a is replaced by acceleration due to gravity g .

$$W = mg$$

Like forces, weight is also a vector and is measured in Newton. If we measure the weight of the same body on earth's and moon's surface, we will observe that the weight on earth will be seven times as compared to weight on moon although mass is same. If we go out into space where there is no celestial body, then our weight will be zero there as there will be no acceleration due to gravity, although mass will be non-zero and remain the same.

Difference between mass and weight:

<u>Mass</u>	<u>Weight</u>
Mass is a property of matter. The mass of an object is the same everywhere.	Weight depends on the effect of gravity. Weight increases or decreases with higher or lower gravity.
Mass can never be zero.	Weight can be zero if no gravity acts upon an object, as in space.
Mass does not change according to location.	Weight varies according to location.
Mass is a scalar quantity. It has magnitude.	Weight is a vector quantity. It has magnitude and is directed toward the center of the Earth or other gravity well.
Mass may be measured using an ordinary balance.	Weight is measured using a spring balance.
Mass usually is measured in grams and kilograms.	Weight often is measured in newtons, a unit of force.

Newton's third law of motion:

Newtown's third law says that the action of every force produces a negative force. The magnitude of these two forces is the same and direction is opposites (antiparallel).

Example: Two boxing gloves A and B collide and come in contact with each other while punching. A Pushes on B and B pushes on A. So, these are action and reaction forces are in opposite direction but have same magnitude (negative sign on the right-hand side of equation in figure 4.4 shows opposite direction)

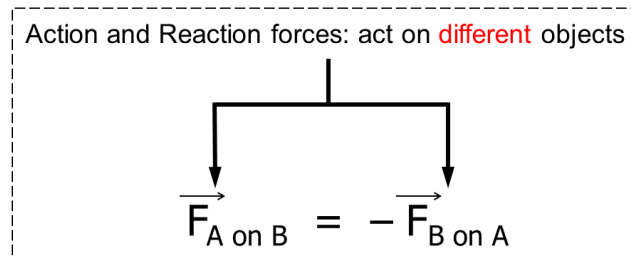


Figure 4.4: Action reaction forces.

Note that $F_{A \text{ on } B}$ is equal to negative $F_{B \text{ on } A}$, so the effect of A is on B & B is on A. A is not affecting C, D or anything else, nor is B on anything else. It is so that A is on B & B is on A.

Example: An apple is lying on the table, so this is the force of the earth on the apple. We will call it action, and this is due to gravity. Its reaction is not the force of the table on the earth, but the force of the apple on the earth. Earth pulls this apple down, and on the other hand this apple pulls the earth to itself. We assume earth is not pulled to apple which is wrong. Apple is light and very small in comparison to the earth, but the apple also pulls the earth toward itself. Now because the earth is so heavy its acceleration is very less. But apple definitely exerts a force on earth. Table also exerts as a force on apple and its reaction is that apple also exerts a force on table. Now you can ask how the table exerts a force on the apple. How does the force exerted by the table affect it? It's obvious that if there was no apple there would be no force. But if you keep the apple on the table there is a slight bend produced in the table as a result of which a force is produced in upward direction.

To make this discussion interesting we will narrate a story of a very educated horse. He studied Newton's law in his spare time and especially Newton's third law. One day, while he was studying, his owner got angry. His owner attached a cart to him and ordered horse to pull it. The horse said that there is no use of it because he has just read Newton's third law and according to

that if he pulls the cart then the cart will also pull him back with same force. He will not move forward hence it is useless to pull the cart. The owner got angry, he gave a pat at the back and what happened? The cart started moving forward. Why did it happen? If the horse is right then the cart would not have moved but there is a mistake in it. So, let's see where the mistake is.

If we look at this student in figure 4.5 walking forward on the ground. Similar to the student depicted in figure we also push ground backwards with our feet. The ground pushes us forward as a reaction. Newton's third law says that if we push something back, it pushes us forward. In this way we move forward. So whenever one object exerts a force on a second object, the second object exerts an equal and opposite force on the first.



Figure 4.5: Person 'P' walking on ground 'G'. F_{GP} is a backward force exerted on ground by a person. F_{PG} is forward force exerted on a person by ground.

Now look at Figure 4.6 of assistant boy 'A' boy is moving forward on ground 'G' and at the same time he is also pulling a sledge 'S' forward. When he pushes ground backward (F_{GA}), the ground pushes him forward (F_{AG}). The tension in this rope (F_{AS}) also pulls this boy backwards. But if the total force produced in the forward direction is more than the backward pull, then this sledge will move forward.

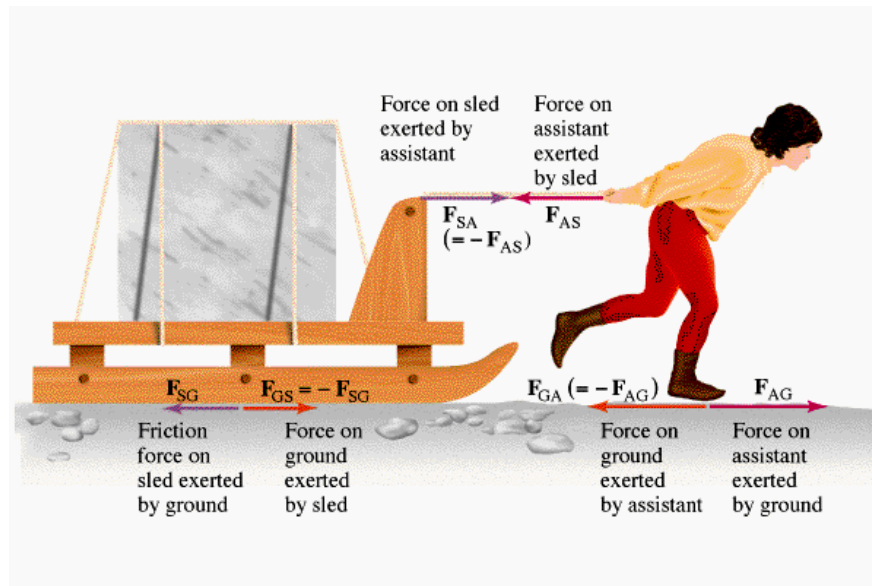


Figure 4.6. Assistant boy ‘A’ moving forward on ground ‘G’ and also pulling sledge ‘S’ forward.

Now one more **example** is depicted in figure 4.7. Your hand is on the table in front of you, and with your hand, you are applying pressure on the table and pushing it downwards but table is pushing your hand upward in opposite direction.

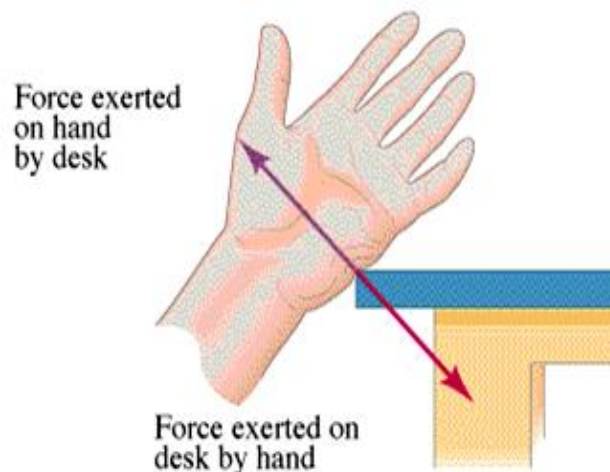


Figure 4.7: Action-reaction pair for hand and table.

Let’s consider another **example** shown in figure 4.8. The person is standing and pushing against the wall and the wall pushes the person back with same force. If the action-reaction forces are equal and opposite in direction, why they do not cancel out each other?



Figure 4.8: Action-reaction pair for person pushing wall.

Figure 4.9 elaborates forces involved with more details. A person is leaning against a wall. Two forces are acting on this person. One is his weight and written it as $F_{m \text{ on } f}$ meaning the force of mass 'm' on the floor 'f'. This force act in downward direction. Reaction to this force i.e., the force of floor on the man $F_{f \text{ on } m}$ acts upward and cancel out each other. The force exerted on the left side is force of man on the wall $F_{m \text{ on } w}$ and the force of wall on the man $F_{w \text{ on } m}$ is on opposite direction. Both these forces are equal in magnitude and in opposite direction.

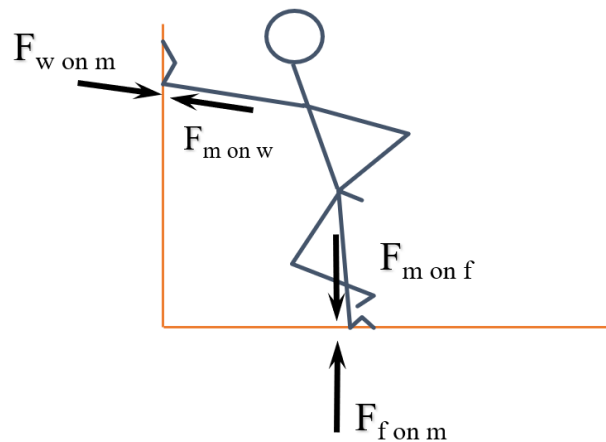


Figure 4.9: Forces involved for physical scenario of man 'm' leaning against wall 'w'.

Now if we push a body with much greater force then there is possibility that it will start moving in the direction of applied force. Such **an example** is shown in figure 4.10. A block/box is on

frictionless (ice) surface. Man pushes block/box to the left (negative x-axis) by a force $F_{m \text{ on } b}$ (force of man on block). Reaction to this is force of block on man ($F_{b \text{ on } m}$) is towards right side (positive x-axis). Although both forces ($F_{m \text{ on } b}$ and $F_{b \text{ on } m}$) are equal and opposite in direction but block moves to left side. So how is it possible? Why are these forces not cancelling each other?

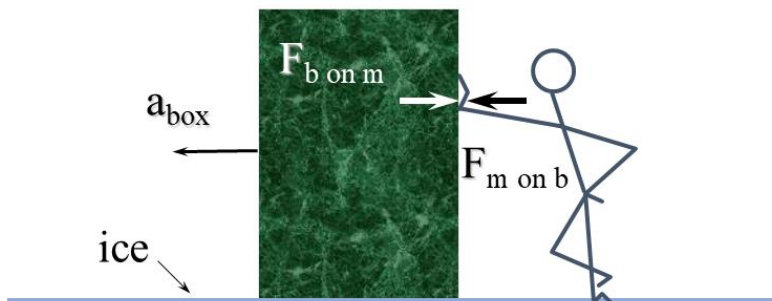


Figure 4.10: A block/box on frictionless (ice) surface being pushed by man.

The answer to why these forces is not cancelling each other out is that a body moves depending upon the total external force acting on that body (having mass m). The force that will act on a body will create acceleration in it. By considering ONLY the block/box as the whole system we can answer above mentioned questions. The force on block/box ($F_{\text{on box}}$) is equal to force of man on block/box ($F_{m \text{ on } b}$) which by newton's second law is equal to product of mass m and acceleration of block/box (a_{box}). Mathematically it can be written as

$$F_{\text{on box}} = F_{m \text{ on } b} = ma_{\text{box}}$$

Acceleration of block/box (a_{box}) will be,

$$a_{\text{box}} = \frac{F_{m \text{ on } b}}{m}$$

Misconception:

Table 4.1: This table enlist few misconceptions related newton's laws

Sr. No.	Claim	True or False. Explanation
1.	If something is moving, there must be a net force on it.	This is a false claim. A body moving at constant velocity has no net force on it. An accelerating body must have a net force on it.

2.	All equal and opposite forces are action-reaction pairs.	This claim is false. The weight of a book sitting on a tabletop and the normal force of the table acting on the book are equal and opposite, but they are not an action-reaction pair!
3.	If there is a force on an object, it must be accelerating.	This claim is false. Only a net force on the object leads to acceleration.

In our daily lives, we observe the validation of Newton's law on many occasions, especially the first and second laws. There is a concept about these which we call inertia that a body tends to maintain its state of motion or rest and its speed doesn't change. This resistance to change of state is proportional to its mass. An example involving glass and paper is discussed in the video lecture followed by the stretching of rubber band as a function of applied force (amount of water).

We will pay close attention to the applications of Newton's law during the next few lectures. The application of Newton's laws is not limited or localized. These laws are applicable in the whole universe (also termed as universal set of laws). Interestingly concepts at the base of Newton's laws do not exist in the world. They only exist in our minds. For example, if we talk about free particles (as mentioned earlier that free particle is the one on which no force acts on it) but is there any free particle in reality? The answer is NO. There is no such thing as a free particle. If we even take a particle/body million and millions of miles away from Earth, there will be a minute force acting on it due to gravity. We have also talked about this earlier there is such a thing as a point mass or point body. Despite these facts, we are related to the abstract concept that Newton's laws related to things that do not exist yet this is the achievement of human thinking. We will discuss the application of these laws and how phenomenon are explained.

Physics-PHY101-Lecture#05

APPLICATIONS OF NEWTON'S LAWS-I

In today's lecture, we will be focusing on Newton's laws of motion and how they can be applied to different situations and circumstances. We will be solving various examples to gain a better understanding of how important Newton Laws are in understanding the world around us. Living in the 21st century, we can sometimes forget the significant difference between the latest and old concepts/laws. Before Newton's laws, it was generally believed that every physical body or particle naturally wanted to remain at rest. However, Newton's first two laws of motion state that every physical body or particle tends to maintain its state of motion, whether it's at rest or in motion. If the state of motion of a body changes, such as a change in speed, it's because some force has acted upon it.

Equilibrium

If a body or a particle is in a state of rest, then we refer to it as an equilibrium state. Equilibrium means that in the state of a system, there is no force acting upon it, i.e., also called the state of rest (no motion). Newton's law means that if the sum of all the forces is equal to zero on that body or combination of bodies, then we get a state of rest. As per the definition, "State of a system when the sum of all forces acting upon it vanishes" is called 'Equilibrium'.

What is meant by the system here? The system here refers to many bodies which are connected to each other or isolated from each other. When you consider all these together then it is called a system.

Examples of equilibrium:

- 1) The gravity acting on the apple continuously pulls it downwards but when placed on the table as shown in Figure 5.1, the apple remains at rest. It doesn't move further down. Which forces are acting on this apple? Gravity is one as mentioned earlier but there must be at least one more force.

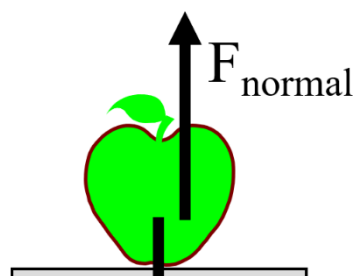


Figure 5.1: Apple placed on the horizontal top surface of the table. Normal force F_{normal} and weight of apple ($W_{\text{apple}} = m_{\text{apple}}g$).

Newton's law governs that whenever a force is applied to a body, the body starts accelerating. The table applies a force opposite to the gravitational force which stops this apple from falling. This force is called the Normal Force (F_{normal}). The word "normal" has two meanings in English, one is "as per usual" and the other is "perpendicular" as in geometry. In Figure 5.1, both these forces are indicated, the normal force (F_{normal}) and the $m_{\text{apple}}g$ which is the force due to gravity (also called weight). Mathematically Newton's second law (for y-axis or vertical direction) can be written as,

$$\sum F = m a_y = F_{\text{Normal}} - m_{\text{apple}} g$$

Normal force (F_{Normal}) is written as positive and gravitational force ($m_{\text{apple}}g$) is written with the negative sign as it is opposite to the direction of F_{Normal} . All forces along the x-axis (horizontal direction) are zero as gravity does not act along this direction and also table is not applying any force sideways. As the apple is at rest, acceleration (a_y) is also zero. The above equation can be simplified to

$$F_{\text{Normal}} - m_{\text{apple}} g = 0 \text{ (as } a_y = 0 \text{)}$$

$$F_{\text{Normal}} = m_{\text{apple}} g$$

So normal force applied to the apple by the table is equal to its weight under the equilibrium (at rest/no motion). It is normally supposed that equilibrium means there is no force acting on the body which is a wrong presumption. Equilibrium or a state of rest arises when the total net force is zero (sum of all forces).

2) Equilibrium Example: Rubber band as slingshot

A rubber band can be considered a system and when pulled tension is created in it. This tension increases as it is pulled more and more. If a piece of paper in the form of a pellet is attached to a rubber band and stretched only and kept in place (at rest, equilibrium) like a slingshot, the tension in the rubber pulls the paper pellet forward. If released, the paper pellet will go forward like a projectile. So, it remains in equilibrium only under the force applied by us which is equal and opposite to tension in the rubber band. Now the question is whether this

paper pellet was in the state of equilibrium when released or when it reached its maximum height considering that it is projectile motion. The answer is no, it was not in the equilibrium state. Although its velocity along the y-direction was zero momentarily at the highest point, its acceleration at this point was not zero as acceleration due to gravity was continuously acting on it. There was no force acting other than force due to gravity so it was not in the state of equilibrium. The total net force acting on it was force due to gravity (weight) which is nonzero.

3) Equilibrium Example: Aircraft flying

Consider an aircraft flying at a constant speed and same elevation. This is an example of equilibrium motion as there is no acceleration (constant speed) indicating net force (sum of all forces) acting on the aircraft is zero.

Now we will solve another problem considering another aircraft of mass m has a position vector given by $\vec{r} = (at + bt^3)\hat{i} + (ct^2 + dt^4)\hat{j}$. What force is acting upon it? Is it in equilibrium?

As velocity $\vec{v}$ is the time derivative of position $\vec{r}$ by

$$\begin{aligned}\vec{v} &= \frac{d\vec{r}}{dt} = \frac{d}{dt} \left((at + bt^3)\hat{i} + (ct^2 + dt^4)\hat{j} \right) \\ \vec{v} &= (a + 3bt^2)\hat{i} + (2ct + 4dt^3)\hat{j}\end{aligned}$$

Now as acceleration $\vec{a}$ itself is time derivative of $\vec{v}$, so

$$\begin{aligned}\vec{a} &= \frac{d\vec{v}}{dt} = \frac{d}{dt} \left((a + 3bt^2)\hat{i} + (2ct + 4dt^3)\hat{j} \right) \\ \vec{a} &= (6bt)\hat{i} + (2c + 12dt^2)\hat{j}\end{aligned}$$

Now force $\vec{F}$ is a product of mass m and acceleration $\vec{a}$, so

$$\vec{F} = m\vec{a} = m((6bt)\hat{i} + (2c + 12dt^2)\hat{j}) = 6m bt\hat{i} + m(2c + 12dt^2)\hat{j}$$

The above equation shows the force acting on the aircraft. Force along with acceleration is nonzero so the aircraft is not in equilibrium.

Tension

Now consider a string tied to a body as shown in Figure 5.2(a). The body has a certain weight (downward force) due to gravity which causes tension in the string. This tension force in the

string is in an upward direction as it balances the weight. If we cut the string which is horizontally stretched, we will perceive tension towards the right as well as towards the left as shown in Figure 5.2 (b).

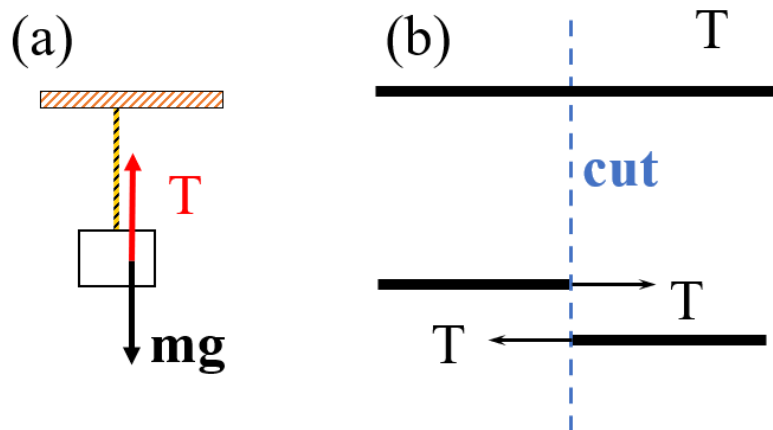


Figure 5.2: (a) A body having mass m tied to a string. The other end of the string is tied to a fixed support. Mass hanging vertically. (b) Tension T is considered in horizontal string. Blue dashed line indicating cut-line.

Now the big advantage of ropes is that we can pull a body from a distance and the direction of the force can be changed. Consider a horizontal string under tension as shown in figure 5.3. the yellow part represents a small part of the string. T_2 acts to the right-side direction and T_1 to the left-side direction of that small part of the string. According to Newton's law, total net force action on that small part of the rope is given by

$$\sum F = T_2 - T_1 = ma \dots\dots\dots 5.1$$

Where m is the mass and a is the acceleration of that small part of the string. Considering that string is very light in terms of mass, the mass of the small part of the string can be neglected and assumed to be zero.

$$m = 0 \text{ (small part of string whose mass is very less)}$$

Eq 5.1 becomes

$$T_2 - T_1 = 0 * a$$

$$T_2 - T_1 = 0$$

$$T_2 = T_1$$

As T_2 and T_1 are the same, we can equate both quantities to be T . This means that tension in a string has very little mass and is constant throughout the string.

An ideal (massless) string has constant tension

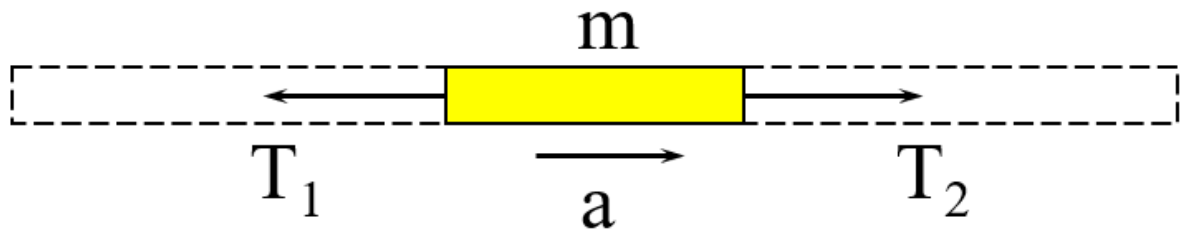


Figure 5.3: A string having horizontal tension. The central yellow part shows a small part of the string having mass m and accelerating by a being pulled to the left and right by tensions T_1 and T_2 , respectively.

Newton's third law applies as forces act on all segments of a string, but these action and reaction forces are equal to each other. As shown in Figure 5.2(a), the body having mass is under equilibrium and is at rest. Newton's second law can be applied as

$$\sum F = T - mg = ma_y$$

Where a_y is acceleration along the y-axis and is zero ($a_y = 0$).

$$T - mg = 0$$

$$T = mg$$

So, for this mass system, the tension is equal to the weight and the value of this tension will be equal everywhere inside the string. Now for string having as considerable mass such as rope, will the tension be equal in all parts of this rope if it is allowed to hang vertically without any additional body? The answer is NO, because the lower end of the rope has to support less mass as compared to the higher part of the rope. So, tension at the higher end is greater as compared to tension at the lower end.

A massive string (rope) does not have constant tension

Changing the direction of Tension

We can change the direction of tension as per requirement. As mentioned earlier, if the string is considered massless tension will be constant throughout the string even for the physical

configurations shown in Figure 5.4. Figure 5.4(a) shows mass-II m_{II} hanging vertically from an ideal string and the direction of tension is changed using a pulley. Now tension is acting horizontally on mass-I m_I .

Figure 5.4 (b) shows another example of the change of tension direction. The weight ($W = mg$) is being lifted. A worker is pulling a rope and with it, the weight is rising upwards. So, F_T is shown on two sides and both of them are adding up. Under equilibrium, the net force upward is two times F_T and it is equal to weight ($W = mg$).

$$\sum F = F_T + F_T - mg = ma_y \dots\dots\dots 5.2$$

Under equilibrium $a_y = 0$

$$F_T + F_T - mg = 0$$

$$2F_T = mg$$

If $2F_T$ is greater than weight ($W=mg$) such as $2F_T > mg$ then eq. 5.2 can be written as

$$2F_T - mg = ma_y$$

$$ma_y > 0$$

$a_y > 0$ (Upward direction or lifting up)

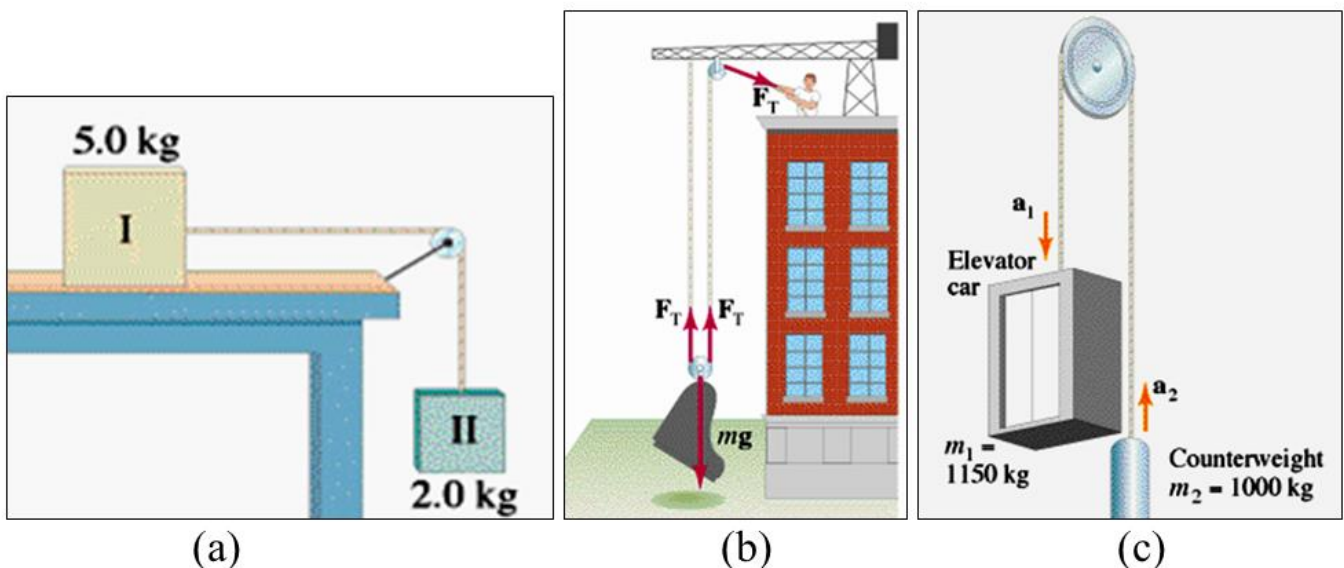
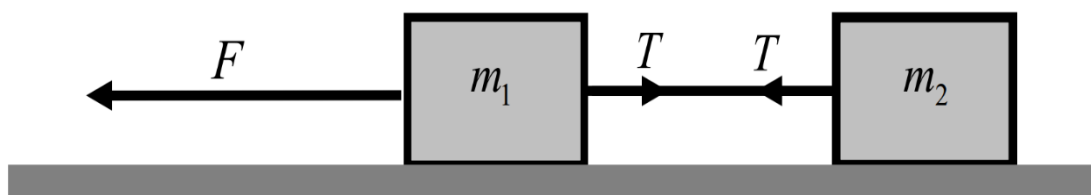


Figure 5.4: Examples of change of tension direction by pulley.

Figure 5.3(c) shows another example elevator car. The tension direction has been changed through the pulley. The mass of the elevator car is 1150 kg and on the other side, there is a counterweight which is equal to 1000 kg. Now the acceleration a_1 is downward as the elevator is more massive as compared to counterweight. The acceleration of the counterweight a_2 is downward.

Problem:

Question: Consider the two blocks shown below on a frictionless surface. Determine the tension and acceleration.



Solution:

F on m_1 :

$$\sum F = m_1 a$$

$$F - T = m_1 a \dots\dots\dots 1)$$

F on m_2 :

$$\sum F = T$$

$$T = m_2 a \dots\dots\dots 2)$$

To find "a"

Plugging eq. 2) in eq. 1)

$$F - m_2 a = m_1 a \Rightarrow F = m_1 a + m_2 a$$

$$a = \frac{F}{m_1 + m_2} \dots\dots\dots 3)$$

To find "T"

Plugging eq. 3) in eq. 2)

$$T = m_2 a = \frac{m_2 F}{m_1 + m_2} \dots\dots\dots 4)$$

Friction

This is another important type of force that we encounter most frequently in our daily lives and it is called force of friction. The force of friction opposes every kind of motion. For example, if you place a body having a certain weight on the table and pull that body, then the

force which is produced due to applied force and contact between the body and table is called frictional force as shown in Figure 5.5. There are countless examples of this in our everyday life. Suppose there is a plate lying on a book on this table, and it will not move on its own unless some force is applied to it. Now when we pull it (as per demonstration in video lecture), we will observe that it moves forward, because we have offset the frictional force. We pulled it with more force and it started moving. The normal force N is upwards and equal to its weight mg . Simultaneously, the greater the weight, the greater will force that opposes its motion. If we double the weight of this plate, then the force required to move the plate is also doubled. As depicted in Figure 5.5, friction results in a force parallel to the surface in the direction opposite to the direction of motion. So whichever direction we pull the body (F_{Applied}), the force of friction will be in the opposite direction. Additionally, the frictional force is perpendicular to the normal force N which is always upward.

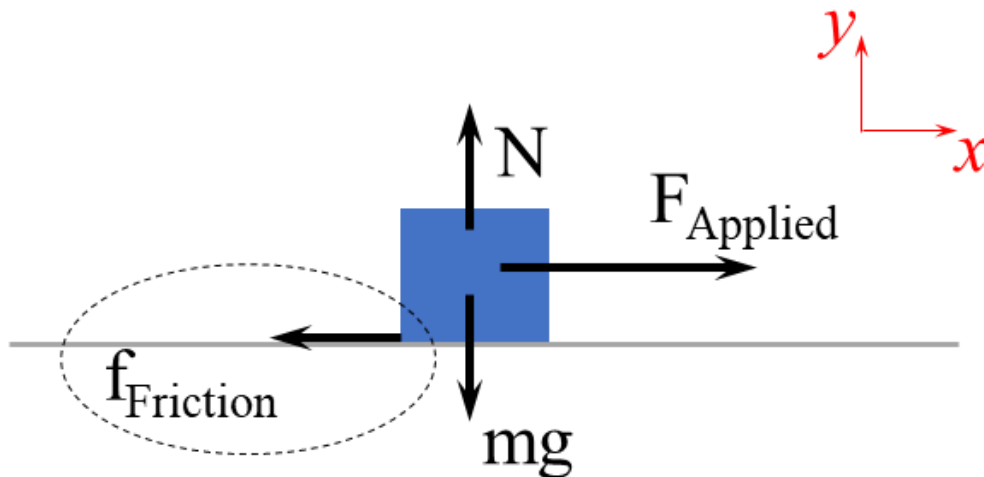


Figure 5.5: Mass m placed on a horizontal surface. Normal force N and weight mg balanced along the y -axis. Frictional force f_{Friction} produced along the negative x -axis as response to external force F_{Applied} which is along the positive x -axis.

Now the pertinent question that why does the friction occur? We observe friction in various situations, for example, when we rub our hands, then we can feel opposing force due to contact of hands. To understand the underlying reason for friction, we need to observe the two surfaces in contact under a microscope. Any surface if seen under the microscope has very fine ups and downs in it and due to this unevenness, friction is created. The more the ups and downs, the more the roughness, then the friction will be higher accordingly. If we look at contact between surfaces of bodies at a microscopic level as shown in Figure 5.6, we will be able to observe the roughness caused by ups and downs which fundamentally causes the

frictional force. If we can reduce this roughness then the frictional force or simply friction will reduce, and the fun will reduce.

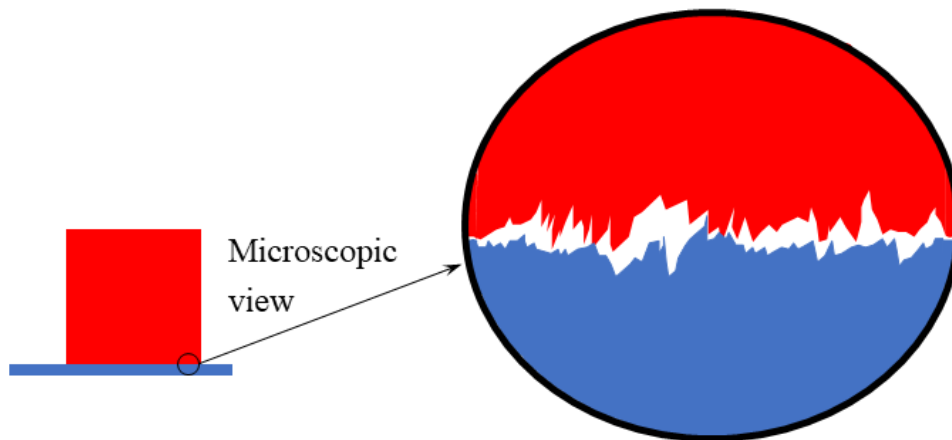


Figure 5.6: Microscopic view of two surfaces in contact which causes friction.

Characteristics of Frictional force f_F :

There are some special characteristics of frictional force. When the body is in motion, the friction that is produced is called kinetic friction. Friction force is proportional to the normal force N .

$$f_F \propto N$$

This sign of proportionality can be removed by the introduction of constant “ μ_k ” of proportionality and is called the “coefficient of kinetic friction”.

$$f_F = \mu_k N$$

$$f_F = \mu_k mg \text{ (as } N = W = mg \text{)}$$

This equation shows that the “heavier” the object or a body, the greater the friction will be. This is an example of empirical law where $f_F = \mu_k N = \mu_k mg$. Can it be compared to fundamental laws such as Newton’s second law? This is an example of an empirical law, it means that it is based on the observation that if the weight of the body/object is doubled then you have to apply double the force, if it is three times then, three times more force is required and so on. These statements are correct to some extent but it is obvious that if we keep increasing the weight of this body/object, then eventually a situation will arise when this law will not be applicable. If the body/object becomes too heavy then the surface (table) under the

body/object might break and friction will not arise. So, this is an example of an Empirical Law which means that that law is based on observations and their scope is limited. Unlike empirical laws, the scope of Newton's law is very wide as we can apply it everywhere under every situation such as on the earth, on the moon, on small bodies and heavy bodies etc. Frictional law is not a such fundamental law. The frictional force is independent of the contact area. It does not matter whether this contact area is more or less. The force is just proportional to the Normal force, not to the area. We can say that it is correct to a great extent but we cannot claim that that is completely true. If the contact area is too large or too small, it can make a difference. Imagine that the entire weight of the plate is concentrated to something finer than a needle then it is quite possible that it will pierce the surface below and frictional force law would not be applicable. These things are very important to keep in mind that there is a difference between empirical laws and fundamental laws.

Frictional Dynamics

Now we will discuss the frictional dynamics, that is, how a body moves in the presence of friction. Now consider the forces along the x-axis (as shown in Figure 5.5).

$$\sum F_x = F_{Applied} - f_{Friction} = ma_x$$

$$\sum F_x = F_{Applied} - \mu_k N = ma_x \dots\dots\dots 5.3$$

Where a_x is the acceleration body having mass m along the x direction. Similarly, for forces along the y -axis

$$\sum F_y = N - mg = ma_y = 0 \text{ (as } a_y = 0 \text{)}$$

$$N = mg$$

So eq. 5.3 can be written as

$$F_{Applied} - \mu_k mg = ma_x$$

By this equation we can determine the acceleration for any given force and any given mass provided we are given the coefficient of Friction μ_k . Friction acts whenever the body is in motion but if a body is at rest then friction is also present. There is a difference between friction at rest and kinetic friction. Kinetic friction means the friction that exists when the body is in motion. Its value ($\mu_k N = \mu_k mg$) is a constant. But when the body is at rest, the

value of friction force is not fixed. As we start pulling this plate it does not move (as per demonstration in the video lecture). We pulled it by applying a small force and its reaction is also a small frictional force accordingly. Only when we pull it with more than a certain limit force then it will move by balancing and overcoming frictional force. This means that static friction adjusts its value depending upon applied force, so static friction is different from kinetic friction in this regard. Now as the body is at rest, a_x will also be zero. Forces along the x-axis will be

$$\sum F_x = F_{Applied} - f_{Friction} = 0 \text{ (as } a_x = 0)$$

$$F_{Applied} = f_{Friction}$$

and for the y-axis, it will be similar to the previous (as in both kinetic and static cases, there is no motion along the y-axis)

$$\sum F_y = N - mg = ma_y = 0 \text{ (as } a_y = 0)$$

$$N = mg$$

The maximum possible force that can be produced by static frictions is f_{MAX} and is given by

$$f_{MAX} = \mu_s N$$

Here μ_s is the coefficient of static friction. The higher its value, the greater will be the static frictional force accordingly. There is a difference between kinetic friction and static friction. **Kinetic friction** is a constant force that means whether you go fast or slow, you have to counter that friction but static friction is different as it is not constant. For example, in the case of pulling a plate on the surface (as demonstrated in the video lecture), static friction adjusts depending on how hard we pull (apply the force) in that direction. The values μ_s may vary. If it is a frictionless surface, μ_s will be zero. But if there is a lot of friction, the value of μ_s would probably be one.

However, it is dependent on materials as different materials have different friction with each other. We can increase or decrease the friction between two materials, for example, if there are two materials and we smoothen them then friction gets reduced. If we lubricate any surface by using oil, then a layer of oil is formed which reduces the friction. When a body starts moving, the coefficient of friction is relatively less. If a body is at rest, the force

required to initiate the motion is higher as compared to the force required to counter the friction when the body is in motion.

In simple words, the kinetic frictional force is less than the static frictional force. This is also depicted in Figure 5.7. We have to apply more force to make the body move and less force to keep it in motion. We talked about static and kinetic/sliding friction but there is a third one. There is also a type of friction which is called sliding friction. For example, when a wheel passes over a railway track, there is friction there too, but it is the rolling friction. The value of rolling friction is very low in comparison to static and kinetic friction.

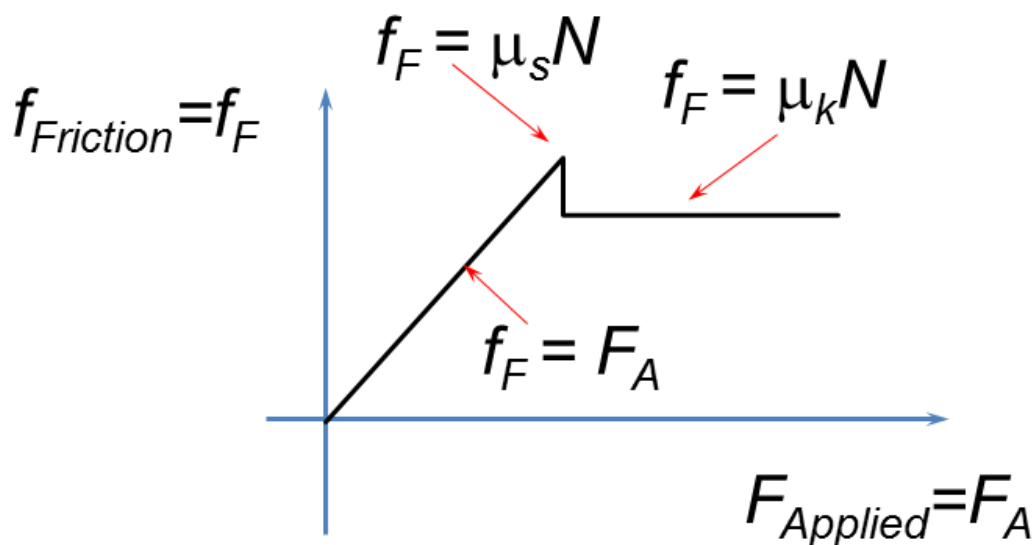
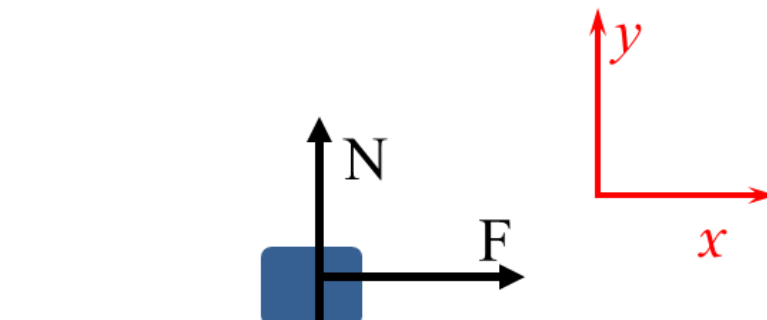


Figure 5.7: Frictional force ($f_{\text{Friction}} = f_F$) against applied force ($F_{\text{Applied}} = F_A$) showing that initially magnitude of frictional force is equal to the applied force but in the opposite direction thus body remains at rest. Once f_{MAX} is achieved the body starts moving and frictional force becomes constant i.e. $\mu_k N = \mu_k mg$ which is mostly less than f_{MAX} .

Problem:

Question: A box of mass $m = 2 \text{ kg}$ slides on a frictionless floor. A force $F_x = 10\text{N}$ pushes on it in the x direction. What is the acceleration of the box? What forces acting on the box?

Solution:



We will solve it by applying Newton's second law along the x-axis and y-axis as shown in the figure below.

$$\sum F_x = F = ma_x$$

$$a_x = \frac{F}{m} = \frac{10N}{2kg} = 5 \frac{m}{s^2}$$

$$\sum F_y = N - mg = ma_y$$

$$a_y = 0 \text{ (as no motion in y-direction) and } N = mg$$

Body on an inclined plane and Angle of Repose:

Now we will consider a physical configuration where the body (plate shown in the video lecture) is on a surface/book which can be tilted. As we tilt the surface, the normal component acting on the body, because of the surface below will decrease. If we continue tilting the surface (raising one end of the surface while keeping the second end fixed), the body will start moving/sliding at a certain angle. This angle is called the Angle of Repose as shown by ' α ' in Figure 5.8(a).

Now we will determine the value of the angle of repose for the physical configuration shown in Figure 5.8(a), where the body having mass M is placed on an inclined plane having angle α with horizontal which can be increased. The coefficient of static friction μ_s is also provided. The angle of repose can be calculated by determining the point till where the equilibrium condition persists. Figure 5.8(a) shows force Mg which is the weight of the body. Its components are also shown.

An unconventional xy-plane coordinate system (shown in by red inset of Figure 5.8(a)) is utilized for the sake of convenience, where weight Mg (which is in the vertical direction) is now considered making an angle α with the x-axis. Mg is resolved into its x and y components as per elaboration in Figure 5.8(b). Normal force N is perpendicular to the inclined surface and static frictional force f_s is parallel to the inclined surface. Under equilibrium, forces parallel to the plane (f_s and $Mg\sin\alpha$) are balanced. At an angle of repose (the angle at which the body starts sliding), acceleration due to weight ($Mg\sin\alpha$) will barely overcome static frictional force. Forces perpendicular to the plane (N and $Mg\cos\alpha$) will always be balanced.

$$f_s = Mg \sin \alpha \dots\dots 5.4$$

$$N = Mg \cos \alpha \dots\dots 5.5$$

Dividing eq. 5.4 by eq. 5.5 will give

$$\frac{f_s}{N} = \frac{Mg \sin \alpha}{Mg \cos \alpha} = \tan \alpha$$

As $f_s = \mu_s N$, above equation can be written as

$$\tan \alpha = \frac{\mu_s N}{N} = \mu_s$$

$$\alpha = \tan^{-1}(\mu_s)$$

Conclusion: The result we have derived is very reasonable and is also very easy to understand. If the friction is eliminated then the angle of repose will become zero.

$$\alpha = \tan^{-1}(\mu_s) \quad \therefore \mu_s = 0$$

$$\alpha = \tan^{-1}(0) = 0$$

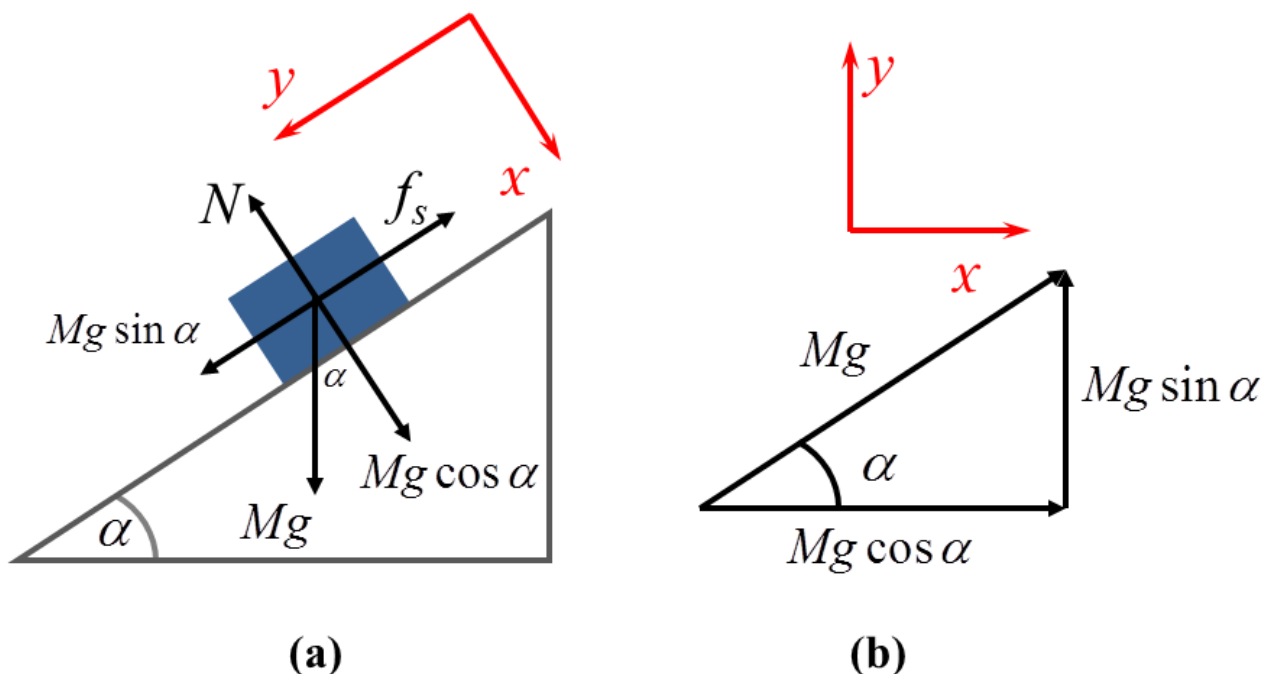


Figure 5.8: (a) Body having mass M on an inclined plane having an angle of inclination α . Note: xy -plane orientation differs from conventional representation where x and y correspond to

horizontal and vertical, respectively. (b) Components of weight Mg are shown as per normal xy-plane orientation.

Question: Why are the brakes of a car not so effective on an inclined road as compared to a levelled road?

The reason is that the normal force inclined road gets reduced. Remember that that component is $Mg\cos\alpha$. If α is zero then it is Mg and if α is 90° then it becomes zero. The brakes of a car do not work so well on an inclined road.

Question: Why a four-wheeler can use the brakes better and why does it grip better on the road?

The normal force is the same for that given coefficient of friction but instead of multiplying by two, if we multiply by four then obviously the frictional force increases. And still many concepts related to friction are still to be discussed, especially that if we move through a fluid or liquid then why does friction occur? How does it occur? Can we include all effects in equations? We will discuss all these scenarios in our next lecture.

Physics-PHY101-Lecture#06

APPLICATIONS OF NEWTON'S LAWS – II

Introduction:

In previous lecture, we have applied Newton's law on frictional problems for solid objects. As we know, frictional force is produced because of the rubbing of two objects together. But this is not necessary for only solid objects, fluid friction is also possible.

Example:

1. If you wave your hand in the air, apparently you will not feel any frictional force. But if you take an umbrella in your hand and then wave your hand you will feel a frictional force because of air resistance with umbrella.
2. If you pass your hand from water, then you will feel a frictional force easily.

Importance of fluid friction:

Newton's law is very important in fluid friction also because when an object is passing through a fluid, object has to make some space to move in fluid, where the use of pair of forces (action-reaction force) is important. For Example, as a body moves through a body it displaces the fluid. It has to exert a force on the fluid to push it out of the way. By Newton's third law, the fluid pushes back on the body with an equal and opposite force.

The two main conclusions of fluid friction are.

- The direction of the fluid resistance force on a body is always opposite to the direction of the body's velocity relative to the fluid.
- The magnitude of the fluid resistance force usually increases with the speed of the body through the fluid

Q. Think, what will happen without fluid friction in sea water?

Ans: Off course fishes would die there. Fishes would not swim in water and eventually leads to death without fluid friction.

Dependence of fluid power:

The power of fluid to resist the object is depends on,

- The speed of object moving in fluid
- The higher the speed of object usually, maximizes to resist the motion of object.

Fluid frictional force and velocity of object:

According to a linear law, we can say that “The fluid resistance force is proportional to the velocity but if the velocity of the object has increased then fluid resistance force becomes proportional to the square or cube of the velocity”.

Limitations of this law:

This law would not be applicable to the case where the speed of the object becomes faster. For solid objects, we use empirical laws (which can be calculated experimentally) but with some limitations.

Example of equilibrium under two forces:

To see how friction/resistance affects different bodies, scientist Galileo performed an experiment at the leaning tower of Pisa. He dropped two balls of different masses (M and m , where $M > m$) from the top of Pisa tower. Both balls hit the ground at the same time regardless of their masses. These results shocked him for a while because before his experiment, it was thought that the heavier ball hit the ground before the lighter ball, but the conclusion was different. Then Galileo again performed the experiment with a ball and a feather. This time he finds that the ball hit the ground before feather even though both have same gravitational force act on them. This experiment leads to the conclusion that there is some other resistive force (frictional force) which restricts their motion. Now in order to see, how resistance affects acceleration and velocity of feather, we have the following,

Let's, the magnitude of fluid resistance ' f ' is almost proportional to the speed of the body through the fluid given as,

$$f \propto v$$

$$f = kv$$

Where, “k” is called the proportionality constant. As, the upward resistive force on feather when it's falling downward is always in opposite direction to the motion of object. Hence,

$$\text{Upward resistive force} = F_r = -kv$$

$$\text{Downward weight force} = F_w = mg$$

The net force acting on the object is,

$$F_w + F_r = F$$

$$mg + (-kv) = ma$$

After certain interval, acceleration ‘a’ of the feather becomes constant (a=0) and attains a velocity called as terminal velocity (V_T) which is given as,

$$mg - kv_T = 0$$

$$v_T = \frac{mg}{k}$$

This was a simple example of equilibrium between two forces.

Problem solving steps:

In general, while solving actual problems one should do the following steps,

- Draw a free body diagram.
- Define an origin for a system of coordinates.
- Define x, y coordinate system.
- Identify all forces (tension, normal, friction, weight, etc.) and their x & y-components.
- Apply Newton's 2nd law separately on x-axis and y-axis.

$$\sum F_x = ma_x$$

$$\sum F_y = ma_y$$

- Find the unknown quantities i.e.,
- Acceleration, velocity, and displacement:

$$F = ma_x$$

$$V = V_o + a_x t$$

$$X = X_o + V_o t + \frac{1}{2} a_x t^2$$

- Forces:

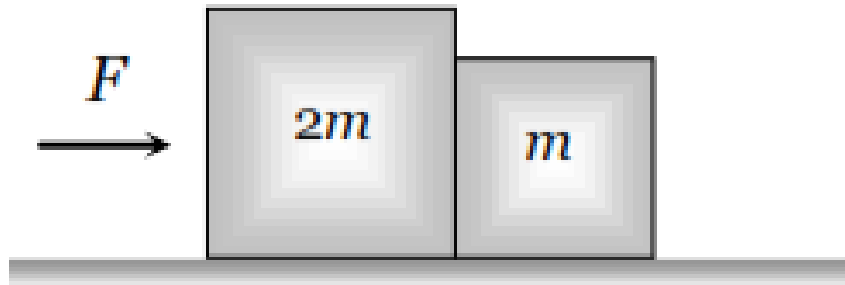
$$F_w = mg \text{ (gravitational force)}$$

$$f = \mu N \text{ (frictional force)}$$

Applications to solve frictional problems:

Problem 1:

Consider two blocks of mass $2m$ and m on a frictionless surface and a force F is applied on $2m$ block from left side. As the two blocks act as one body, hence, begins to move towards right side. We have to calculate force on mass ' m ' due to mass ' $2m$ '. As shown in figure.



Solution:

Acceleration of the blocks can be calculated as,

$$F = ma$$

$$\text{Total mass} = 2m + m$$

$$F = (2m + m)a \Rightarrow F = 3ma \Rightarrow a = \frac{F}{3m}$$

Now isolate block ' m ' and then apply newtons 2nd law,

$$\sum F = ma \quad \therefore a = \frac{F}{3m}$$

$$F_{2on1} = m \left(\frac{F}{3m} \right) \Rightarrow F_{2on1} = \frac{F}{3}$$

This is the force exerted on mass m from mass $2m$. Similarly, the force on mass $2m$ from mass m is,

$$F_{1on2} = 2m \left(\frac{F}{3m} \right) = \frac{2F}{3}$$

Conclusion: Results show that the force acts on m block because the force of $2m$ block is just one third ($1/3^{\text{rd}}$) of the total force.

Limitation of newtons law:

Newtons laws are only applicable to inertial frame of reference.

Difference between inertial and non-inertial frame of reference:

Inertial frame	▪ The frame of reference which is moving with uniform velocity and does not accelerate ($a=0$). ▪ Obeys Newton's law of motion. ▪ For example, a train moving with uniform velocity is an inertial frame of reference.
Non-inertial frame	▪ The frame of reference which is accelerating ($a \neq 0$) is called non-inertial frame of reference. ▪ Does not obey Newton's law of motion. ▪ For example, a freely falling elevator is taken as non-inertial frame of reference.

Problems in non-inertial frame of reference:

Problem 2:

Suppose you are in a lift. Here we have to deal with different cases:

- **Case-1:** where, you are at rest or moving at constant velocity ($a=0$) then normal force N (i.e., the force with which the floor of the lift is pushing on you) and the force due to gravity are exactly equal,

$$N - Mg = 0$$

$$N = Mg$$

The relation shows the apparent weight is equal to the true weight just for the condition that the acceleration must be equal to zero.

- **Case-2:** if the lift is accelerating upwards then,

$$N - Mg = Ma$$

$$N = M(g + a)$$

These relations show that the apparent weight is more than the true weight (you feel heavier) as there is an added gravitational (g) force of 9.8 m/s^2 . Acceleration would add an extra gravitational force, making you feel twice as heavy.

- **Case-3:** if the lift is accelerating downwards then,

$$N - Mg = Ma$$

$$N = M(g - a)$$

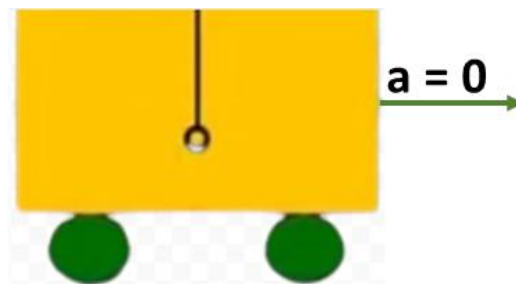
This relation shows that the apparent weight is less than the true weight (you feel lighter).

- **Case-4:** if the cable, supporting the lift is breaks, the lift falls downward with $a = g$. then

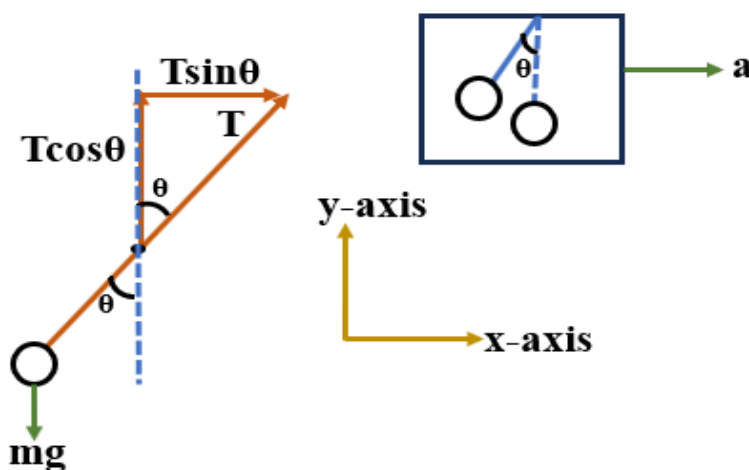
$$N = M(g - g) = 0$$

This shows the apparent weight under the free fall is zero and you will experience weightlessness just like astronauts in space do.

Problem 3: Imagine that you are in a railway wagon and want to know how much you are accelerating. You are not able to look out of the windows. A mass is hung from the roof. Find the acceleration of the car from the angle made by the mass.



Solution:



According to the balancing of forces,

Forces acting horizontally,

$$\sum F_x = T \sin \theta$$

$$ma = T \sin \theta \rightarrow 1)$$

Forces acting vertically,

$$\sum F_y = T \cos \theta - mg$$

$$0 = T \cos \theta - mg$$

$$T \cos \theta = mg \rightarrow 2)$$

On dividing equation 1) and 2), we get

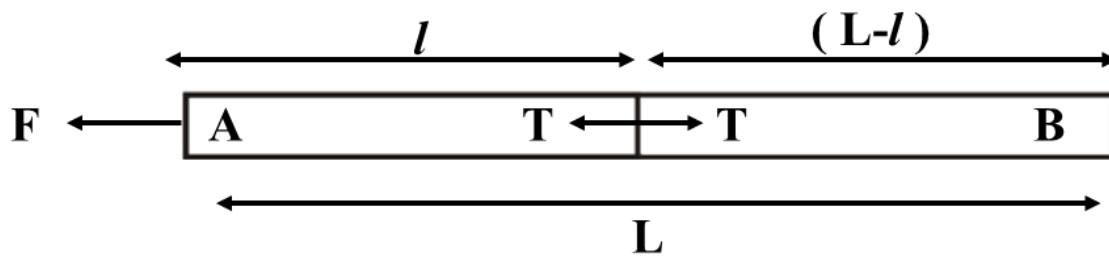
$$\frac{T \sin \theta}{T \cos \theta} = \frac{ma}{mg}$$

$$\tan \theta = \frac{a}{g}$$

Conclusion: Result shows that the mass doesn't matter during the motion.

Problems in inertial frame of reference:

Problem 4: A uniform rope of length L , lying on a horizontal smooth floor, is pulled by a horizontal force F . What is the tension in the rope at a distance l from the end where the force is applied?



Solution:

Massive rope have different value of tension at different position, $\sum F_x = F - T$

(mass of rope upto length 'l') $a = F - T$

$(ml)a = F - T \Rightarrow T = F - mla \rightarrow 1)$

$\therefore m = \frac{M}{L} = \frac{\text{mass}}{\text{length}} = \text{linear mass density}$

$\therefore F = Ma, \therefore F = mL a \rightarrow 2)$

plug in equation 2) in 1), we get,

$T = mL a - mla = ma(L - l)$

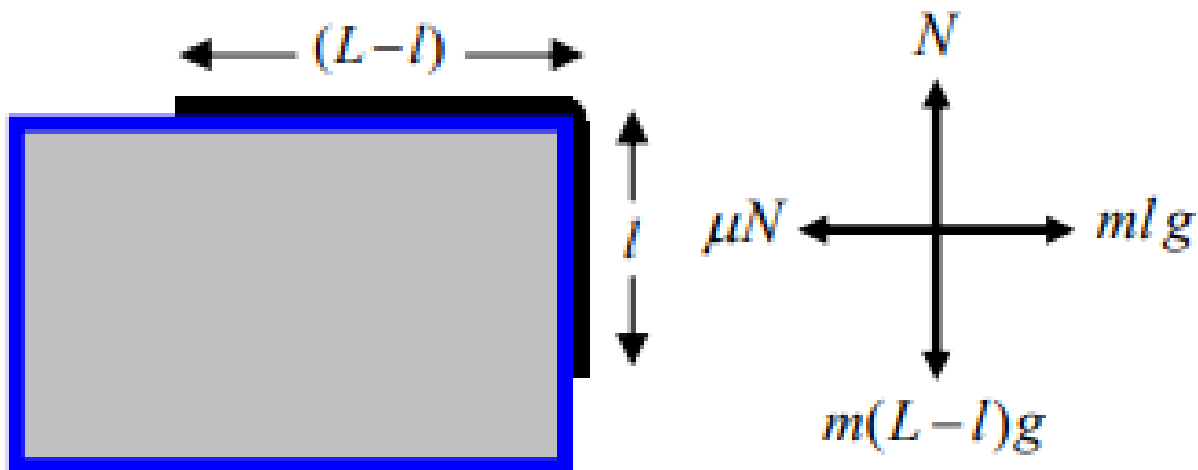
$T = maL \left(1 - \frac{l}{L}\right) = F \left(1 - \frac{l}{L}\right) \rightarrow 3$

Checking:

- If $l = 0$ then, $T = F$ (as, this is the point where we have applied force)
- If $l = L$ then, $T = 0$ (as, at the end of the rope, there is nothing (no force) to act upon)

Problem 5: A rope of total length L and mass per unit length m is put on a table with a length l hanging from one edge. What should be length l such that the rope just begins to slip?

Solution: To solve this, look at the balance of forces in the diagram.



Total length=L, Total mass=M

$$\text{mass per unit length}=m=\frac{M}{L}$$

$\therefore$ Force of gravity on hanging rope=(mass of hanging rope) g

Force of gravity on hanging rope = mlg

Force of gravity on rope lies on the table=(mass of rope on table) g

Force of gravity on rope lies on the table= $m(L-l)g$

Forces acting on the rope lies on the table;

$$\sum F_x = \mu N - mgl$$

$$0 = \mu N - mgl \Rightarrow \mu N = mgl \rightarrow 1$$

$$\sum F_y = N - mg(L-l)$$

$$0 = N - mg(L-l) \Rightarrow N = mg(L-l) \rightarrow 2$$

Putting eq.2 in eq.1, we get

$$\mu N = mgl$$

$$\mu(mg(L-l)) = mgl$$

$$\mu mgL - \mu mgl = mgl$$

$$\mu mgL = \mu mgl + mgl$$

$$\mu mgL = mg(l + \mu l) \Rightarrow \mu L = (\mu + 1)l$$

$$l = \frac{\mu L}{(\mu + 1)}$$

Conclusion:

- Note that if μ is very small ($\mu=0$) then, even a small piece of string that hangs over the edge will cause the entire string to slip down.
- And if you take μ to be a very large number then, a very small portion of the length of rope is enough to stay on table.

Problem 6: A massive rope of total length l is passing through a pulley as shown in figure.

The larger length of rope is moving downward with some acceleration which we have to calculate here.

Solution:

Total length = l

mass per unit length = m

on the right side of fig.

$$\sum F_y = m(l-x)g - mgx$$

(total mass of the rope) $a = mlg - mxg - mxg$

$$mla = mlg - 2mxg$$

$$mla = m(lg - 2xg) \Rightarrow la = (lg - 2xg)$$

$$a = \frac{lg}{l} - \frac{2xg}{l} \Rightarrow g - \frac{2xg}{l}$$

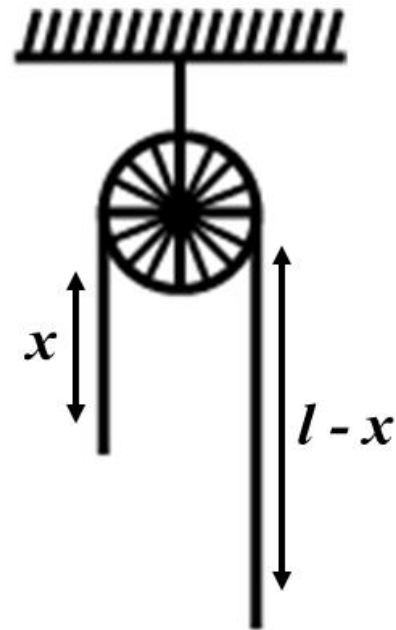
$$a = g \left(1 - \frac{2x}{l} \right)$$

Checking; if,

$$x = \frac{l}{2} \Rightarrow a = g \left(1 - \frac{2 \cdot \frac{l}{2}}{l} \right)$$

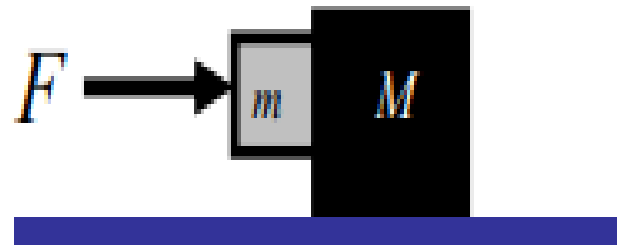
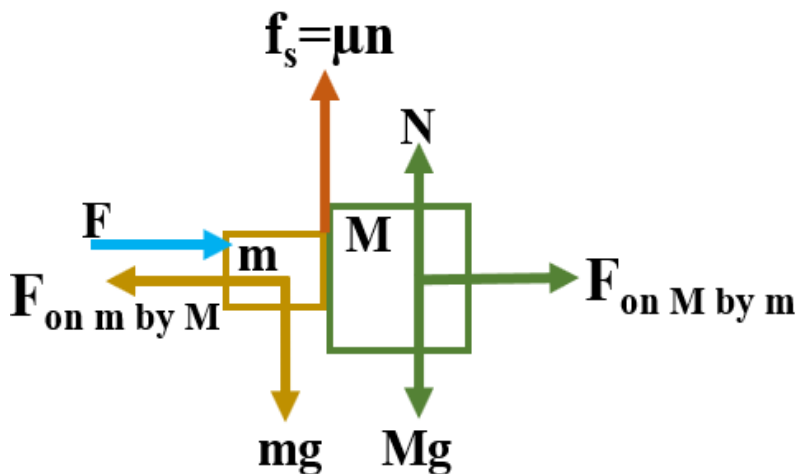
$$a = g$$

Both sides of the rope will have equal acceleration.



Problem 7: In this problem, two masses M and m attached together. We would like to calculate the minimum force such that the small block does not slip down.

Solution:



Since masses moves together so, $\sum F_x = F$

$$(m + M)a = F \Rightarrow a = \frac{F}{m + M} \rightarrow 1$$

At contact surface, $\sum F_y = f_s - mg$

$$0 = f_s - mg \Rightarrow f_s = mg \Rightarrow mg \leq f_s \rightarrow 2$$

Where, $f_s = \mu n \Rightarrow mg \leq \mu n \rightarrow 3$

$n =$ force by M on m $\Rightarrow n = ma$

$n =$ (mass of box on which force act)(acceleration of the system)

$$n = m \left(\frac{F}{m + M} \right) \rightarrow 4 \quad \because a = \frac{F}{m + M}$$

Using eq. 4 in eq.3, we get,

$$mg \leq \mu m \left(\frac{F}{m + M} \right) \Rightarrow g \leq \mu \left(\frac{F}{m + M} \right)$$

$$\mu \geq \frac{g(m + M)}{F} \rightarrow 5$$

Where, μ is a dimensionless quantity as it's the ratio of two forces.

Conclusion:

- We want the friction ' μn ' to be at least as large as the downwards force ' mg ' so that, it minimizes the horizontal force to prevent it from slipping.

Problem 8: Suppose two bodies m_1 and m_2 attached to a string and pass over the pulley. Mass m_1 is on the frictionless table and the forces acted on the masses are shown in figure. We have to calculate the tension and acceleration produced in the string.

Solution: If the string is massless then the tension in rope would be same at all points. So according to the balancing of forces, one can calculate as following:

Lets calculate the forces on m_1 & m_2 first,

Forces on m_1 ;

$$\sum F_x = T \Rightarrow m_1 a = T \rightarrow 1$$

$$\sum F_y = N - m_1 g$$

$$0 = N - m_1 g \Rightarrow N = m_1 g$$

Forces on m_2 ;

$$\sum F_y = m_2 g - T \Rightarrow m_2 a = m_2 g - T \rightarrow 2$$

To find "a", putting eq. 1 in eq. 2,

$$m_2 a = m_2 g - m_1 a$$

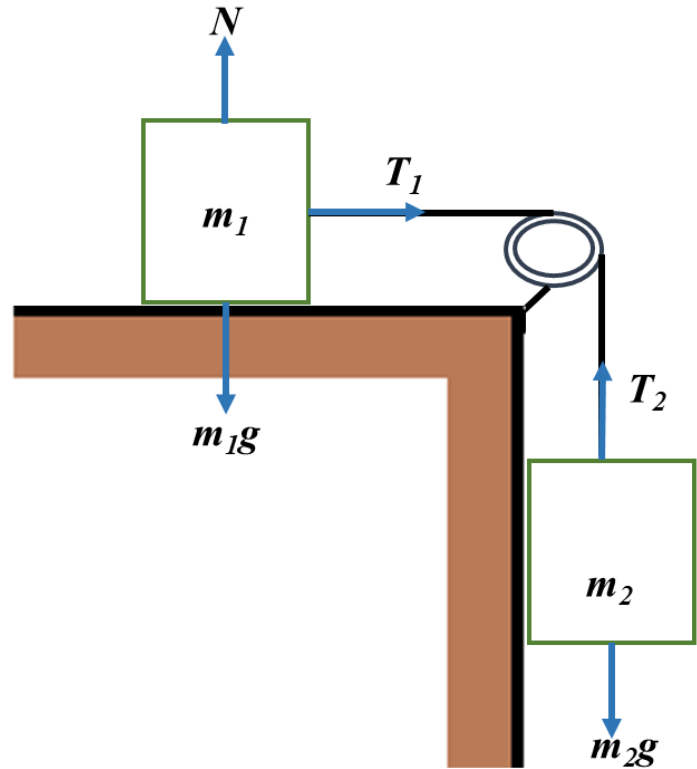
$$(m_1 + m_2) a = m_2 g \Rightarrow a = \left(\frac{m_2}{m_1 + m_2} \right) g \rightarrow 3$$

Now, to calculate "T", using eq.1 in eq.2,

$$m_2 \frac{T}{m_1} = m_2 g - T \Rightarrow m_2 T = m_1 m_2 g - m_1 T$$

$$(m_1 + m_2) T = m_1 m_2 g$$

$$T = \left(\frac{m_1 m_2}{m_1 + m_2} \right) g \rightarrow 4$$



Checking:

- If we put $m_1 = m_2$ in eq.3 and consider both masses as one mass, then it concludes that half of the mass is accelerated downward, and rest of the half is not. We can say that half of the mass is responsible for this acceleration.
- If m_1 or $m_2 = 0$ in eq.4, then T (tension in string) becomes zero.

Problem 9: Consider two bodies of unequal masses m_1 and m_2 connected by the ends of a string, which passes over a frictionless pulley as shown in the diagram. We have to calculate the tension in the string and acceleration of the masses.

Solution:

Forces on m_1 ;

$$\sum F_y = T - m_1 g$$

$$m_1 a = T - m_1 g \rightarrow 1$$

Forces on m_2 ;

$$\sum F_y = m_2 g - T$$

$$m_2 a = m_2 g - T \rightarrow 2$$

On subtracting, eq.1 & eq.2, we will get acceleration as,

$$m_1 a = T - m_1 g$$

$$m_2 a = m_2 g - T$$

$$(m_1 + m_2) a = (m_2 - m_1) g$$

$$a = \left(\frac{m_2 - m_1}{m_1 + m_2} \right) g \rightarrow 3$$

To calculate 'T', multiply eq.1 to m_2

and multiply eq.2 to m_1 and then on subtracting, we get

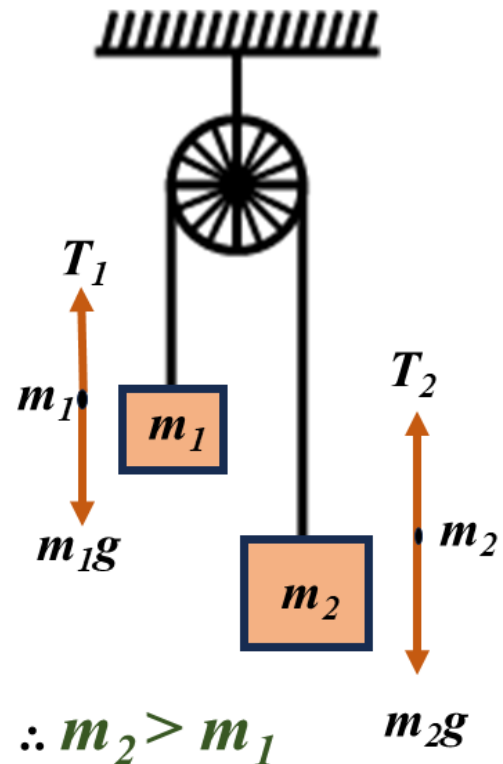
$$m_1 m_2 a = m_2 T - m_1 m_2 g$$

$$m_1 m_2 a = m_1 m_2 g - m_1 T$$

$$0 = (m_1 + m_2) T - 2m_1 m_2 g$$

$$(m_1 + m_2) T = 2m_1 m_2 g$$

$$T = \left(\frac{2m_1 m_2}{m_1 + m_2} \right) g \rightarrow 4$$



Conclusion: If $m_1 = m_2$ then, the body would not be able to accelerate because at both sides, masses are equal so, $a = 0$.

Physics-PHY101-Lecture 7

WORK AND ENERGY

Definition of Work:

“Force applied in direction of displacement x displacement”.

This means that the force F acts at an angle θ with respect to the direction of motion as shown in figure 7.1.

$$W = \vec{F} \cdot \vec{d} = Fd\cos\theta$$

- a) Work is a scalar - it has magnitude but no direction.
- b) Work has dimensions: $(MLT^{-2} \cdot L = ML^2 T^{-2})$
- c) Work has units: Newton · Meter $\equiv$ Joule (J)

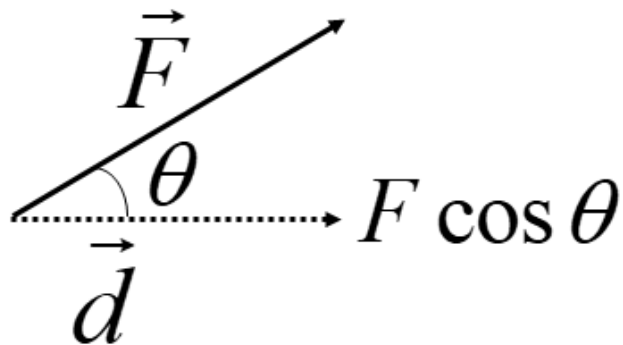


Figure 7.1: Work done by a constant force F .

Nature of work:

- **Positive work:** if the applied force displaces the object in its direction, then the work done is known as positive work. $\theta = 0^\circ$
- **Negative work:** if the force and displacement work in the opposite direction, then the work done is known as negative work. $\theta = 180^\circ$
- **Zero work;** If the force and displacement act perpendicular to each other, then the work done is known as zero work. $\theta = 90^\circ$

Forces do work on objects:

If a crate is pulled along the floor, only the force component parallel to the displacement d contributes to the work as shown in figure 7.2.

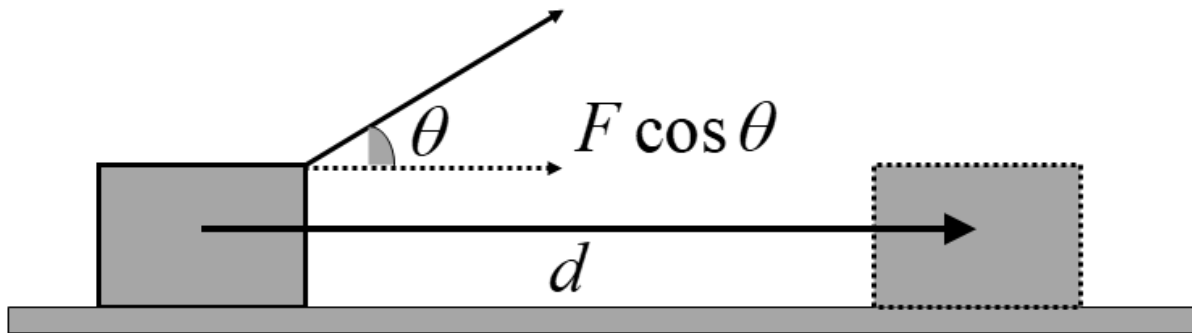
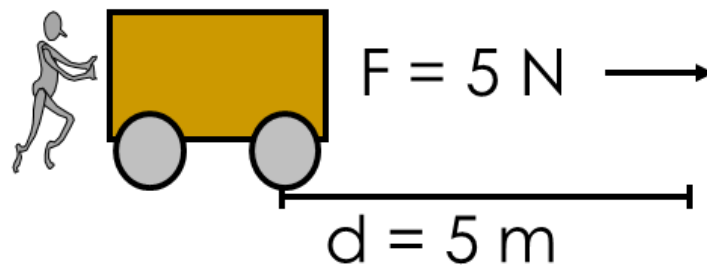


Figure 7.2: Work done by a force F in displacing object through displacement d .

We will justify this by following problem.

Problem 1: Suppose a man pushing a cart. A 5N force acts on it and displaces the cart through a displacement of 5m. As the force and displacement both are in same direction (parallel) to each other, calculate the work done on cart.



Solution: As we know,

$$W = F \cdot d \cos \theta = 5 \cdot 5 \cos 0 = 25 (1) \text{ Nm} = 25 \text{ J}$$

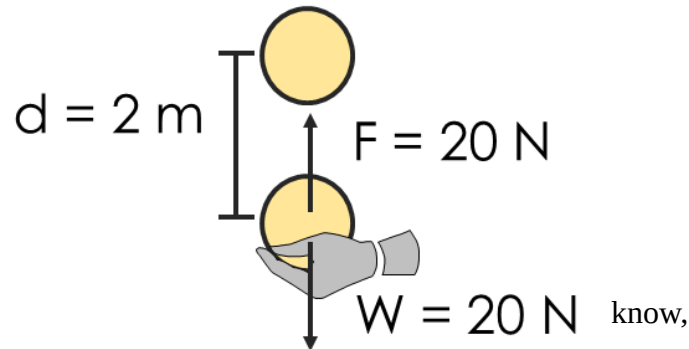
Problem 2: Suppose an apple falling from a tree. Due to its weight (1N), a force of gravity acts on it. Let suppose if it covers a displacement of 3m, then calculate the work done on apple due to gravity.



Solution: As we know,

$$W = F \cdot d \cos\theta = 1\text{ N} \times 3\text{ m} = 3\text{ Nm} = 3\text{ J}$$

Problem 3: A heavy weight of 20 N is lifted from a height of 2m. calculate the work done on the heavy weight.



Solution: As we

$$W = F \cdot d \cos\theta = 20\text{ N} \times 2\text{ m} \cos\theta$$

Here, we have to deal with two cases:

Work on ball by $F_{\text{hand}} = 20\text{ N} \times 2\text{ m} \cos 0 = 40\text{ J}$ (+ve work done)

Work on ball by $F_{\text{gravity}} = 20\text{ N} \times 2\text{ m} \cos 180 = -40\text{ J}$ (because force and displacement are anti-parallel to each other i.e., -ve work done)

Conclusion:

- If a mass is attached to a string and is moving in a circular motion, tension T is produced in a string. But this tension is not responsible for work as, T is perpendicular to the d (displacement).
- If the angle between two quantities is 90° , then it contributes towards no work.

Work done by a variable force:

Let the force $F(x)$ act in the x direction, and let it vary in magnitude with x according to the function $F(x)$ as shown in figure 7.3. What is the work done when the body moves from some initial position to some final position?

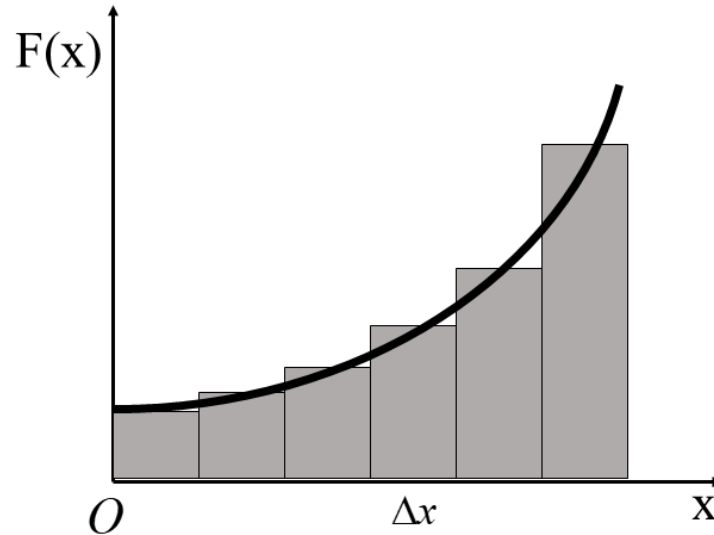


Figure 7.3. Variation of Force with Respect to x . The graph illustrates the dynamic nature of the force applied, showing fluctuations in response to changes in x .

What if the force varies with distance (say, a spring pulls harder as it becomes longer). In that case, we should break up the distance over which the force acts into small pieces so that the force is approximately constant over each bit. As we make the pieces smaller and smaller, we will approach the exact result.

$$\Delta W_1 = F_1 \Delta x$$

$$\Delta W_2 = F_2 \Delta x$$

$$\Delta W_3 = F_3 \Delta x$$

Now add up all the little pieces of work:

$$\begin{aligned} W &= \Delta W_1 + \Delta W_2 + \cdots + \Delta W_N \\ &= F_1 \Delta x + F_2 \Delta x + \cdots + F_N \Delta x \end{aligned}$$

To get the exact result let $\Delta x \rightarrow 0$ and the number of intervals $N \rightarrow \infty$:

$$W = \sum_{n=1}^N F_n \Delta x$$

$$W = \lim_{\Delta x \rightarrow 0} \sum_{n=1}^{\infty} F_n \Delta x$$

$$\text{Definition: } \lim_{\Delta x \rightarrow 0} \sum_{n=1}^{\infty} F_n \Delta x \equiv \int_{x_i}^{x_f} F(x) dx$$

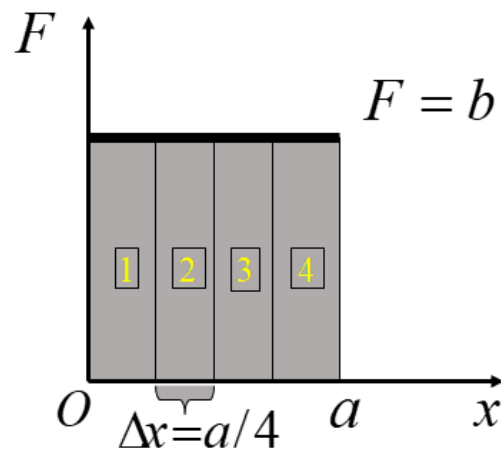
is the integral of F with respect to x from x_i to x_f .

This quantity is the work done by a force, constant or non-constant. So, if the force is known as a function of position, we can always find the work done by calculating the definite integral.

Example (for constant force):

Let suppose we have a rectangle, which is divided into 4 parts.

Area =



$$\frac{1}{4} a(b) + \frac{1}{4} a(b) + \frac{1}{4} a(b) + \frac{1}{4} a(b) = ab$$

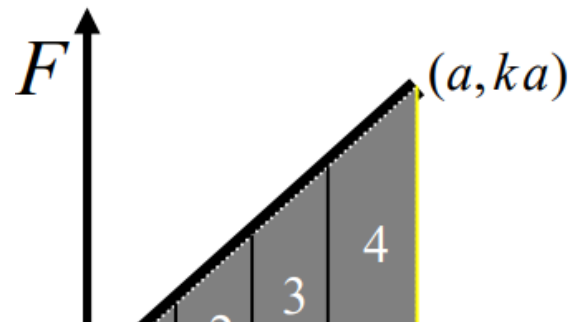
$$W = \int_0^a F dx$$

$$\text{For a constant force, } W = F \int_0^a dx = F \left| x \right|_0^a$$

$$\therefore F = b$$

$$W = b(a - 0) = ab$$

Example (for variable force):



Let suppose we have a triangle, which is divided into 4 parts. Here, $F = kx$, force is proportional to x i.e., the force increases linearly with x .

$$\text{Area} = \frac{1}{2} * \text{base} * \text{height}$$

From the fig. base is "a" and height is "F", $\therefore F = ka$

$$\text{Area of shaded region} = \frac{1}{2}(a)(ka) = k \frac{a^2}{2}$$

Mathematically, work done is given as, $W = \int F dx$

$$W = \int_0^a F dx = \int_0^a kx dx = \frac{k}{2} \left[x^2 \right]_0^a = \frac{k}{2} (a^2 - 0) = k \frac{a^2}{2}$$

Example: Calculation of area under the curve:

Suppose we have a parabolic function $y = x^2$ as shown in figure 7.4. The parabola is divided into 4 parts so, $\Delta x = 1/n = 1/4$ (as maximum value on x and y -axis is "1") and evaluated at four different points as shown in figure.

Area under the curve between 0 and 1 is approximately, $\sum_{i=1}^n y_i \Delta x$

As, number of rectangles, $n=4$, hence, $\Delta x = \frac{1.00}{n} \rightarrow 1$

$x_i = \left(i - \frac{1}{2}\right) \Delta x \Rightarrow$ putting the value of $\Delta x = \frac{1.00}{n}$

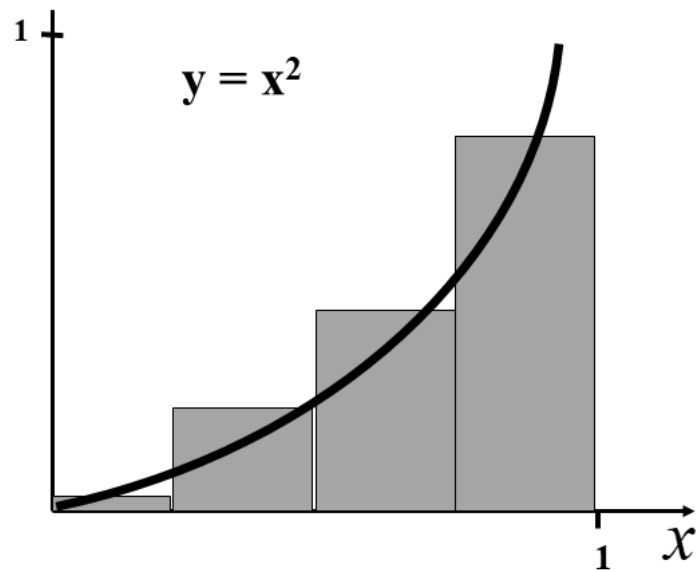


Figure 7.4: Graph of the Parabolic Function $y = x^2$. The curve represents the quadratic relationship between the independent variable x and the dependent variable y , forming a symmetric parabola centered at the origin.

$$x_i = \left(i - \frac{1}{2}\right) \frac{1.00}{n} = \frac{\left(i - \frac{1}{2}\right)}{n}$$

As, the function is, $y_i = x_i^2$, hence, $y_i = \frac{\left(i - \frac{1}{2}\right)^2}{n^2} \rightarrow 2$

Since, area under the curve is, $\sum_{i=1}^n y_i \Delta x$, so

$$\sum_{i=1}^n y_i \Delta x = \sum_{i=1}^n \frac{\left(i - \frac{1}{2}\right)^2}{n^2} \cdot \frac{1.00}{n} = \sum_{i=1}^n \frac{\left(i - \frac{1}{2}\right)^2}{n^3}$$

Now lets solve R.H.S of the equation,

$$\sum_{i=1}^n \frac{\left(i - \frac{1}{2}\right)^2}{n^3} = \sum_{i=1}^n \frac{1}{n^3} \left(i^2 + \frac{1}{4} - i\right) \Rightarrow = \frac{1}{n^3} \left[\sum_{i=1}^n i^2 + \frac{1}{4} \sum_{i=1}^n 1 - \sum_{i=1}^n i \right] \rightarrow 3$$

Using properties of summation,

$$\because \sum_{i=1}^n 1 = n, \because \sum_{i=1}^n i = \frac{n(n+1)}{2}, \because \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

Now 3 equation becomes,

$$\begin{aligned} &= \frac{1}{n^3} \left[\frac{n(n+1)(2n+1)}{6} + \frac{1}{4}n - \frac{n(n+1)}{2} \right] \Rightarrow = \frac{1}{n^3} \left[\frac{(n^2+n)(2n+1)}{6} + \frac{n}{4} - \frac{(n^2+n)}{2} \right] \\ &= \frac{1}{n^3} \left[\frac{2n^3 + n^2 + 2n^2 + n}{6} + \frac{n}{4} - \frac{(n^2+n)}{2} \right] \Rightarrow = \frac{1}{n^3} \left[\frac{4n^3 + 2n^2 + 4n^2 + 2n + 3n - 6n^2 - 6n}{12} \right] \\ &= \frac{1}{n^3} \left[\frac{4n^3 - n}{12} \right] = \frac{n(4n^2 - 1)}{12n^3} = \frac{(4n^2 - 1)}{12n^2} = \frac{4n^2}{12n^2} - \frac{1}{12n^2} = \frac{1}{3} - \frac{1}{12n^2} \end{aligned}$$

$$\sum_{i=1}^n \frac{\left(i - \frac{1}{2}\right)^2}{n^3} = \frac{1}{3} - \frac{1}{12n^2}$$

$$\text{putting } n = 4, \sum_{i=1}^n \frac{\left(i - \frac{1}{2}\right)^2}{n^3} = 0.32812$$

Now if we divide the parabola into 8 parts ($n = 8$) then the result will be: $\sum_{i=1}^n \frac{\left(i - \frac{1}{2}\right)^2}{n^3} = 0.33203$

And if $n = 16$, then,
$$\sum_{i=1}^n \frac{\left(i - \frac{1}{2}\right)^2}{n^3} = 0.33301$$

And if $n = 32$, then,
$$\sum_{i=1}^n \frac{\left(i - \frac{1}{2}\right)^2}{n^3} = 0.33325$$

Now if we divide the parabola into infinite parts ($n = \infty$) then the result will be :

$$\sum_{i=1}^n y_i \Delta x = \lim_{\Delta x \rightarrow 0} \sum_{i=1}^n y_i \Delta x = \int_0^1 x^2 dx = \lim_{n \rightarrow \infty} \left[\frac{1}{3} - \frac{1}{12n^2} \right] = \frac{1}{3} = 0.3333$$

In general, if we integrate a function $\int_0^1 x^n dx$ then the result would be $= \frac{1}{n+1}$. In this way we can

calculate the total work done by a variable force.

Introduction to Energy:

In physics work and energy is of great importance. Energy is the capacity of a physical system to do work.

- It comes in many forms – mechanical, electrical, chemical, nuclear, etc.
- It can be stored
- It can be converted into different forms
- It can never be created or destroyed

Types of Energy:

1) Elastic energy:

Elastic energy is stored in an elastic object - such as a coiled spring or a stretched elastic band. Elastic objects store elastic energy when a force causes them to be stretched.

2) Gravitational energy:

Gravitational energy associated with gravity or gravitational force. In other words, the energy held by an object when it is in a high position compared to a lower position.

3) Electrical energy:

Electrical energy is the movement of electrons. Lightning is an example of electrical energy.

4) Chemical energy:

Chemical energy is stored in the bonds of atoms and molecules. It is the energy that holds these particles together. Stored chemical energy is found in food, biomass, petroleum, and natural gas.

5) Thermal energy:

Thermal energy is created from the vibration of atoms and molecules within substances. The faster they move, the more energy they possess and the hotter they become. Thermal energy is also called heat energy.

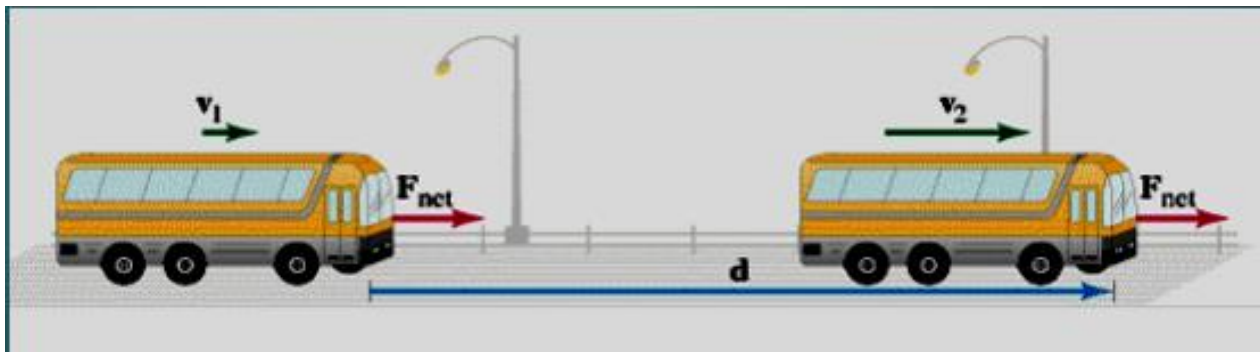
6) Nuclear energy:

Nuclear energy is stored in the nucleus of atoms. This energy is released when the nuclei are combined (fusion) or split apart (fission). Nuclear power plants split the nuclei of uranium atoms to produce electricity.

7) Sound energy:

Sound energy is the energy produced due to vibrations. There is usually much less energy in sound than in other forms of energy.

Problem 4: A constant force accelerates a bus (mass m) from speed v_1 to speed v_2 over a distance d . What work is done by the engine?



Solution:

$$\text{Recall: } v_2^2 - v_1^2 = 2a(x_2 - x_1)$$

where: v_2 = final velocity, x_2 = final position

v_1 = initial velocity, x_1 = initial position

$$\therefore a = \frac{v_2^2 - v_1^2}{2d}$$

To calculate work:

$$\begin{aligned}
 W &= F d \\
 &= m a d \\
 &= m \frac{v_2^2 - v_1^2}{2d} d \\
 &= \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2
 \end{aligned}$$

Define Kinetic Energy:

$$KE = \frac{1}{2} m v^2$$

so, work done is equal to the change in kinetic energy.

$$W = \Delta K.E$$

Problem 5: A truck weighs 20 times more than a rickshaw but moves 5 times slower. Which has more kinetic energy?

Solution:

$$\begin{aligned}
 KE \text{ (rickshaw)} &= \frac{1}{2} m v^2 \\
 KE \text{ (truck)} &= \frac{1}{2} (20m) (v/5)^2 \\
 &= \frac{20}{25} \cdot \frac{1}{2} m v^2
 \end{aligned}$$

As, the speed of truck is 5 times less than the rickshaw, hence, the kinetic energy of truck is less than the rickshaw.

Work Energy Principle:

Net work done on object = Change in *K.E* of object

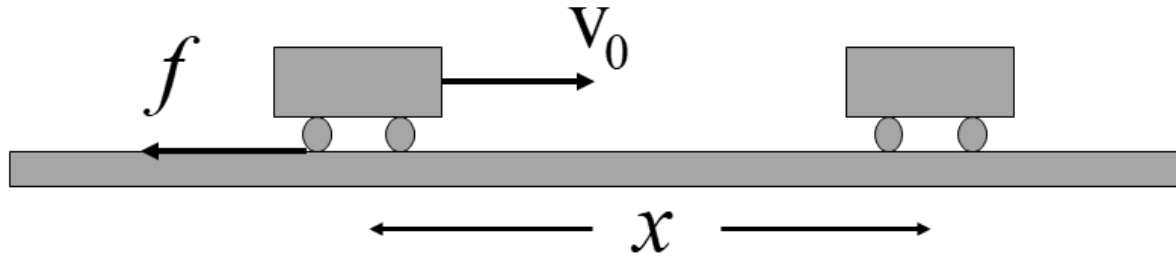
Work can be:

- Positive (K.E increases)
- Negative (K.E decreases)

Units of energy: Energy has the same units as work:

Joule = Newton. Meter = Nm

Problem 6: A car is moving on a road, and a frictional force acts on it. We want to know how far the car travel before it comes to rest.



Solution: According to Work Energy Principle,

$$W_{net} = K_f - K_i = 0 - K_i$$

$$-f x = -\frac{1}{2} m v_0^2 \quad \text{but } f = \mu N = \mu mg$$

$$\text{k.E consumed by the frictional work is, } f x = \frac{1}{2} m v_0^2$$

$$\mu mg x = \frac{1}{2} m v_0^2 \Rightarrow x = \frac{v_0^2}{2\mu g}$$

Checking:

- If $\mu = 0$, then $x = \infty$, which means, in the absence of friction, car does not stop and move through an infinite distance.
- If $\mu = \infty$, then $x = 0$, which means, car would not move or cover a distance because of infinite friction.

Introduction to power:

The work done by a force is just the force multiplied by the distance – it does not depend upon time. But suppose that the same amount of work is done in half the time. We then say that the power is twice as much.

Power is the “rate of doing work”

$$\text{Power} = \frac{\text{Work done}}{\text{Time taken}}$$

If the force does not depend on time:

$$\frac{\text{Work}}{\text{Time}} = \frac{F \Delta x}{\Delta t} = F v$$

$$\therefore \text{Power} = F v$$

Units of power: J/sec = Watts

Old units: horsepower (hp)

$$1 \text{ hp} = 746 \text{ W} = 0.746 \text{ kW}$$

Problem 7: A 2000 kg trolley is pulled up a 30° hill at 20 mi/hr by a rope. How much power is the machine providing?

Solution:

- The power is $P = F v = T v$
- No acceleration $\Rightarrow$ no net force
- Balance forces along and normal to plane

$$\text{In the x direction: } T = mg \sin \theta$$

$$v = 20 \text{ mi/hr} = 8.93 \text{ m/s}$$

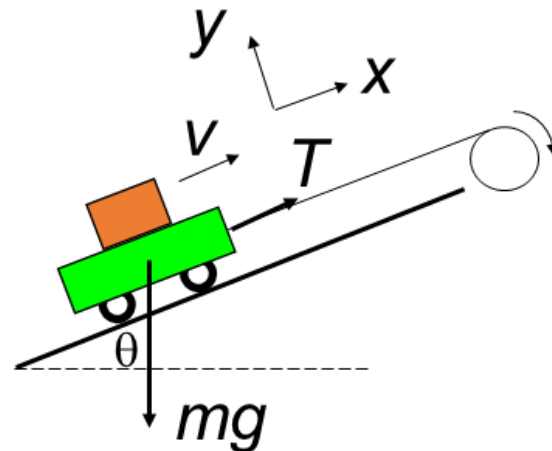
$$g = 9.8 \text{ m/s}^2$$

$$m = 2000 \text{ kg}$$

$$\sin \theta = \sin(30^\circ) = 0.5$$

$$P = (2000 \text{ kg}) \cdot (9.8 \text{ m/s}^2) (8.93 \text{ m/s}) (0.5)$$

$$P = 88,000 \text{ W (power of machine)} = 88 \text{ kW}$$



Physics-PHY101-Lecture 8

Conservation of Energy

Potential energy:

Potential energy is, as the word suggests, the energy “locked up” up somewhere and which can do work. It can be defined as, “**energy possessed by a body due to its position**”.

It has the ability or capacity to do work like other forms of energies. Potential energy can be converted into kinetic energy ($\frac{1}{2}mv^2$) and vice versa.

Formula: Potential energy can be calculated using the following formula

$$P.E = m.g.h$$

Where,

- m is the mass in kilograms
- g is the acceleration due to gravity (10 m/s^2)
- h is the height in meters

SI unit:

$$P.E = m.g.h = \text{kg}.\text{ms}^{-2}.\text{m} = \text{kgm}^2\text{s}^{-2} = \text{Joule (J)}$$

Types of potential energy:

The types of potential energy are,

1. Elastic potential energy
2. Gravitational potential energy

Elastic potential energy:

When an object is compressed or stretched, the energy stored in object is called elastic potential energy. The more the object stretch or compress, the more elastic potential energy is.

Examples:

- A twisted rubber band that powers a toy plane
- An archer's stretched bow
- A bent diver's board just before a diver dive in

Formula: Elastic potential energy can be calculated using the following formula,

$$U = \frac{1}{2} kx^2$$

Where,

- U is the elastic potential energy
- k is the spring force constant
- x is the string stretch length in m

Problem 1:

Suppose you pull on a spring and stretch it by an amount away from its normal (equilibrium) position. How much energy is stored in the spring?

Solution:

Obviously, the spring gets harder and harder to pull as it becomes longer. When it is extended by length x and you pull it a further distance dx, the small amount of work done is $dW = Fdx = kx dx$. Adding up all the small bits of work gives the total work:

$$W = \int_0^x F dx = \int_0^x kx dx = \frac{1}{2} kx^2$$

Gravitational potential energy:

The gravitational potential energy of an object is defined as “the energy possessed by an object that raise to a certain height against gravity. As the object is raised against the force of gravity, some amount of work (W) is done on it.

Work done on the object = force × displacement

$$W = F.h$$

$$W = mg.h$$

(a = g = gravitational acceleration)

$$W = P.E = m.g.h$$

Example: If you lift a stone of mass from the ground up a distance, you have to do work against gravity. By conservation of energy, the work done by you was transformed into gravitational potential energy whose value is exactly equal to mgx . Where is the energy stored?

Answer: it is stored neither in the mass nor in the earth. It is stored in the gravitational field of the combined system of stone + earth.

1. How much work does it take to lift a mass m to height h , as shown in figure?

Answer:

$$\begin{aligned} W_{\text{ext}} &= F_{\text{ext}} d \\ W_{\text{ext}} &= mg h \\ W_{\text{ext}} &= m g (y_2 - y_1) \end{aligned}$$

You did work on the object. Therefore, its energy increased.

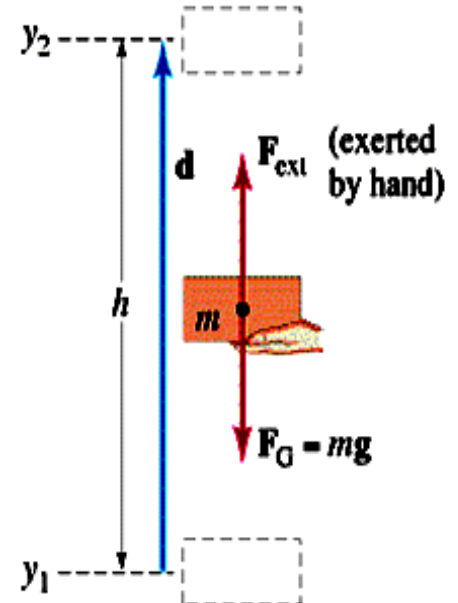
Key points:

- PE is measured with respect to some reference level.
- Only changes in PE actually have physical meaning.
- Changes in PE do not depend on path.
- Energy is a shared property.
- Work is force x distance.
- Energy is the capacity to do work.
- Power is the “rate of doing work”
- $\text{Power} = \frac{\text{Work done}}{\text{Time taken}}$

Problem 2:

Your heart is working as a pump and the volume of blood lifted daily is 8000 liters and the blood is lifted up to a height of 1.5 m. calculate its potential energy and power.

Solution:



Volume of blood lifted daily = 8000 litres

Average height lifted = 1.5 m

$\therefore$ Density of blood ≈ 1 kg/litre

$P.E = mgh$ $\therefore$ density = mass/volume $\Rightarrow$ mass = density * volume

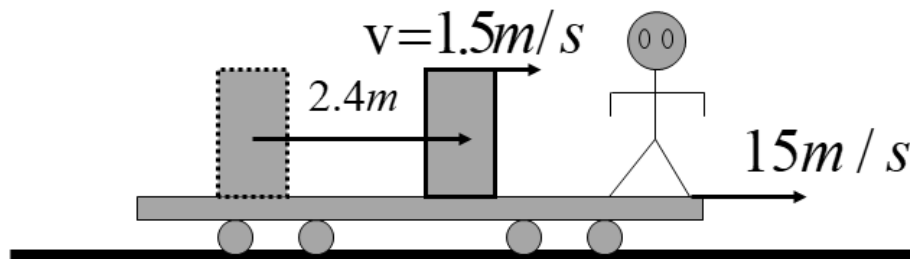
Work done = $P.E = 8000 \times 1 \times 9.8 \times 1.5$

$\approx 120,000$ J in 24 hours

$\therefore$ Power = $\frac{\text{Work done}}{\text{time}} = \frac{120000}{24 \times 60 \times 60} \approx 1.4$ W

Problem 3:

A box of mass 12kg is pushed with a constant force so that its speed goes from zero to 1.5m/sec (as measured by the person at rest on the cart) and it covers a distance of 2.4m. Assume there is no friction.



Solution:

mass of box = 12 kg

$$\Delta K = K_f - K_i = \frac{1}{2}(12\text{kg})(1.5\text{m/s})^2 - 0 = 13.5\text{J}$$

$$a = \frac{v_f^2 - v_i^2}{2(x_f - x_i)} = \frac{(1.5\text{m/s})^2 - 0}{2(2.4\text{m})} = 0.469\text{m/s}^2$$

This acceleration results from a constant net force given by:

$$F = ma = (12\text{kg})(0.469\text{m/s}^2) = 5.63\text{N}$$

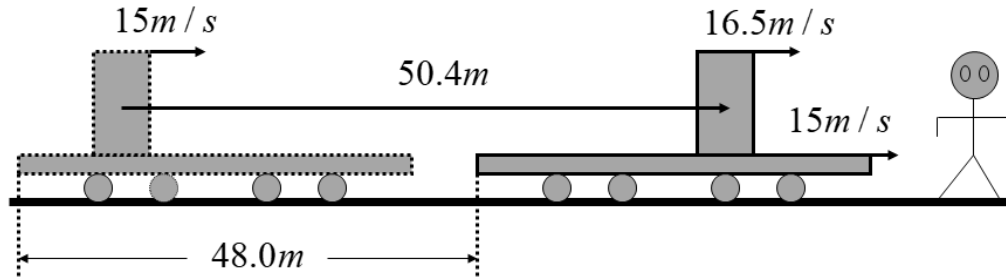
Work done on the crate is:

$$W = F\Delta x = (5.63\text{N})(2.4\text{m}) = 13.5\text{J}$$

(same as $\Delta K = 13.5\text{J}$!)

Problem 4:

Suppose there is somebody standing on the ground, and that the trolley moves at 15 m/sec relative to the ground.



Solution:

Let us repeat the same calculation:

$$\begin{aligned}\Delta K' &= K'_f - K'_i = \frac{1}{2}mv_f'^2 - \frac{1}{2}mv_i'^2 \\ &= \frac{1}{2}(12\text{kg})(16.5\text{ m/s})^2 - \frac{1}{2}(12\text{kg})(15.0\text{ m/s})^2 \\ &= 284\text{ J}\end{aligned}$$

(This is not equal to $\Delta K = 13.5\text{ J}$)

This example clearly shows that work and energy have different values in different frames. Now, if this person calculate the force to be 5.63 N, then calculate work done by the person.

$$\therefore a' = a \therefore F' = F = 5.63\text{ N}$$

$$t = \frac{v_f - v_i}{a} = \frac{1.5\text{ m/s}}{0.469\text{ m/s}^2} = 3.2\text{ s}$$

$$\text{and train moves } (15\text{ m/s})(3.2\text{ s}) = 48\text{ m}$$

$$\begin{aligned}\text{Total displacement} &= \Delta x' \\ &= 48\text{ m} + 2.4\text{ m} = 50.4\text{ m}\end{aligned}$$

The ground-based observer also concludes that the work is:

$$W' = F'\Delta x' = (5.63\text{ N})(50.4\text{ m}) = 284\text{ J}$$

Conclusion:

- If you calculate work or kinetic energy in different frame of reference, the results would be different in both frames.

- Potential energy and kinetic energy depend on the frame you choose to measure it in. If you are running with a ball, it has zero kinetic energy with respect to you. But someone who is standing will see that it has kinetic energy.
- In the absence of friction, the total energy of the system is conserved.
- Total mechanical energy is:

$$E_{\text{mech}} = KE + PE$$
- If no friction then E_{mech} is conserved:

$$\Delta(E_{\text{mech}}) = \Delta(KE) + \Delta(PE) = 0$$
- $E_{\text{mech}} = KE + PE$ is constant

Problem 5:

A ball is thrown upwards at speed v_0 . How high will it go before it stops?

Solution:

The loss of potential energy is equal to the gain of potential energy,

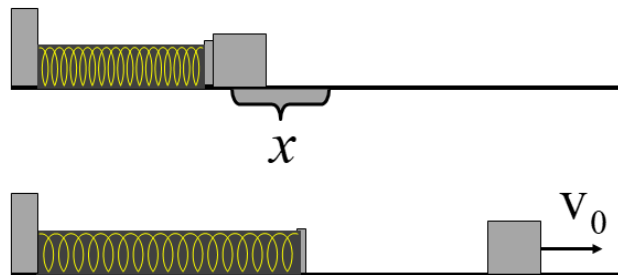
$$K.E = P.E$$

$$\frac{1}{2}mv_0^2 = mgh$$

$$h = \frac{v_0^2}{2g}$$

Problem 6:

A mass is attached to a spring, this spring is compressed, and elastic potential energy is stored in it. The extension created in this string is x . When mass m is released from the spring, it moves with velocity V_0 . Here, we want to calculate that velocity.



Solution:

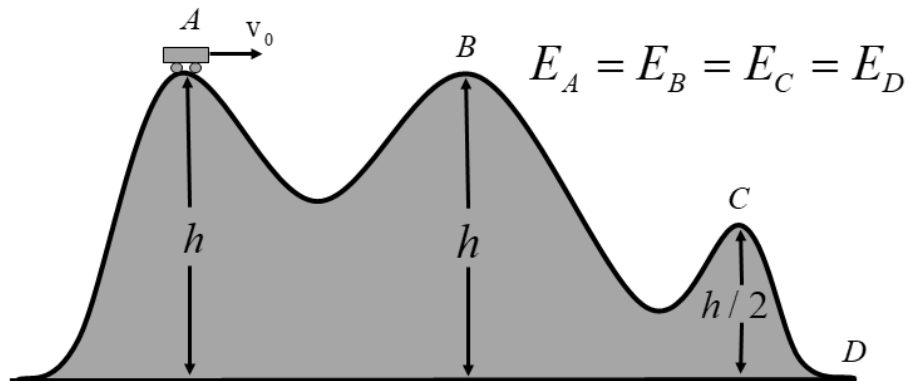
As there is no friction, the kinetic energy of the spring becomes equal to the elastic potential energy.

$$\frac{1}{2}mv_0^2 = \frac{1}{2}kx^2$$

$$v_0 = \sqrt{\frac{k}{m}}x$$

Problem 7:

Now look at the smooth, frictionless motion of a car over the hills as shown below:



What will be the speed of the car,

- a) at point B,
- b) at point C,
- c) at point D?

Solution: As, in the absence of friction, energy at every point would be the same.

$$E_A = E_B = E_C = E_D$$

According to the law of conservation of energy,

From A to B;

$$\frac{1}{2}mv_A^2 + mgh = \frac{1}{2}mv_B^2 + mgh$$

$$v_A = v_B = v_0 \rightarrow 1$$

From B to C;

$$\frac{1}{2}mv_B^2 + mgh = \frac{1}{2}mv_C^2 + mg\frac{h}{2}$$

$$\frac{1}{2}mv_B^2 + mgh = \frac{1}{2}(mv_C^2 + mgh)$$

$$mv_B^2 + mgh = (mv_C^2 + mgh)$$

$$v_B^2 + gh = v_C^2$$

$$v_C = \sqrt{v_0^2 + gh} \rightarrow 2$$

From C to D;

$$\frac{1}{2}mv_C^2 + mg\frac{h}{2} = \frac{1}{2}mv_D^2$$

$$v_D^2 = v_C^2 + gh \rightarrow 3$$

putting eq.2 in eq.3

$$v_D^2 = v_0^2 + gh + gh$$

$$v_D = \sqrt{v_0^2 + 2gh} \rightarrow 4$$

Remember that potential energy has meaning only for a force that is conservative. A conservative force is that for which the work done in going from point A to point B is independent of the path chosen. Friction is an example of a non-conservative force, and a potential energy cannot be defined. For a conservative force,

$$F = -\frac{dV}{dx}$$

$$\text{as, } V = \frac{1}{2}kx^2$$

$$F = -\frac{d}{dx}\left[\frac{1}{2}kx^2\right] = -\frac{1}{2}k \cdot \frac{d}{dx}[x^2]$$

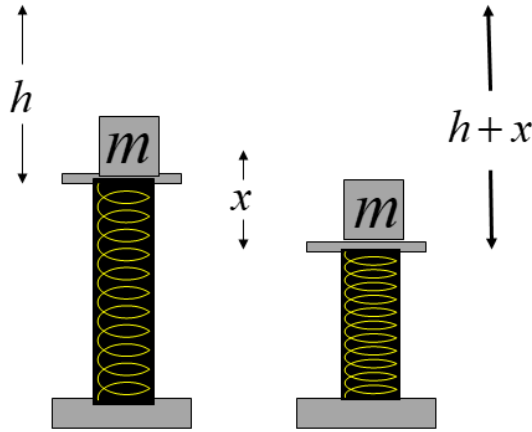
$$F = -\frac{1}{2}k \cdot 2x = -kx$$

Problem 8:

A mass m is taken on height h above spring. The mass compresses the spring upon falling on it as shown in the figure. The extension in the spring is x . We have to calculate that x .

Solution:

According to the conservation of energy, initial K.E becomes equal to the P.E stored in the spring,



$$\frac{1}{2} kx^2 = mg(x + h)$$

$$\frac{1}{2} kx^2 = mgx + mgh \Rightarrow \frac{1}{2} kx^2 - mgx - mgh = 0$$

$$a = \frac{1}{2} k, \quad b = -mg, \quad c = -mgh$$

On applying quadratic formula, we get,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{mg \pm \sqrt{(mg)^2 - 4\left(\frac{1}{2}k\right)(-mgh)}}{k} = \frac{mg \pm \sqrt{m^2 g^2 + (2k)(mgh)}}{k}$$

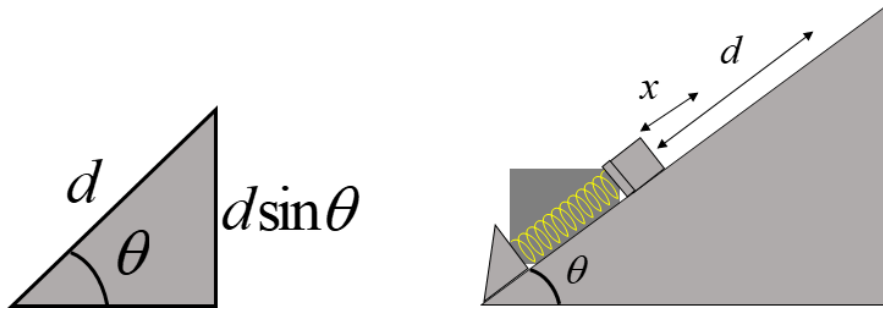
$$x = \frac{mg \pm mg \sqrt{1 + \frac{2kh}{mg}}}{k}$$

$$\Rightarrow x = \frac{mg}{k} (1 \pm \sqrt{1 + 2hk / mg})$$

Here, we find the two solutions for this problem. The two solutions prove the oscillation of the spring (up & down) when mass falls on it.

Problem 9:

Consider an inclined plane. A spring with attached mass is taken on it. The mass is stretched through a distance of d , which compresses the spring through x . We want to calculate the value of x .



Solution: According to the conservation of energy, K.E becomes equal to the P.E stored in the spring,

$$\frac{1}{2} kx^2 = mgd \sin \theta$$

$$\Rightarrow d = \frac{kx^2}{2mg \sin \theta}$$

$$mgd \sin \theta = \frac{1}{2} mv^2$$

$$\Rightarrow v = \sqrt{2gd \sin \theta}$$

To find d , we have,

$$mg(x + d) \sin \theta = \frac{1}{2} kx^2 \Rightarrow mgx \sin \theta + mgd \sin \theta = \frac{1}{2} kx^2$$

$$mgd \sin \theta = \frac{1}{2} kx^2 - mgx \sin \theta$$

$$d = \frac{kx^2}{2mg \sin \theta} - x$$

To find x , we have,

$$\frac{1}{2} kx^2 - mgx \sin \theta - mgd \sin \theta = 0$$

$$a = \frac{1}{2} k, \quad b = -mg \sin \theta, \quad c = -mgd \sin \theta$$

on applying quadratic formula, we get,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{mg \sin \theta \pm \sqrt{(mg \sin \theta)^2 - 4\left(\frac{1}{2}k\right)(-mgd \sin \theta)}}{k}$$

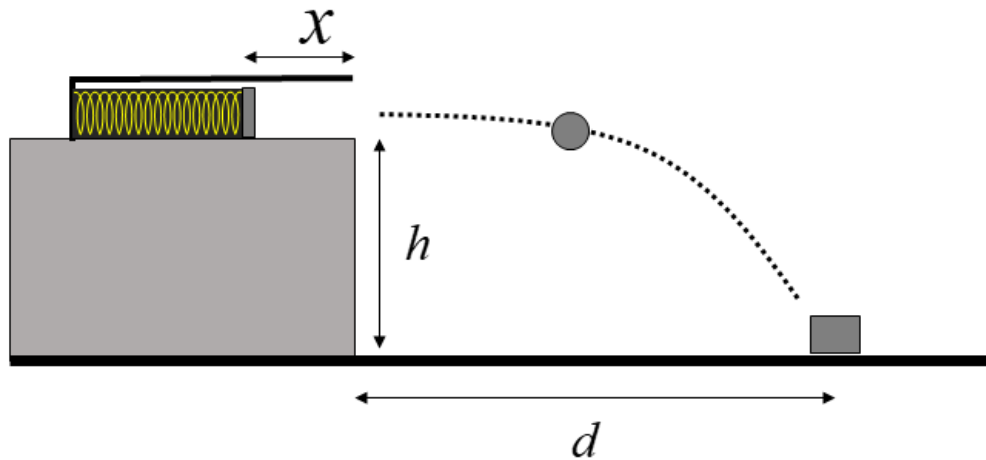
$$x = \frac{mg \sin \theta \pm \sqrt{(mg \sin \theta)^2 + 2k(mgd \sin \theta)}}{k}$$

$$x = \frac{mg \sin \theta \pm mg \sin \theta \sqrt{1 + \left(\frac{2kd}{mg \sin \theta}\right)}}{k}$$

$$x = \frac{mg \sin \theta}{k} \left(1 \pm \sqrt{1 + \frac{2kd}{mg \sin \theta}} \right)$$

Problem 10:

A spring is taken at height h from ground level and a ball is attached to it. When the spring is compressed, the ball falls down to earth and make a parabolic path and touches the mass which is on ground at distance d . We want to know the value of x (extension of the spring) as shown in figure.



Solution:

According to the conservation of energy, K.E becomes equal to the P.E stored in the spring,

$$\frac{1}{2}mv_0^2 = \frac{1}{2}kx^2$$

$$\Rightarrow x = \sqrt{\frac{m}{k}}v_0$$

$$v_0 = v_{ix} = \sqrt{\frac{kx}{m}}$$

y – component;

$$S_y = v_{iy}t + \frac{1}{2}a_yt^2$$

$$S_y = h, \quad a_y = -9.8$$

$$h = 0 - \frac{1}{2}gt^2 = -\frac{1}{2}gt^2$$

$$\Rightarrow t = \sqrt{\frac{2h}{g}}$$

x – component;

$$S_x = v_{ix}t + \frac{1}{2}a_xt^2$$

$$d = \sqrt{\frac{kx}{m}}t + 0$$

$$d = \sqrt{\frac{kx}{m}} \cdot \sqrt{\frac{2h}{g}} = \sqrt{\frac{2kh}{mg}}x$$

$$\Rightarrow x = d\sqrt{\frac{mg}{2hk}}$$

Problem 11:

A ball of mass m is falling vertically upward which performs a projectile motion under constant acceleration. We want to calculate the maximum height of the ball reaches during its motion.

Solution: According to the conservation of energy, initial K.E and P.E of the ball becomes equal to the final K.E and P.E,

$$K_0 + U_0 = K + U$$

$$\frac{1}{2}mv_{0x}^2 + \frac{1}{2}mv_{0y}^2 = \frac{1}{2}mv_{0x}^2 + mgh$$

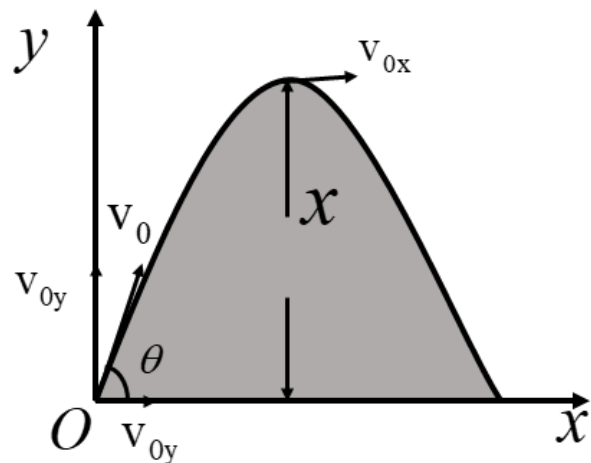
$$\frac{1}{2}mv_{0y}^2 = mgh$$

$$\therefore v_{0y} = v_0 \sin \theta$$

$$v_{0y}^2 = 2gh$$

$$v_0^2 \sin^2 \theta = 2gh$$

$$h = \frac{v_0^2 \sin^2 \theta}{2g}$$



At different angles, the ball reaches to different height but it reaches its maximum height at 90 degree.
 $\therefore \sin 90 = 1$

$$h_{\max} = \frac{v_0^2}{2g}$$

Conservative force:

Conservative force means, work does not depend on the path taken.

- Gravitational force
- Electrostatic force
- Elastic force

Note: Remember that potential energy has meaning only for a force that is conservative. A conservative force is that for which the work done in going from point A to point B is independent of the path chosen.

Non-Conservative force:

Non-Conservative force means work is dependent on the path taken.

For Example:

- Frictional force
- air resistance force
- velocity dependent forces

Potential energy cannot be defined for non-conservative forces.

Problem 12:

Suppose a particle has potential energy as $V(x)$, and the particle followed path as shown in figure.

Solution:

If the particle moves Δx

Then change in PE is ΔV

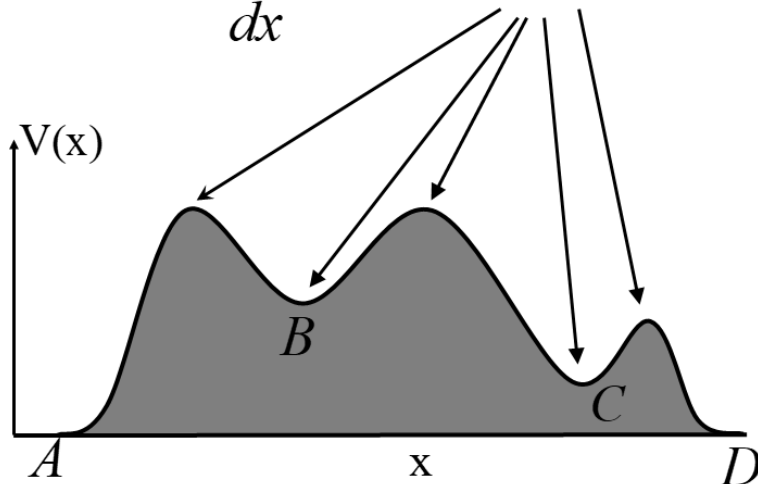
Where: $\Delta V = -F \Delta x$

$$\Rightarrow F = - \frac{\Delta V}{\Delta x}$$

Now let $\Delta x \rightarrow 0$

$$F = - \lim_{\Delta x \rightarrow 0} \frac{\Delta V}{\Delta x} = - \frac{dV}{dx}$$

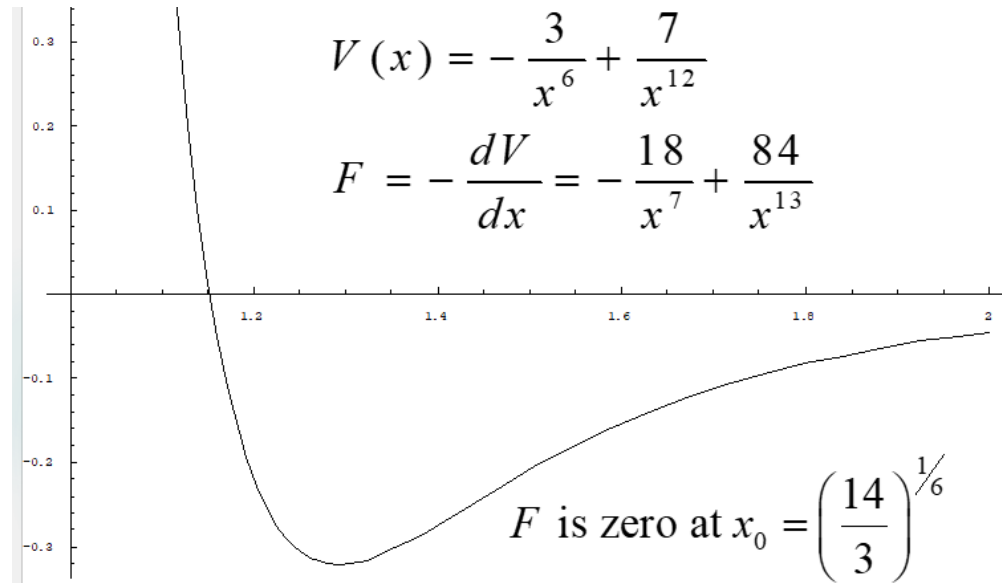
$$F = - \frac{dV}{dx} \text{ is zero here}$$



The tangent shows that the derivative of potential is zero at the top point of the curve, where the tangent is flat to curve, force is also zero there.

Example:

Suppose we have a potential function as shown in the figure.



The potential function is given as,

$$V(x) = -\frac{3}{x^6} + \frac{7}{x^{12}}$$

$$\text{As, } \Rightarrow F = -\frac{dV}{dx} = -\frac{d}{dx} \left[-\frac{3}{x^6} + \frac{7}{x^{12}} \right],$$

Using power rule nx^{n-1} , and taking differential w.r.t "x",

$$F = 3(6.x^{-6-1}) + 7(12.x^{-12-1}) = \left[-\frac{18}{x^7} + \frac{84}{x^{13}} \right]$$

F is zero at x_0 , Hence,

$$0 = \left[-\frac{18}{x_0^7} + \frac{84}{x_0^{13}} \right]$$

$$\frac{x_0^{13}}{x_0^7} = \frac{84}{18} \Rightarrow x_0^{13-7} = \frac{28}{6}$$

$$x_0^6 = \frac{14}{3} \Rightarrow x_0 = \left(\frac{14}{3}\right)^{1/6}$$

Conclusion: We can conclude that potential can be defined only for the conservative forces. We cannot define potential for non-conservative forces.

Because non-conservative forces do not satisfy the path-independence property. The work done by non-conservative forces depends on the specific path taken, and as a result, it is not possible to define a potential energy associated with non-conservative forces in the same way. Examples of non-conservative forces include friction and air resistance.

Physics-PHY101-Lecture 9

MOMENTUM

Momentum is the "quantity of motion" possessed by a body. More precisely, it is defined as,

“The product of mass and velocity of a body”.

$$p = mv$$

Dimensions of momentum:

The dimensions of momentum are MLT^{-1} .

Unit of momentum:

The units of momentum are kg-m/s.

Momentum is a vector quantity and has both magnitude and direction.

Newton's Second Law and Momentum:

Newton's Second Law can be expressed in terms of momentum. It is defined as,

“The rate of change of momentum of a body is equal to the resultant force acting on the body and is in the direction of that force”.

$$m\vec{a} = \vec{F} \text{ (old form)}$$

$$\frac{d\vec{p}}{dt} = \vec{F} \text{ (new form of Newton's Law)}$$

They are the same:

$$\frac{d\vec{p}}{dt} = \frac{d(m\vec{v})}{dt} = m \frac{d\vec{v}}{dt} = m\vec{a} = \vec{F}$$

Newton's 2nd law for several particles:

When there are many particles, then total momentum is,

$$\begin{aligned}
\vec{\mathbf{P}} &= \vec{\mathbf{p}}_1 + \vec{\mathbf{p}}_2 + \cdots \vec{\mathbf{p}}_N \\
\frac{d}{dt} \vec{\mathbf{P}} &= \frac{d}{dt} \vec{\mathbf{p}}_1 + \frac{d}{dt} \vec{\mathbf{p}}_2 + \cdots \frac{d}{dt} \vec{\mathbf{p}}_N \\
\frac{d}{dt} \vec{\mathbf{P}} &= \vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 + \cdots \vec{\mathbf{F}}_N \\
\frac{d}{dt} \vec{\mathbf{P}} &= \sum_{i=1}^{i=N} \vec{\mathbf{F}}_i = \text{total external force}
\end{aligned}$$

This shows that when there are several particles, the rate at which the total momentum changes is equal to the total force.

Conservation of Linear Momentum:

If the sum of the total external forces vanishes, then the total momentum is conserved.

$$\begin{aligned}
\sum \vec{\mathbf{F}}_{ext} &= 0 \text{ Then,} \\
\frac{d\vec{\mathbf{P}}}{dt} &= 0 \\
\vec{\mathbf{P}} &= \text{constant}
\end{aligned}$$

Momentum is conserved for an isolated system. This is quite independent of what sort of forces act between the bodies i.e., electric force, gravitational force, etc. - or how complicated these are. We shall see why this is so important from the following examples.

Problem 1:

Two balls, which can only move along a straight line, collide with each other as shown in figure 9.1. The initial momentum is $p_i = m_1 u_1 + m_2 u_2$ and the final momentum is $p_f = m_1 v_1 + m_2 v_2$. Obviously one ball exerts a force on the other when they collide, so their momentum changes. But, from the fact that there is no external force acting on the balls,

$$p_i = p_f$$

$$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2$$

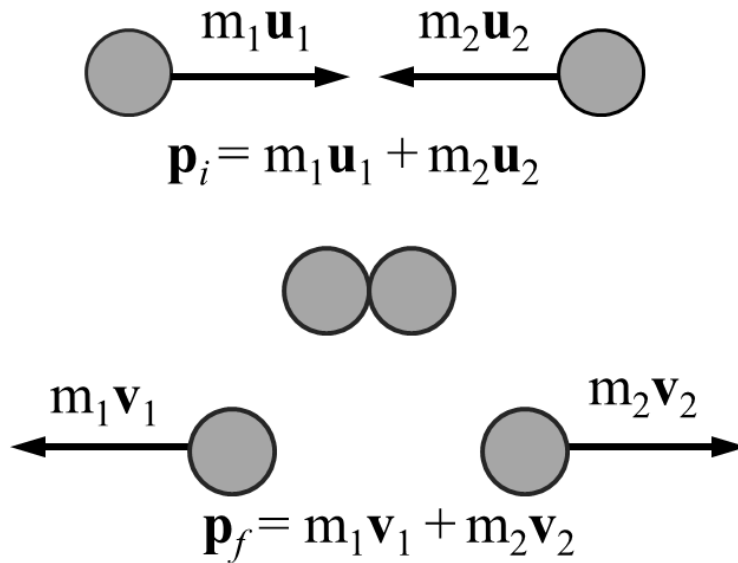


Figure 9.1. Momentum Transfer in a one-dimensional collision between the two balls along a Straight Path.

Problem 2:

A bomb at rest explodes into two fragments as shown in figure 9.2. Before the explosion the total momentum was zero. So obviously it is zero after the explosion as well.

$$\begin{aligned} \mathbf{P}_i &= \mathbf{0} \quad \text{and} \\ \mathbf{P}_f &= \mathbf{0} \quad \text{but } \mathbf{P}_f = m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2 \\ \therefore m_1 \mathbf{v}_1 &= -m_2 \mathbf{v}_2 \end{aligned}$$

During the time when the explosion happens, the forces acting upon the pieces are very complicated and change rapidly with time. But when all is said and done, there are two pieces flying away with a total zero final momentum. In other words, the fragments fly apart with equal momentum but in opposite directions. The center-of-mass stays at rest. So, knowing the velocity of one fragment permits knowing the velocity of the other fragment.

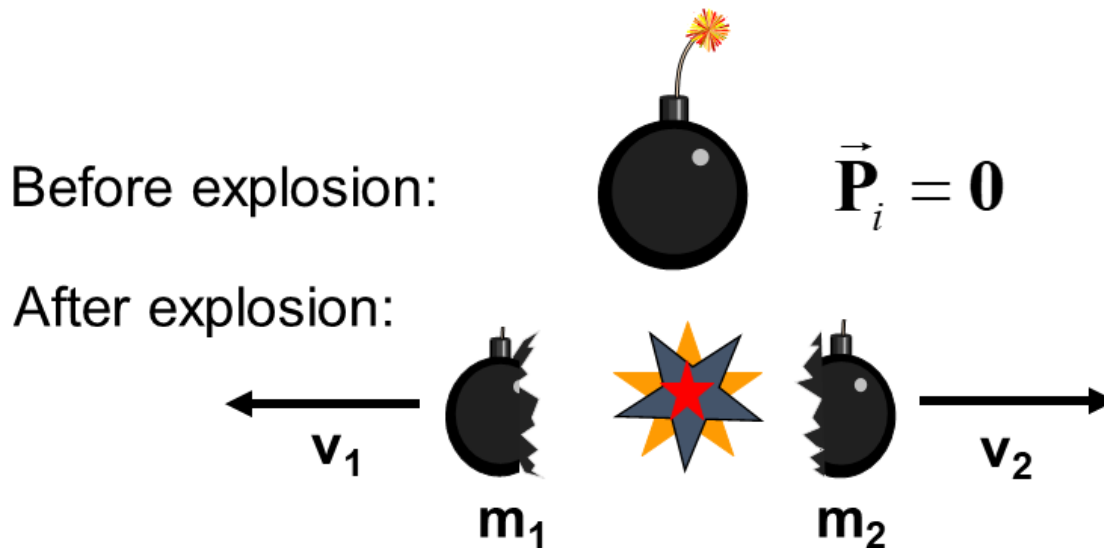


Figure 9.2. The physics of explosions as a bomb at rest divides into two fragments, shows momentum conservation.

Problem 3:

A bomb of mass 10 kg, initially at rest, explodes into two pieces of masses 4 kg and 6 kg. If the speed of the 4 kg piece is 12 m/s, find the speed of the 6 kg piece.

Solution:

Mass of bomb = $M = 10\text{ kg}$

After explosion,

$$m_1 = 4\text{ kg}$$

$$V_1 = 12\text{ m/s}$$

$$m_2 = 6\text{ kg}$$

$$V_2 = ?$$

Using formula,

$$m_1 V_1 = m_2 V_2$$

$$V_2 = \frac{m_1 V_1}{m_2} = \frac{4 \times 12}{6} = 8\text{ m/s}$$

$$V_2 = 8\text{ m/s}$$

Problem 4:

A rocket conserves momentum through the principle of action and reaction, as described by Newton's third law of motion: "For every action, there is an equal and opposite reaction." In the context of a rocket, this law explains how momentum is conserved during the propulsion process as shown in figure 9.3.

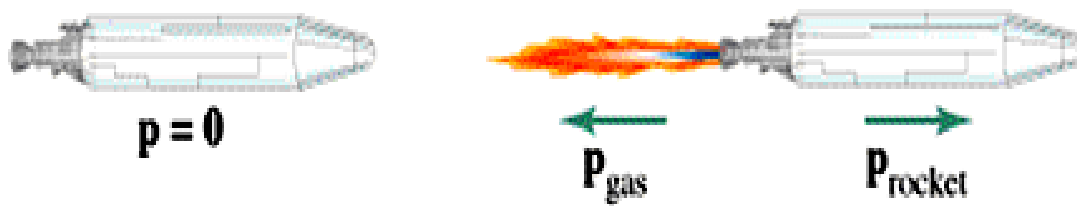


Figure 9.3. Momentum Conservation in the Propulsion Process.

When a rocket engine expels exhaust gases at high velocity in one direction, it generates a force in the opposite direction according to Newton's third law. This force is what propels the rocket forward. The expelled gases have mass and velocity, contributing to a significant momentum in the opposite direction. Since momentum is conserved in an isolated system, the rocket gains an equal and opposite momentum. This results in the rocket moving forward. Mathematically, this can be expressed using the equation for momentum:

$$\begin{aligned}
 p_i &= p_f \\
 p_{\text{gas}} + p_{\text{rocket}} &= 0 \\
 p_{\text{gas}} &= -p_{\text{rocket}}
 \end{aligned}$$

Since the initial momentum is small or zero, the final momentum of the rocket is primarily determined by the momentum of the expelled exhaust gases. The larger the momentum of the gases, the larger the momentum gained by the rocket.

Problem 5:

When a gun is fired, there are two main components involved: the gun itself and the bullet (projectile) being expelled from the gun as shown in figure 9.4. The gun and the bullet are initially at rest, so their initial momentum is zero. When the gun is fired, the bullet accelerates out of gun and gains momentum in one direction (forward), the gun and the shooter gain an equal and opposite momentum in the opposite direction (backward).

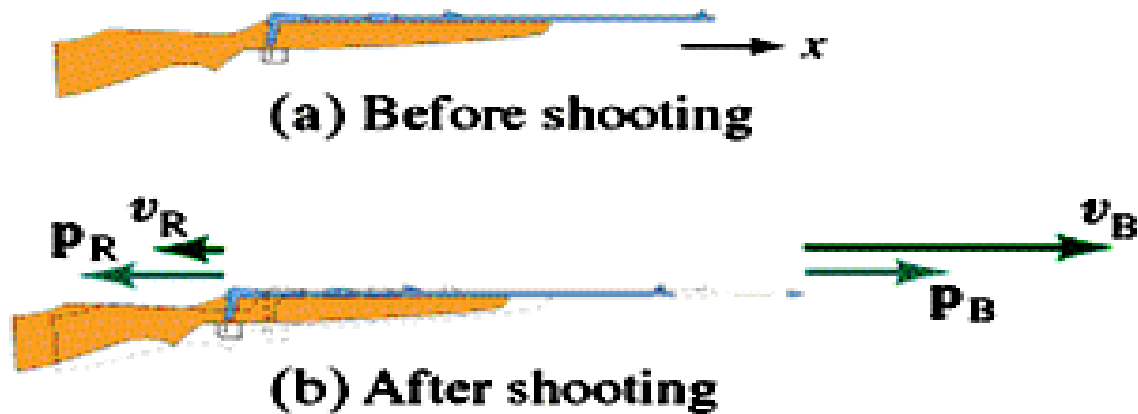


Figure 9.4. The dynamics of gun firing, where bullet acceleration and recoil impart equal and opposite momentum to maintain conservation.

$$p_i = p_f$$

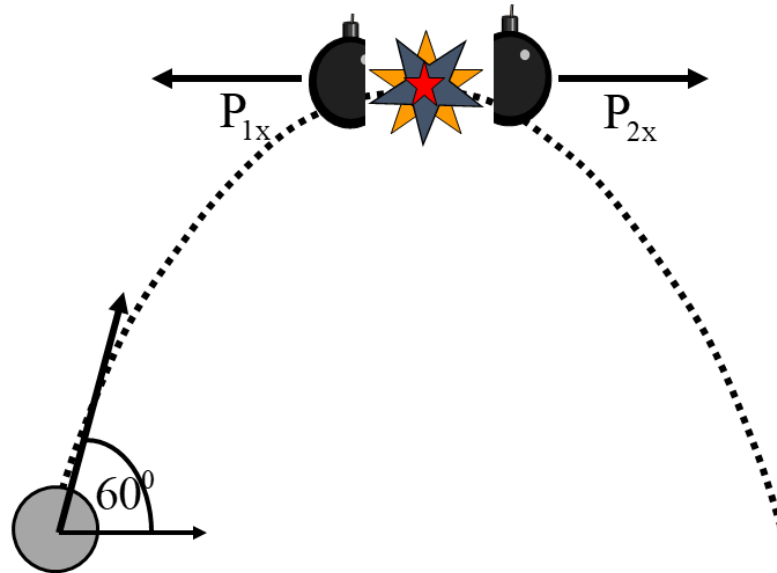
$$p_{\text{rifle}} + p_{\text{bullet}} = 0$$

$$p_{\text{rifle}} = -p_{\text{bullet}}$$

This conservation of momentum ensures that the total momentum before and after the gunshot remains zero, as no external forces are acting on the system.

Problem 6:

A shell is fired from a cannon with a speed of 10 m/s at an angle 60° with the horizontal. At the highest point in its path, it explodes into two pieces of equal masses. One of the pieces retraces its path to the cannon. Find the velocity of the other piece immediately after the explosion.



Solution:

Before explosion:

$$V_{ix} = V_i \cos \theta = 10 \cos 60 = 5 \text{ m/s}$$

$$P_x = M \cdot V_x = 5 \text{ M kg m/s}$$

After explosion:

$$MV_{ix} = -\frac{M}{2}V_{1x} + \frac{M}{2}V_{2x}$$

$$\therefore V_1 = V_{ix}, \text{ and } V_2 = V_{2x}$$

$$V_{ix} = -\frac{1}{2}V_{ix} + \frac{1}{2}V_{2x}$$

$$2V_{ix} = -V_{ix} + V_{2x}$$

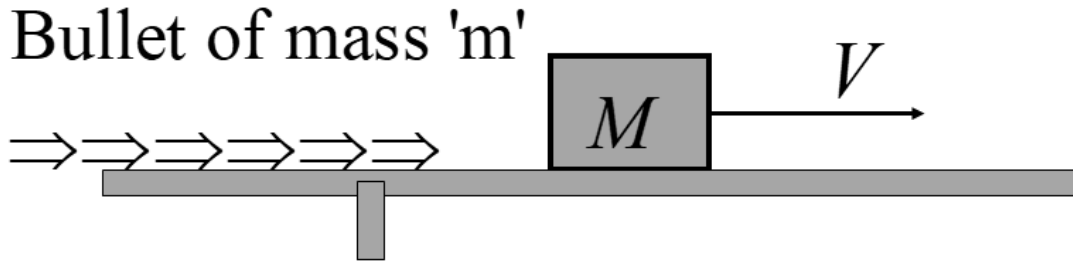
$$2V_{ix} + V_{ix} = V_{2x}$$

$$V_{2x} = 3V_{ix}$$

$$V_{2x} = 3 \times 5 = 15 \text{ m/s}$$

Problem 7:

A stream of bullets, each of mass m , is fired horizontally with a speed v into a large wooden block of mass M that is initially at rest on a horizontal table. If the block is free to slide without friction, what speed will it acquire after it has absorbed N bullets?



Solution:

Linear momentum is conserved:

$$P_f = P_i$$

$$(M + Nm)V = N(mv)$$

$$V = \frac{Nm}{(M + Nm)} v$$

Problem 7:

A cannon with mass M equal to 1300 kg fires a 72 kg ball in horizontal direction with a muzzle speed v of 55 m/s. The cannon is mounted so that it can recoil freely.

- What is the velocity V of the recoiling cannon with respect to the earth?
- What is the initial velocity V_E of the ball with respect to the earth?

Solution:

a) Linear momentum is conserved:

$$P_f = P_i$$

$$MV + m(v + V) = 0$$

$$V = -\frac{mv}{m + M} = -\frac{(72\text{kg})(55\text{m/s})}{1300\text{kg} + 72\text{kg}}$$

$$V = -2.9 \text{ m/s}$$

b) $V_E = v + V$

$$V_E = 55\text{m/s} + (-2.9 \text{ m/s})$$

$$V_E = 52\text{m/s}$$

Problem 8:

Consider a body at point A which is at height h from the ground. The body has no friction, when it is at point A, but when it comes down straight to earth it has friction which renders the body to move again to the curve as shown in fig. For that case, we want to know,

- Is the momentum conserved?
- Where does the particle finally come to rest?

**Solution:**

Suppose the total distance moved on the flat part before it comes to rest is 'x',

$$\begin{aligned}
 mgh &= \frac{1}{2}mv^2 \\
 \therefore \frac{1}{2}mv^2 &= f \cdot x \\
 \frac{1}{2}mv^2 &= \mu Nx = \mu mgx \\
 mgh &= \mu mgx \\
 x &= \frac{h}{\mu}
 \end{aligned}$$

Impulse and Momentum:

When you hit your thumb with a hammer it hurts, doesn't it? Why? Because a large amount of momentum has been destroyed in a short amount of time. If you wrap your thumb with foam, it will hurt less. To understand this better, remember that force is the rate of change of momentum,

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}$$

$$d\mathbf{p} = \mathbf{F}dt = I$$

Impulse:

It is defined as, “force acting over time to change the momentum of an object”.

If the force changes with time between the limits, then one should define impulse as,

$$I \equiv \int_{t_1}^{t_2} \mathbf{F}dt$$

$$\text{Since } \int_{t_1}^{t_2} \mathbf{F}dt = \int_{p_i}^{p_f} d\mathbf{p}$$

$$I = \mathbf{p}_f - \mathbf{p}_i$$

In words, the change of momentum equals the impulse, which is equal to the area under the curve of force versus time. Even if you wrap your thumb in foam, the impulse is the same. But the force is definitely not.

Sometimes we only know the force numerically (i.e., there is no expression like $F=\text{something}$). But we still know what the integral means: it is the area under the curve of force versus time. The curve here is that of a hammer striking a table. Before the hammer strikes, the force is zero, reaches a peak, and goes back to zero.

Graphical representation:

1. Let us draw a graph between force $F(t)$ and time (Δt) as shown in figure 9.5. From this graph, one can see the area under the curve gives impulse and the value of force is maximum at t_2 but minimum at t_1 and t_3 (also, $F(t)$ is same at t_1 and t_3). Below t_1 and above t_3 , $F(t)$ begins to zero.

$$I = F_{av} \Delta t$$

Area under the curve = impulse

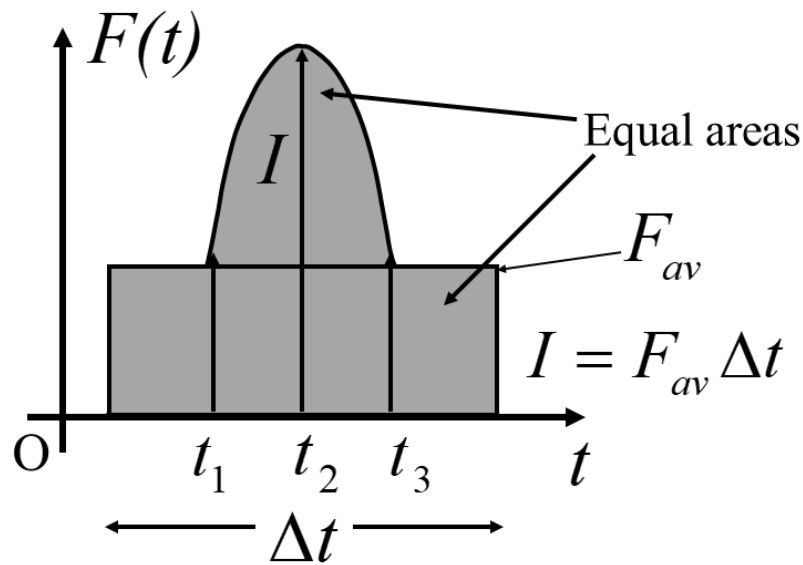


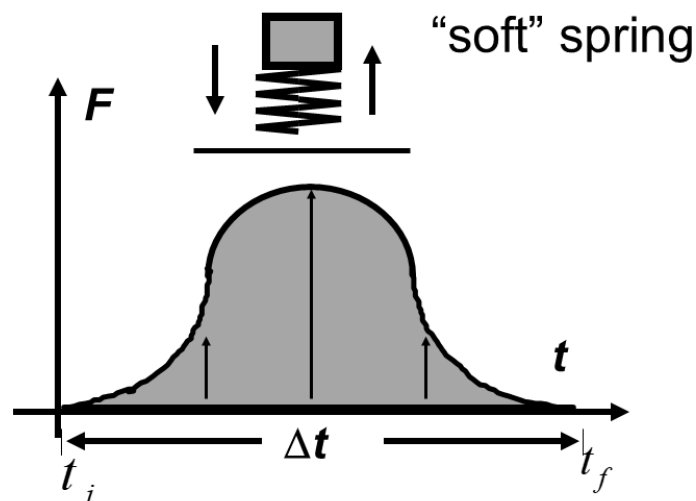
Figure 9.5. Graphical representation of $F(t)$ and Δt , unraveling impulse as the area under the curve, with maximal force at t_2 and minimum force at t_1 and t_3 .

2. Force may be of any type. Let us now consider a case where a soft spring is taken upon which when force is applied, it shows less resistance as being soft in nature. When a weight is taken over a soft spring it shows,

Δt is big, F is small

But the product of force and time gives us a constant value which is called impulse.

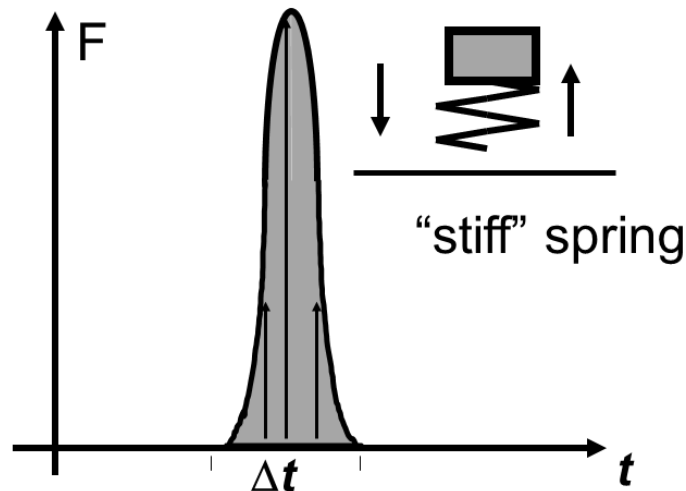
$$I = F \Delta t$$



3. Now consider a case of a stiff spring. When a small force is applied to that spring, it may create a significant resistance.

Δt is small, F is big

But the product of force and time again gives us a constant value which is called impulse.



Conclusion:

Momentum is a concept that is useful because of Newton's 2nd law. Physics is a quantitative field. It turns out that momentum conservation still holds even when we go beyond Newton's laws. But momentum is not just mv .

Practice Questions:

- Q: Would you rather land with your legs bending or stiff?
- Q: Why do cricket fielders move their hands backwards when catching a fast ball?
- Q: Why do railway carriages have dampers at the front and back?

Physics-PHY101-Lecture 10

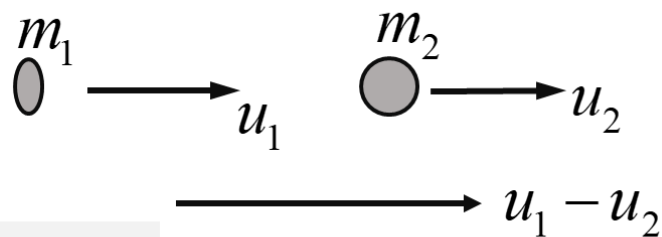
COLLISIONS

Collisions are extremely important to understand because they happen all the time - electrons collide with atoms, a bat with a ball, cars with trucks, star galaxies with other galaxies. In every case, the sum of the initial momenta equals the sum of the final momenta. This follows directly from Newton's Second Law, as we have already seen.

Elastic collision in one dimension:

Let us take the simplest collision. Consider two bodies of mass m_1 and m_2 moving with velocities u_1 and u_2 . After the collision they are moving with velocities v_1 and v_2 as shown in figure 10.1. For elastic collision, the total linear momentum, and kinetic energies of the two bodies before and after collision must remain the same.

Before collision:



After collision:

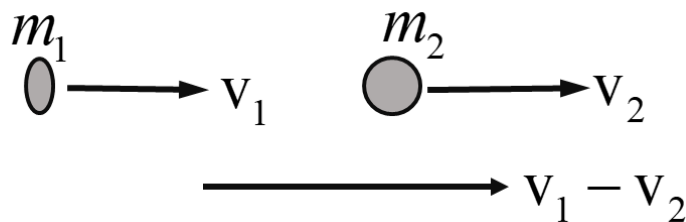


Figure 10.1. Illustrating one-dimensional elastic collision dynamics, where masses m_1 and m_2 exchange velocities u_1, u_2 to v_1 , and v_2 after collision while preserving total linear momentum and kinetic energies.

From the law of conservation of linear momentum,

$$\text{Total momentum before collision } (p_i) = \text{Total momentum after collision } (p_f)$$

$$m_1 u_1 + m_2 u_2 = m_1 v_1 + m_2 v_2$$

$$m_1 (u_1 - v_1) = m_2 (v_2 - u_2) \rightarrow 1$$

For elastic collision,

Total kinetic energy before collision (K.E_i) = Total kinetic energy after collision (K.E_f)

$$\frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$$

$$\frac{1}{2} m_1 (u_1^2 - v_1^2) = \frac{1}{2} m_2 (v_2^2 - u_2^2)$$

$$m_1 (u_1^2 - v_1^2) = m_2 (v_2^2 - u_2^2) \rightarrow 2$$

Using the formula, $a^2 - b^2 = (a + b)(a - b)$, we can rewrite the above equation as

$$m_1 (u_1 + v_1)(u_1 - v_1) = m_2 (v_2 + u_2)(v_2 - u_2) \rightarrow 3$$

Dividing equation (3) by (1) gives,

$$\frac{m_1 (u_1 + v_1)(u_1 - v_1)}{m_1 (u_1 - v_1)} = \frac{m_2 (v_2 + u_2)(v_2 - u_2)}{m_2 (v_2 - u_2)}$$

$$u_1 + v_1 = v_2 + u_2$$

$$u_1 - u_2 = v_2 - v_1$$

$$u_1 - u_2 = -(v_1 - v_2) \rightarrow 4$$

This means that for any elastic head on collision, the relative speed of the two elastic bodies after the collision has the same magnitude as before collision but in opposite direction. Further note that this result is independent of mass.

Rewriting the equation 4 for v_1 and v_2 ,

$$v_1 = v_2 + u_2 - u_1 \rightarrow 5$$

$$v_2 = u_1 + v_1 - u_2 \rightarrow 6$$

To find the final velocities v_1 and v_2 :

Substituting equation (5) in equation (1) gives the velocity of m_1 as,

$$m_1 (u_1 - v_1) = m_2 (v_2 - u_2)$$

$$m_1 (u_1 - v_1) = m_2 (u_1 + v_1 - 2u_2)$$

$$m_1 u_1 - m_1 v_1 = m_2 u_1 + m_2 v_1 - 2m_2 u_2$$

$$(m_1 - m_2) u_1 + 2m_2 u_2 = (m_1 + m_2) v_1$$

$$v_1 = \frac{(m_1 - m_2)}{(m_1 + m_2)} u_1 + \frac{2m_2}{(m_1 + m_2)} u_2 \rightarrow 7$$

Similarly, by substituting equation (7) in equation (6), we get the final velocity of m_2 as,

$$v_2 = u_1 + \frac{(m_1 - m_2)}{(m_1 + m_2)}u_1 + \frac{2m_2}{(m_1 + m_2)}u_2 - u_2$$

$$v_2 = \frac{2m_1}{(m_1 + m_2)}u_1 + \frac{(m_2 - m_1)}{(m_1 + m_2)}u_2 \rightarrow 8$$

These equations holds true in all inertial frames.

Case-I:

When bodies have same mass i.e., $m_1 = m_2$,

$$v_1 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)u_1 + \left(\frac{2m_2}{m_1 + m_2}\right)u_2$$

$$v_1 = (0)u_1 + \frac{2m_2}{2m_2}u_2$$

$$v_1 = u_2$$

$$v_2 = \left(\frac{2m_1}{m_1 + m_2}\right)u_1 + \left(\frac{m_2 - m_1}{m_1 + m_2}\right)u_2$$

$$v_2 = \frac{2m_1}{2m_1}u_1 + (0)u_2$$

$$v_2 = u_1$$

The equations show that in one dimensional elastic collision, when two bodies of equal mass collide after the collision their velocities are exchanged.

Case-II:

When bodies have the same mass i.e., $m_1 = m_2$ and second body (usually called target) is at rest ($u_2 = 0$), By substituting $m_1 = m_2$ and $u_2 = 0$ in equations we get,

$$v_1 = 0$$

$$v_2 = u_1$$

Equations show that when the first body comes to rest the second body moves with the initial velocity of the first body.

Case-III:

$$m_1 \ll m_2 \text{ and } u_2 = 0$$

$$v_1 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)u_1 + \left(\frac{2m_2}{m_1 + m_2}\right)u_2$$

$$v_1 = -u_1$$

$$v_2 = \left(\frac{2m_1}{m_1 + m_2}\right)u_1 + \left(\frac{m_2 - m_1}{m_1 + m_2}\right)u_2$$

$$v_2 = 0$$

The equations implies that the first body which is lighter, returns back in the opposite direction with the same initial velocity as it has a negative sign. The second body which is heavier in mass continues to remain at rest even after collision.

For example, if a ball is thrown at a fixed wall, the ball will bounce back from the wall with the same velocity with which it was thrown but in opposite direction.

Case-IV:

$$m_2 \ll m_1 \quad u_2 = 0$$

$$v_1 = \left(\frac{m_1 - m_2}{m_1 + m_2}\right)u_1 + \left(\frac{2m_2}{m_1 + m_2}\right)u_2$$

$$v_1 = u_1$$

$$v_2 = \left(\frac{2m_1}{m_1 + m_2}\right)u_1 + \left(\frac{m_2 - m_1}{m_1 + m_2}\right)u_2$$

$$v_2 = 2u_1$$

The equations implies that the first body which is heavier continues to move with the same initial velocity. The second body, which is lighter, will move with twice the initial velocity of the first body. It means that the lighter body is thrown away from the point of collision.

Elastic and inelastic collision:

Elastic Collision	Inelastic Collision
--------------------------	----------------------------

The total kinetic energy is conserved	The total kinetic energy of the bodies at the beginning and the end of the collision is different.
Momentum is conserved	Momentum is conserved
No conversion of energy takes place	Kinetic energy is changed into other energy such as sound or heat energy.
Highly unlikely in the real world as there is almost always a change in energy.	This is the normal form of collision in the real world.
An example of this can be swinging balls or a spacecraft flying near a planet but not getting affected by its gravity in the end.	An example of an inelastic collision can be the collision of two cars.

Sometimes we wish to slow down particles by making them collide with other particles. In a nuclear reactor, neutrons can be slowed down in this way.

A completely inelastic collision:

A collision that is completely inelastic in physics is one in which the two colliding objects stick together to form a single mass as shown in figure 10.2. The total momentum is conserved in such collisions but kinetic energy is not conserved; instead, part or all of it is converted into deformation energy or internal kinetic energy.

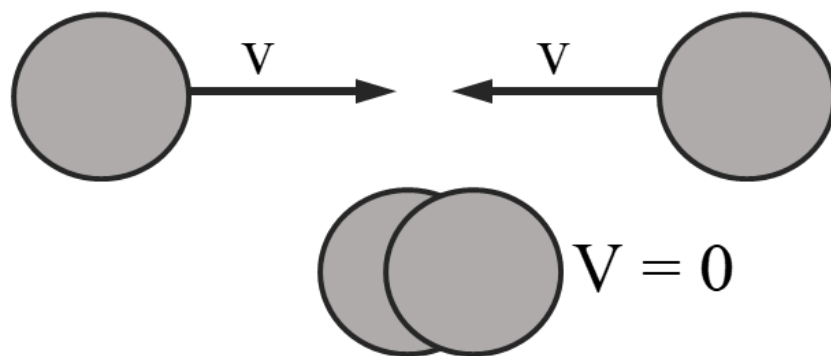


Figure 10.2. Illustrating one-dimensional in-elastic collision dynamics, where two masses moves with the same velocity V , conserving only the total linear momentum but the kinetic energies are not conserved.

Mathematically,

$$\text{Initial momentum} = \text{Final momentum} = 0$$

$$\text{Initial kinetic energy} = 2 \frac{1}{2} mv^2 \text{ (for two masses)}$$

$$\text{Final kinetic energy} = 0 \text{ (as } V = 0 \text{)}$$

A common example of a totally inelastic collision is the result of two balls colliding and sticking to one another. After the collision, the combined mass that results travels at the same speed.

Problem 1: By what fraction is the kinetic energy of a neutron (mass m_1) decreased in a head-on collision with an atomic nucleus (mass m_2) initially at rest?

Solution:

$$m_1 \ll m_2, \quad v_{2i} = 0$$

$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2} \right) u_{1i} + \left(\frac{2m_2}{m_1 + m_2} \right) u_{2i}$$

$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2} \right) u_{1i} \Rightarrow \frac{v_f}{u_i} = \left(\frac{m_1 - m_2}{m_1 + m_2} \right)$$

$$\left(\frac{v_f}{u_i} \right)^2 = \left(\frac{m_1 - m_2}{m_1 + m_2} \right)^2$$

Fractional decrease in neutron K.E :

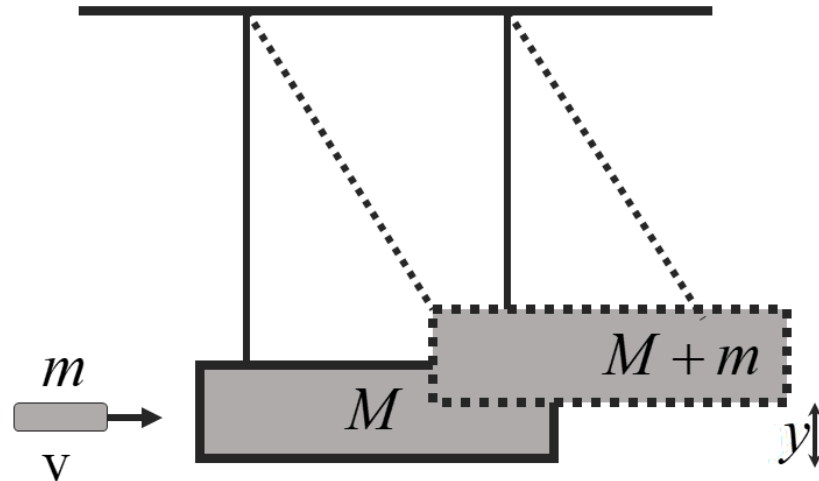
$$\frac{K_i - K_f}{K_i} = 1 - \frac{K_f}{K_i} = 1 - \frac{v_f^2}{v_i^2}$$

$$\frac{K_i - K_f}{K_i} = 1 - \left(\frac{m_1 - m_2}{m_1 + m_2} \right)^2$$

$$\frac{K_i - K_f}{K_i} = \frac{(m_1 + m_2)^2 - (m_1 - m_2)^2}{(m_1 + m_2)^2}$$

$$\frac{K_i - K_f}{K_i} = \frac{4m_1 m_2}{(m_1 + m_2)^2}$$

Problem 2: A bullet with mass m is fired into a block of wood with mass M , suspended like a pendulum and makes a completely inelastic collision with it. After the impact, the block swings up to a maximum height y . What is the initial speed of the bullet?



Solution:

$$mv + 0 = (m + M)V$$

$$mv = (m + M)V$$

$$v = \frac{(m + M)}{m}V \rightarrow 1$$

Conservation of energy gives,

$$\frac{1}{2}(m + M)V^2 = (m + M)gy$$

$$y = \frac{(m + M)V^2}{2(m + M)g}$$

$$y = \frac{V^2}{2g}$$

$$V = \sqrt{2gy} \rightarrow 2$$

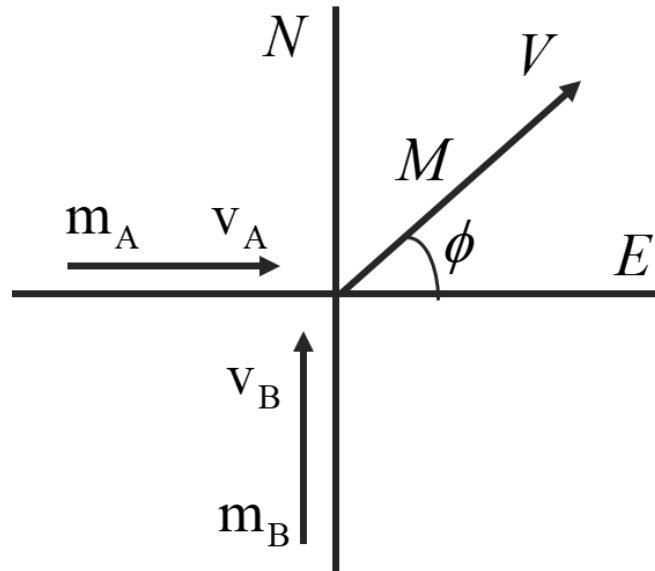
On putting equation (2) in equation (1),

$$v = \frac{(m + M)}{m}\sqrt{2gy}$$

Conclusion: Equation shows the direct relation between the initial speed of bullet and the wooden block's height ' y '. The wooden block gains height as much as speedily the bullet enters the wooden block.

Problem 3: A car 'A' of mass 1000 kg is traveling north at 15 m/s collides with another car B of mass 2000 kg traveling east at 10 m/s. After a collision they move as one mass. Find the total momentum just after the collision.

Solution:



$$P_x = p_{Ax} + p_{Bx} = m_A v_{Ax} + m_B v_{Bx}$$

$$P_x = 0 + 2000 \times 10 = 20,000 \text{ kg m/s}$$

$$P_y = p_{Ay} + p_{By} = m_A v_{Ay} + m_B v_{By}$$

$$P_y = 1000 \times 15 + 0 = 15000 \text{ kg m/s}$$

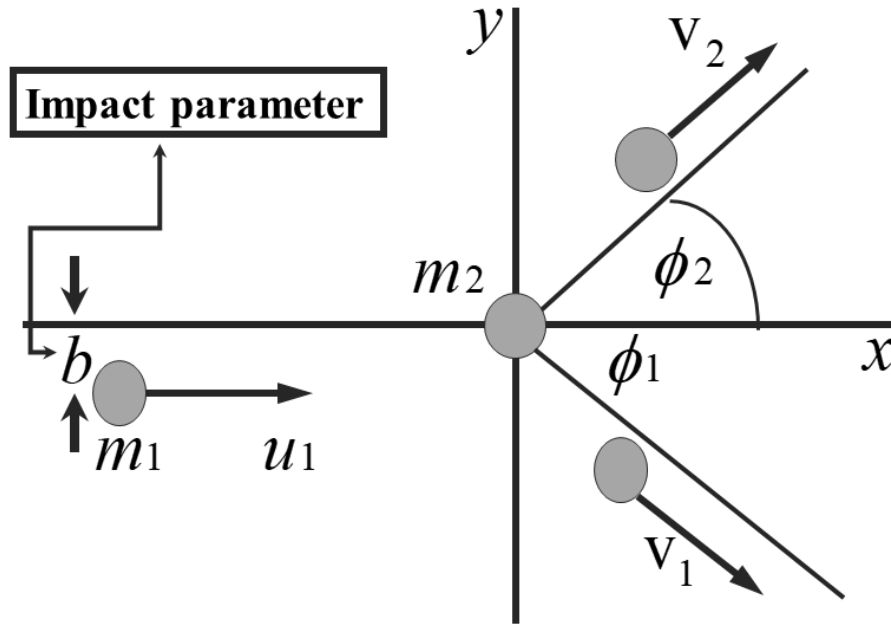
$$P = \sqrt{P_x^2 + P_y^2} = \sqrt{20,000^2 + 15,000^2} = 25000 \text{ kg m/s}$$

$$\tan \theta = \frac{P_y}{P_x} = \frac{15,000}{20,000} = 0.75 \Rightarrow \theta = 37^\circ$$

Problem 4:

Consider two masses m_1 moving with velocity u_1 and m_2 at rest ($u_2=0$) as shown in figure. This is not a central collision (i.e., m_1 directly collide to m_2), instead m_1 slightly touches m_2 by one of its sides. Apply the law of conservation of momentum to this case.

Solution:



$$1) p_{ix} = p_{fx}$$

$$\Rightarrow m_1 u_1 = m_1 v_1 \cos \phi_1 + m_2 v_2 \cos \phi_2$$

$$2) p_{iy} = p_{fy}$$

$$\Rightarrow 0 = m_1 v_1 \sin \phi_1 - m_2 v_2 \sin \phi_2$$

$$3) KE_i = KE_f$$

$$\Rightarrow \frac{1}{2} m_1 u_1^2 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$$

As there are 4-unknowns and 3-equations, they cannot be solved.

Problem 5: Consider two masses m_1 moving with velocity v_1 and m_2 moving with velocity v_2 . Both masses collided with each other. After collision, they become a one body with mass M moving with velocity v_f . Apply the law of conservation of momentum to this case.

Solution:

$$1) p_{ix} = p_{fx}$$

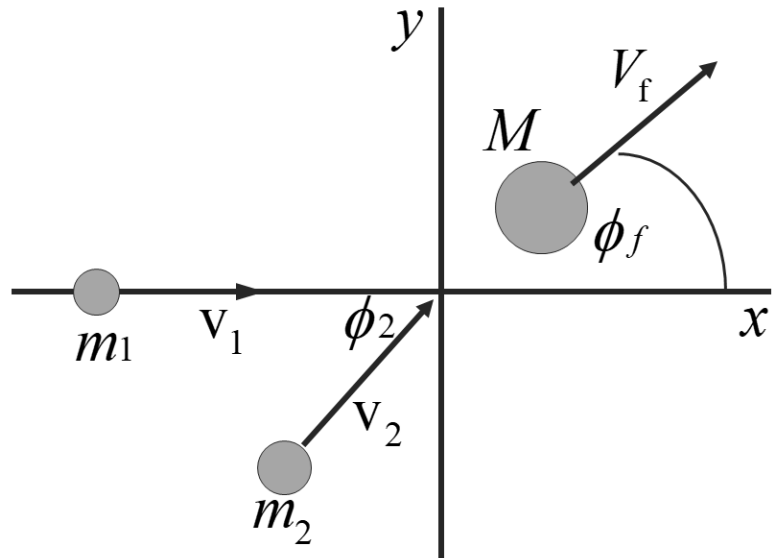
$$\Rightarrow m_1 v_1 + m_2 v_2 \cos \phi_2$$

$$= (m_1 + m_2) V_f \cos \phi_f$$

$$2) p_{iy} = p_{fy}$$

$$\Rightarrow m_2 v_2 \sin \phi_2 = M V_f \sin \phi_f$$

Now there are 2-unknowns and 2-equations, hence, can be solved.



Conclusion:

- Momentum is always conserved in collisions, but energy may or may not be.
- We have come to trust momentum conservation very much: discovery of the neutrino, hints of black holes, discovery of dark matter.

PHY101-Lecture#11

Rotational Kinematics

When an object undergoes rotation or revolution, it means that the distance from a specific point, known as the center, remains constant. This signifies that during rotation, regardless of the object's position, this distance remains unchanged. We refer to this fixed distance as the radius. As the radius vector changes its position, an associated angle ϕ (phi), also changes accordingly. Every rotation entails an angle, ϕ , which defines the rotational position. Unlike Cartesian coordinates (X and Y) used to pinpoint a location, in rotational motion, we simplify this to a single value, ϕ . Furthermore, during rotation, a fixed distance 's' known as arc length is maintained. This arc length corresponds to the radius vector, and as ϕ increases, so does the arc length, denoted by 's'. It's noted that 's' is directly proportional to ϕ , indicating that as ϕ increases, the arc length also increases.

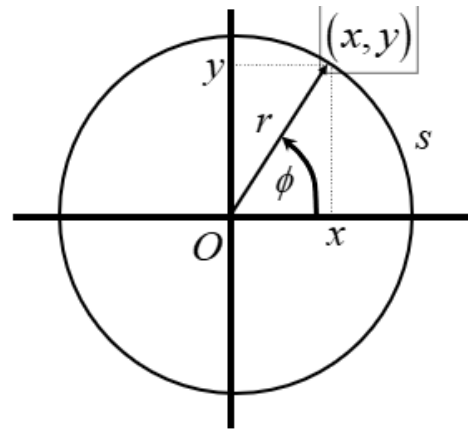


Fig. 11. 1: For a circle of radius r, one radian is the angle subtended by an arc length equal to r.

arc length = radius $\times$ angular displacement

$$s = r\phi$$

$$\text{one revolution} = 2\pi \text{ radians}$$

$$= 360 \text{ degrees}$$

$$1 \text{ radian} = 57.3^\circ$$

$$1 \text{ radian} = 0.159 \text{ revolution}$$

$$s = 2\pi r = \text{total circumference}$$

Angular Speed

Consider an object at this moment, it occupies a certain position, let's denote the fixed distance from the center as 'A'. Initially positioned at ϕ_1 , it then rotates to another position, which we'll refer to as ϕ_2 . Consequently, it maintains a fixed angular difference of $\phi_2 - \phi_1$. Observe its

depiction: 'r' has a complex value. Initially at ϕ_1 and later at ϕ_2 . Suppose it's at ϕ_1 at time t_1 and at ϕ_2 at time t_2 . From this, we derive the concept of angular speed.

$$\bar{\omega} = \frac{\phi_2 - \phi_1}{t_2 - t_1} = \frac{\Delta\phi}{\Delta t}$$

$$\omega = \lim_{\Delta t \rightarrow 0} \frac{\Delta\phi}{\Delta t}$$

$$\omega = \frac{d\phi}{dt}$$

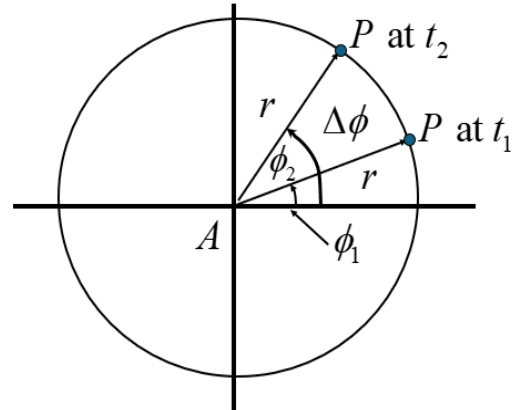


Fig. 11. 2: A point on a rotating circular path for t_1 and t_2 displaced through the angle $\Delta\phi = \phi_2 - \phi_1$.

as $\Delta\phi$ decreases, we assume that $\Delta\phi$ and Δt are very small. This difference occurs over a brief period, causing Δt to decrease. Consequently, ω (Omega),

our angular speed, approaches the value of $\frac{d\phi}{dt}$. We denote this angular speed as ' ω '.

Consider the example of a clock. You'll notice that the second hand moves rapidly, followed by the minute hand, and then the hour hand. Now, let's explore their respective angular speeds. To calculate angular speed, we must remember that each hand covers a fixed angle during every revolution. For the second hand, this angle is traversed within 60 seconds. Therefore, its angular speed is 2π divided by 60 seconds. For the minute hand, this distance is covered within one hour, which means 60 minutes times 60 seconds equals 3600 seconds. For the hour hand, spanning 12 hours, you'd divide this time by 12, then calculate its respective values.

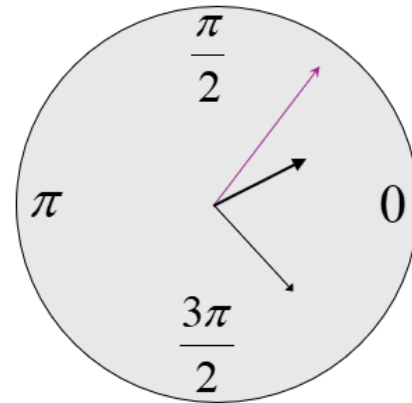


Fig. 11. 3: Three different angular speeds in a wall clock are connected to the radian angles.

$$\omega = \frac{2\pi}{T}$$

$$\omega_{\text{second}} = \frac{2\pi}{60} = 0.105 \text{ rad / s}$$

$$\omega_{\text{minute}} = \frac{2\pi}{60 \times 60} = 1.75 \times 10^{-3} \text{ rad / s}$$

$$\omega_{\text{hour}} = \frac{2\pi}{60 \times 60 \times 12} = 1.45 \times 10^{-4} \text{ rad / s}$$

Problem 11.1: Our sun is 2.3×10^4 light years away from the center of our Milky Way galaxy. It moves in a circle around this center at 250 km/s.

- (a) How long does it take the sun to make one revolution about the galactic center?
- (b) How many revolutions has the sun completed since it was formed about 4.5×10^9 years ago?

Solution:

$$\text{a) } 1 \text{ Light Year} = 9.46 \times 10^{15} \text{ m}$$

$$v = R\omega = R \frac{\theta}{t} = R \frac{2\pi}{T}$$

$$\therefore \text{ for one revolution } T = \frac{2\pi R}{v}$$

$$T = 5.5 \times 10^{15} \text{ s} = 1.74 \times 10^8 \text{ years}$$

$$\text{b) } \frac{4.5 \times 10^9}{1.74 \times 10^8} = 26 \text{ revolutions}$$

Angular acceleration

Angular Acceleration is defined as the time rate of change of angular velocity. It is usually expressed in radians per second per second. Thus,

$$a = \frac{d\omega}{dt}$$

The angular acceleration is also known as rotational acceleration. It is a quantitative expression of the change in angular velocity per unit time.

The average angular acceleration ($\bar{\alpha}$) over a specific time interval. Here, ω represents angular velocity, and t represents time. ω_1 and ω_2 are the initial and final angular velocities respectively, and t_1 and t_2 are the initial and final times respectively.

$$\bar{\alpha} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t}$$

$$\alpha = \lim_{\Delta t \rightarrow 0} \frac{\Delta\omega}{\Delta t}$$

$$\alpha = \frac{d\omega}{dt} = \frac{d}{dt} \frac{d\phi}{dt} = \frac{d^2\phi}{dt^2}$$

Here, ϕ represents angular displacement. This equation shows that angular acceleration (α) can also be expressed as the second derivative of angular displacement with respect to time. It represents how rapidly the angular velocity is changing, which in turn indicates how quickly an object is rotating.

When the angle changes, the distance also changes accordingly. This implies that the length of the arc also undergoes alteration. As I rotate my hand in a certain direction, the angle increases, and there's a corresponding change in the distance, referred to as 's', which equals 'r' multiplied by ' ϕ ', the angle through which it has moved. This ' ϕ ' varies with time, so 's' equals 'r' multiplied by ' ϕ '. Now, let's differentiate it with respect to time. Since 'r' is constant, it becomes 'r' times the derivative of ' ϕ ' with respect to 't', denoted as d/dt of 'r' times ' ϕ ', which is ' ω '. The formula for the speed at which this object is moving is 'r' times ' ω '. If ' ω ' is constant, then differentiating it again is straightforward.

$$s = r\phi$$

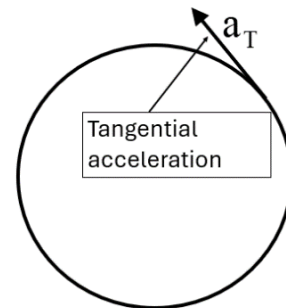
$$\frac{ds}{dt} = r \frac{d\phi}{dt}$$

$$v = r\omega$$

$$\frac{dv}{dt} = r \frac{d\omega}{dt}$$

$$a_T = r\alpha$$

the acceleration ' a_T ' in the tangent direction equals ' r ' times ' α '. The tangent direction refers to the direction tangent to the motion path. I'll illustrate this with a diagram shortly in Fig.11.4. There exists a close relationship between linear motion and angular motion, despite the differences: one involves straight-line movement while the other involves rotation with constant distance. However, their mathematical principles differ, although to a



similar extent. Let's revisit the formulas we derived earlier for linear motion.

Fig. 11. 4: Tangential acceleration is tangent of a circle.

Relationship between linear and angular variables:

Translational Motion	Rotational Motion
$v = v_0 + at$	$\omega = \omega_0 + \alpha t$
$x = x_0 + v_0 t + \frac{1}{2} at^2$	$\phi = \phi_0 + \omega_0 t + \frac{1}{2} \alpha t^2$
$v^2 = v_0^2 + 2a(x - x_0)$	$\omega^2 = \omega_0^2 + 2\alpha(\phi - \phi_0)$

Problem 11.2: A point on the rim of a 0.75 m diameter grinding wheel changes speed from 12 m/s to 25 m/s in 6.2 s. What is the angular acceleration during this interval?

Solution:

$$a = \frac{v_f - v_i}{t} = 2.1 \text{ m/s}^2$$

$$\therefore a = r\alpha$$

$$r = d / 2 = 0.75 / 2 = 0.375$$

$$\alpha = \frac{a}{r} = \frac{2.1}{0.375} = 5.6 \text{ rad/s}^2$$

Problem 11.3: The angular speed of a car engine is increased from 1170 rev/min to 2880 rev/min in 12.6 s.

- Find the average angular acceleration in rev/min^2 .
- How many revolutions does the engine make during this time?

Solution:

$$\omega_i = 1170 \text{ rev/min}$$

$$\omega_f = 2880 \text{ rev/min}$$

$$t = 12.6 \text{ s} = 12.6 / 60 = 0.21 \text{ min}$$

$$\alpha = \frac{\omega_f - \omega_i}{t} = \frac{(2880 - 1170) \text{ rev/min}}{0.21 \text{ min}} = 8140 \text{ rev/min}^2$$

$$\phi = \omega_i t + \frac{1}{2} \alpha t^2 = 425 \text{ rev}$$

Problem 11.4: A diver makes 2.5 complete revolutions on the way from a 10 m platform to the water below shown in Fig.11.5. Assuming zero initial vertical velocity, calculate the average angular velocity.

$$h = v_i t + \frac{1}{2} g t^2$$

$$\therefore v_i = 0$$

$$h = \frac{1}{2} g t^2$$

$$h = 10 \text{ m}, g = 10 \text{ m/s}^2$$

$$t = \sqrt{\frac{2h}{g}} = 1.43 \text{ s}$$

$$\therefore \omega = \frac{\phi}{t} = \frac{2\pi \left(\frac{\text{rad}}{\text{rev}}\right) n}{t}$$

$$n = 2.5 \text{ (rev)}$$

$$\omega = \frac{2\pi \left(\frac{\text{rad}}{\text{rev}}\right) \times 2.5 \text{ (rev)}}{1.43 \text{ s}} = 11 \text{ rad/s}$$

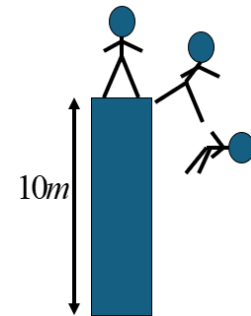


Fig. 11. 5: Path of a diver from a 10 m Platform.

Rotation is a common phenomenon in our surroundings, not limited to just the wheels of a car. If you observe any machinery, whether it's a car, motorcycle, or in a lathe shop, you'll notice the interaction between small and large wheels. Understanding the concepts of angular speed and angular acceleration allows us to comprehend many aspects of our surroundings.

Consider a bicycle as an example. In a bicycle, there are two gears: a large one driving a smaller one. As the large gear rotates once, the smaller gear rotates several times. We can calculate this rotation ratio using their respective radii. If the radius of the larger gear is represented as ' r_c ' and

that of the smaller gear as ' r_a ' as shown in Fig.11.6, then the smaller gear will rotate ' r_c / r_a ' times for every rotation of the larger gear. If there's a threefold difference between their radii, the smaller gear will rotate three times for every rotation of the larger gear. Now, let's focus on the chain mechanism. The chain's purpose is to transmit force effectively. Despite its motion, the chain maintains a constant speed

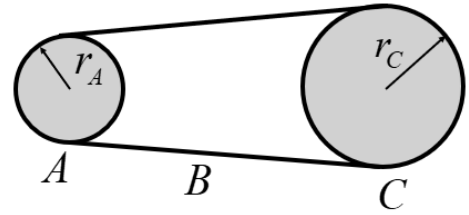


Fig. 11. 6: Angular speed between gears of a bicycle.

throughout its length. This uniformity ensures that as one part of the chain moves forward, the other part moves backward with the same speed, ensuring effective force transmission. When acceleration occurs, it is also transmitted backward through the chain. Now, let's work through a problem related to this concept.

Problem 11.5: Wheel A of radius $r_A = 10.0$ cm is coupled by a chain B to wheel C of radius $r_C = 25.0$ cm as shown in Fig.11.7. Wheel A increases its angular speed from rest at a uniform rate of 1.60 rad/s^2 .

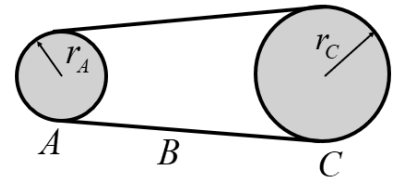


Fig. 11. 7: Angular speed between gears of a bicycle provides certain rotational speed for a specific time.

Determine the time for wheel C to reach a rotational speed of 100 rev/min.

Given:

$$r_A = 10.0 \text{ cm}$$

$$r_C = 25.0 \text{ cm}$$

Angular acceleration of wheel A, $\alpha = 1.60 \text{ rad/s}^2$

Desired rotational speed of wheel C, $\omega_C = 100 \text{ rev/min}$

First, we need to convert the desired rotational speed of wheel C from revolutions per minute to radians per second:

$$\text{Angular speed of wheel C} = \omega_C = \frac{100 \times 2\pi}{60} = \frac{100 \times 2(3.14)}{60} = 10.5 \text{ rad/s}$$

Now, we can use the given equations:

$$v_A = v_C \Rightarrow r_A \omega_A = r_C \omega_C$$

$$1. \omega_A = \frac{r_C \omega_C}{r_A}$$

$$\omega_A = \frac{25.0 \times 10.5}{10.0} = 26.3 \text{ rad/s}$$

$$2. \alpha = \frac{\omega_A - 0}{t}$$

$$3. t = \frac{\omega_A}{\alpha} = \frac{26.3}{1.60} = 16.4 \text{ s}$$

Uniform circular motion

Consider an object rotating in a circle at a constant speed. For instance, imagine I've tied it to a thread and I'm rotating it. Now, the length of the thread remains constant, ensuring that the distance from the center, or the radius, remains unchanged, and the speed of the object also remains constant. However, does this imply that the speed remains constant without any acceleration? We need to delve deeper into this. It's essential to consider the direction of velocity. Initially, as mentioned before, the speed remains constant, denoted as v . However, there's a difference in direction: if v_1 is upward, then v_2 is slightly sideways, creating an angle between them. Moreover, v_2 is directed towards the center, giving rise to what we call centripetal acceleration. Now, let's determine its value. Understanding its direction is crucial; it's directed towards the center, thus termed centripetal acceleration.

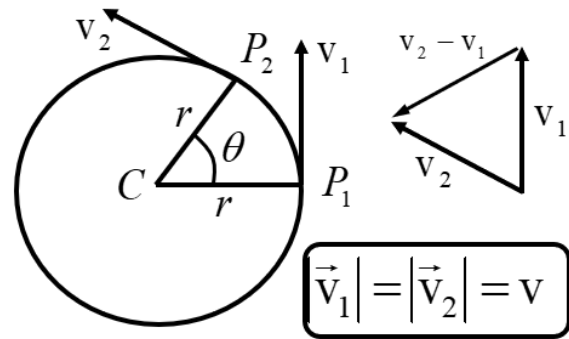


Fig. 11. 8: As object moves from P_1 to P_2 the direction of its velocity changes from v_1 to v_2 . The resultant vector of velocity $v_2 - v_1$ is toward the center of the circle.

To find out the difference, we need to calculate $v_2 - v_1$. Consider this scenario: if the body is initially positioned at P_1 and after some time it moves to the point P_2 , forming an angle θ (theta), Let's denote its length a_R , as the radius, and θ as the angle through

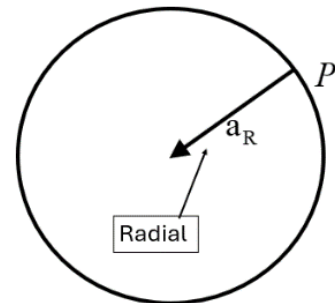


Fig. 11. 9: As the object moves along the circular path the direction of its velocity changes so the object undergoes a radial acceleration.

which it has rotated. By drawing a triangle here, we can notice a minor discrepancy between the triangle and the actual path. This occurs when theta is very small. However, if I adjust it slightly, ensuring theta is small, there will be no discernible difference between the triangle and the actual path. Thus, Δr is approximately equal to $r\theta$.

$$\Delta r = v\Delta t \approx r\theta$$

This same concept directly applies to velocity. Notice that another triangle is formed, where delta and velocity are both influenced by theta.

$$\Delta v \approx v\theta$$

Applying this concept again, we find that alpha, denoted 'a', equals v^2 divided by 'r', representing the average acceleration. However, you'll learn that there's minimal

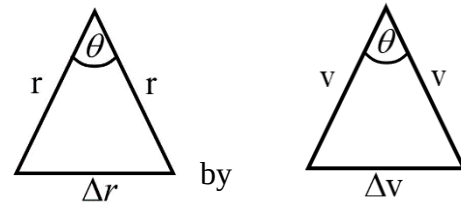


Fig. 11. 10: Triangular approximation for radius r and velocity v.

disparity between average and instantaneous values, given negligible differences in distances and times. Consequently, as we let Δt approach zero, we obtain v^2/r , known as centripetal acceleration.

$$\bar{a} = \frac{\Delta v}{\Delta t} \approx \frac{v\theta}{r\theta/v} = \frac{v^2}{r}$$

$$a = \lim_{\Delta t \rightarrow 0} \frac{\Delta v}{\Delta t} = \frac{v^2}{r}$$

$$\vec{a}_R = -\frac{v^2}{r} \hat{r}$$

Negative sign shows the acceleration is readily inward.

If there's no other acceleration, meaning the object maintains a constant speed, then only radial acceleration exists. However, if you gradually increase its speed, acceleration occurs; likewise, when decelerating. This introduces another acceleration component: tangential acceleration as shown in Fig 11.11. The total acceleration becomes a vector with two components: one being centripetal and the other is tangential. The resultant acceleration of these two components equals the total acceleration 'a'.

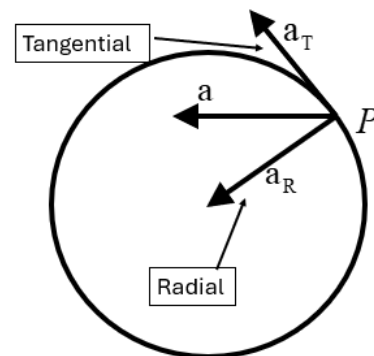


Fig. 11. 11: Total acceleration of the rotating object at a point P.

$$a_T = r\alpha, a_R = \frac{v^2}{r} = r\omega^2$$

$$a = \sqrt{a_T^2 + a_R^2}$$

Problem 11.6: The Moon revolves about the Earth as shown in Fig.11.12, making a complete revolution in 27.3 days.

Assume that the orbit is circular and has a radius of 238,000 miles. What is the magnitude of the acceleration of the Moon towards the Earth?

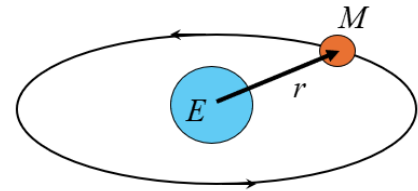


Fig. 11. 12: Illustration of the Moon and Earth during orbital motion.

Given Data :

Radius of the Moon's orbit : $r = 3.81952 \times 10^8$ meters

Period of the Moon's orbit : $T = 27.3$ days

Calculate Orbital Speed (v):

Convert the period of the Moon's orbit from days to seconds :

$$T = 27.3 \times 24 \times 60 \times 60 \text{ seconds}$$

$$T = 2.36 \times 10^6 \text{ seconds}$$

Use the formula for the circumference of a circle to find the orbital speed :

$$v = \frac{2\pi r}{T} = \frac{2 \times 3.14159 \times 3.81952 \times 10^8}{2.36 \times 10^6} \text{ m/s}$$

$$v = \frac{23.95546 \times 10^8}{2.36 \times 10^6} = 1015.421 \text{ m/s}$$

Calculate Acceleration (a):

Use the centripetal acceleration formula :

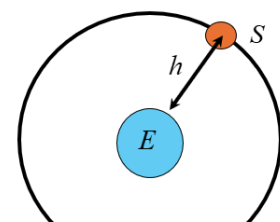
$$a = \frac{v^2}{r}$$

Substitute the calculated values :

$$a = \frac{(1015.421 \text{ m/s})^2}{3.81952 \times 10^8} \text{ m/s}^2 = \frac{1030885.961}{3.81952 \times 10^8} \text{ m/s}^2$$

$$a \approx 2.719 \times 10^{-3} \text{ m/s}^2$$

Problem 11.7: Calculate the speed of an Earth satellite as shown in Fig.11.13 that it is traveling at an altitude h of 210 km where $g = 9.2 \text{ m/s}^2$. The radius R of the Earth is 6370 km.



Solution:

$$a = \frac{v^2}{r}$$

$$a = g \text{ and } r = R + h$$

$$g = \frac{v^2}{R + h}$$

$$v = \sqrt{(R + h)g} = 7780 \text{ m/s}$$

Fig. 11. 13: Artificial satellite 'S' orbiting around the Earth at a height h.

Vector Cross Products

Let's explore another method of defining the product of two vectors, known as the cross product. Imagine drawing lines on a piece of paper. First, we'll create vector A and then position the second vector, 'B', anywhere within the same plane on this paper. Between them, there exists an angle θ . Now, while the dot product involves the sine of theta multiplied by the magnitude of A times B, the cross product takes a different approach. It's perpendicular to both A and B as shown in Fig.11.14, extending in a direction we'll call the 'n' direction. This perpendicularity is similar to the tip of a pencil rooted on the paper. The magnitude of this vector, denoted as 'A cross B', equals the magnitude of A times B times sine theta, with this being its limit.

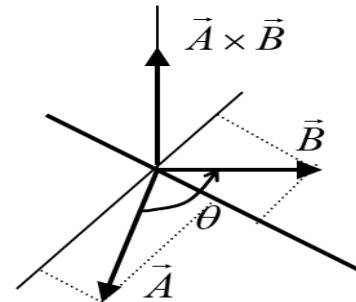


Fig. 11. 14: Vectors A and B and their cross-product $A \times B$ are perpendicular to each other.

$$\vec{A} \times \vec{B} = AB \sin \theta \hat{n}$$

$\hat{n}$ is perpendicular to AB-plane

Now, let's consider unit vectors in three dimensions, represented by $\hat{i}$, $\hat{j}$, and $\hat{k}$ cross products of these unit vectors are well defined. As the angle between any two different unit vectors is always 90 degrees, if we take the cross product of $\hat{i}$ and $\hat{j}$, it results in the third direction, $\hat{k}$, and similarly for other combinations.

$$\begin{aligned}\hat{i} \times \hat{j} &= \hat{k} \\ \hat{k} \times \hat{i} &= \hat{j} \\ \hat{j} \times \hat{k} &= \hat{i}\end{aligned}$$

Important properties

Several important properties characterize the cross product. For instance, if two vectors are parallel, their cross product is zero. When the angle between them is zero, the cross product of a vector with itself is also zero.

$$\begin{aligned}\vec{A} \times \vec{B} &= -\vec{B} \times \vec{A} \\ \vec{A} \times \vec{A} &= 0\end{aligned}$$

There are other properties as well, such as the distributive property.

$$(\vec{A} + \vec{B}) \times \vec{C} = (\vec{A} \times \vec{C}) + (\vec{B} \times \vec{C})$$

When expressed using unit vectors, $\vec{A} \times \vec{B}$ can be represented as a determinant.

$$\begin{aligned}\vec{A} \times \vec{B} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \\ &= (A_y B_z - A_z B_y) \hat{i} + (A_z B_x - A_x B_z) \hat{j} \\ &\quad + (A_x B_y - A_y B_x) \hat{k}\end{aligned}$$

Summary

In this chapter, we explained the fundamental concepts of rotational kinematics, specifically focusing on angular speed, angular velocity, and angular acceleration. These concepts come into play when a body undergoes rotational motion, moving along a circular path. We questioned why objects tend to move in circular paths when, inherently, they tend to move in straight lines. Exploring these principles helps us understand the forces at play that compel objects to rotate. Overall, this chapter provided an overview of rotational kinematics, shedding light on the dynamics of circular motion.

Physics-PHY101-Lecture 12

PHYSICS OF MANY PARTICLES

Introduction to center of mass:

Everybody is made up of particles and every particle has a direction with every other particle, that it either pulls it towards itself or pushes it. There are some quantities about which if we have the information then we can say something about the system as a whole which means they include the center of mass. So where the center of the mass will be, there will be a kind of concentration of mass.

Let's say there is an elephant. An ant moves on top of the elephant. It will not make much difference if the ant moves here or there. The mass of the elephant is so much that the center of mass will be the center of elephant. We would like to define the **center of mass** as,

A system's or object's center of mass is the location where the mass is considered to be concentrated. Simply, it's the average position of the mass distribution.

Mathematically for two masses the center of mass r_{cm} is:

$$\vec{r}_{cm} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$$

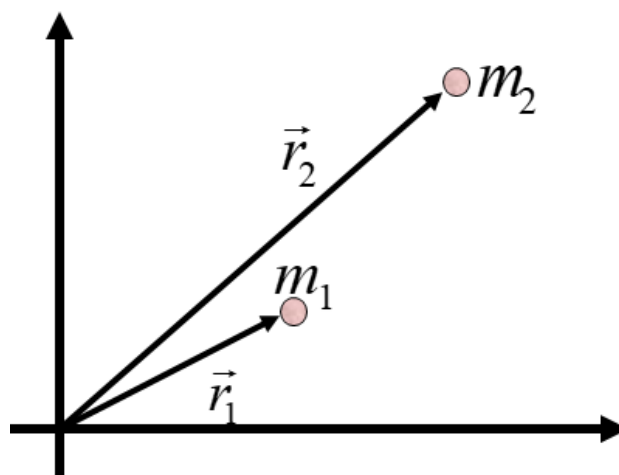


Figure 12.1. Distribution of masses ' m_1 and m_2 ' with their position vectors ' r_1 and r_2 '.

The center of mass equation can be written in two or three dimensions, depending on the given situation in question. In the case of two dimensions, its coordinates will be x and y; in the case of three dimensions, there will be x, y, and z. Equation of center of mass for two dimension is:

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

$$y_{cm} = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2}$$

The center of mass will lie between two masses if their masses are equal ($m_1 = m_2 = m$), and closer to the heavier mass if one mass is significantly greater than the other. If M is heavier than m, the center of mass is near mass M as can be seen from figure 12.2.

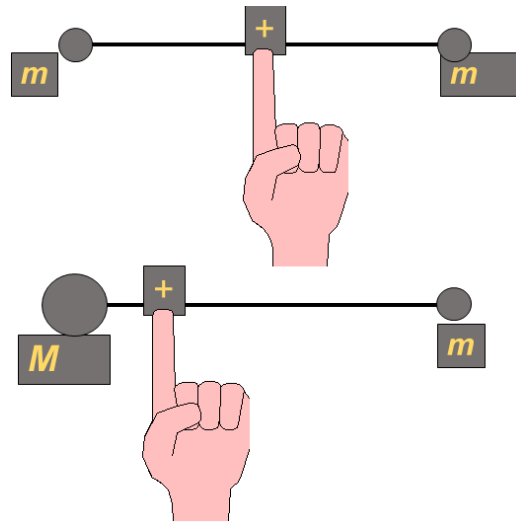


Figure 12.2. Center of mass alignment for unequal masses. Equal masses ($m_1 = m_2 = m$) result in a center of mass equidistant between them. For unequal masses ($M > m$), the center of mass is closer to the heavier mass.

Let's take another example. Consider two bodies of equal masses, one is placed at $x = 2$ m and other is placed at $x = 6$ m as shown in figure 12.3, then center of mass will be ,

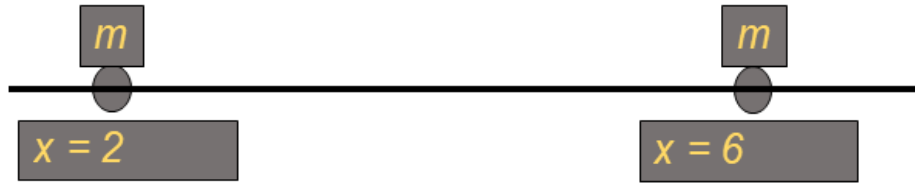


Figure 12.3. Two bodies of equal masses are positioned at $x = 2$ m and $x = 6$ m. The center of mass location is determined by their equilibrium.

$$x_{cm} = \frac{mx_1 + mx_2}{m + m}$$

$$X_{cm} = \frac{m(2 \text{ meter}) + m(6 \text{ meters})}{2m}$$

$$X_{cm} = \frac{8m}{2m} = 4m \text{ (4 meters)}$$

Now consider two bodies of unequal masses. One is a lighter 'mass m ' placed at $x = 6$ m and other is a heavier 'mass $3m$ ' placed at $x = 2$ m as shown in figure 12.4.

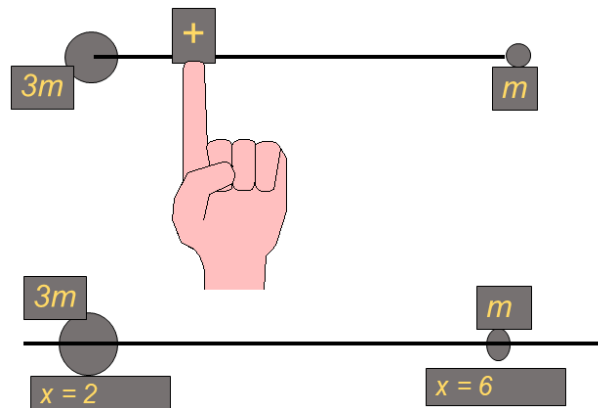


Figure 12.4. A lighter mass ' m ' is positioned at $x = 6$ m and a heavier mass ' $3m$ ' at $x = 2$ m. The center of mass is influenced by the masses and their respective positions.

$$x_{cm} = \frac{(3m)x_1 + mx_2}{3m + m} = \frac{(3m)2 + 6m}{4m} = \frac{12m}{4m} = 3 \text{ meters}$$

For N masses, the center of mass is:

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots + m_N\vec{r}_N}{m_1 + m_2 + \cdots + m_N}$$

$$\vec{r}_{cm} = \frac{1}{M} \left(\sum m_n \vec{r}_n \right)$$

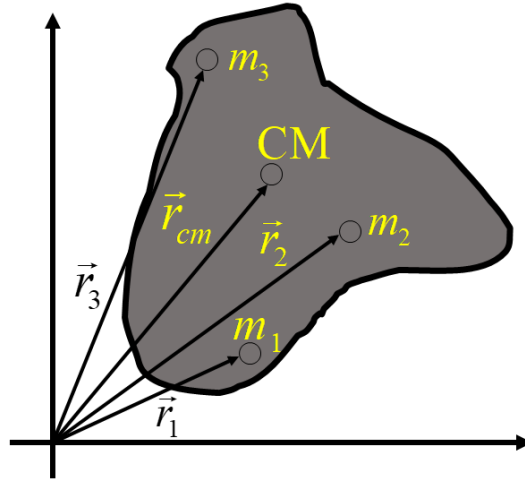
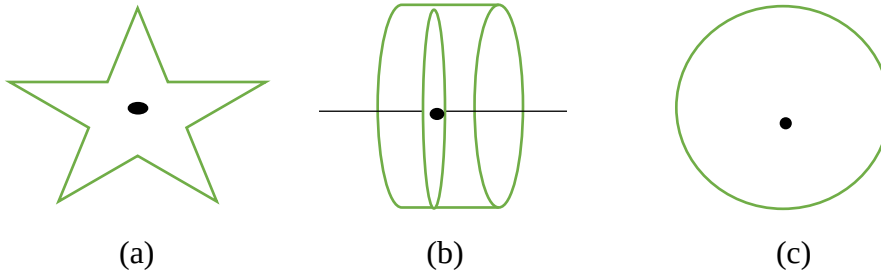


Figure 12.5: There is a point CM which we call center of mass, from the origin O, vector $\vec{r}$ be the position vector.

For symmetrical object, the center of mass is easy to guess as can be seen from the figures below.



In figure (a), the center of mass is in the center of star and in (b) the center of mass is in the center of cylinder if the material of cylinder will be uniform and in figure (c) if it's a uniform sphere then its center of mass is also in the center of sphere.

In order to understand that why the center of mass is useful concept, we have to apply newton's law. The velocity and acceleration of the center of mass can be calculated by the definition of the center of mass and differentiating it w.r.t time.

$$\vec{v}_{cm} = \frac{d\vec{r}_{cm}}{dt} = \frac{1}{M} \left(\sum m_n \vec{v}_n \right) \quad \therefore \vec{r}_{cm} = \frac{1}{M} \left(\sum m_n \vec{r}_n \right)$$

$$\vec{a}_{cm} = \frac{d\vec{v}_{cm}}{dt} = \frac{1}{M} \left(\sum m_n \vec{a}_n \right)$$

It is known as the acceleration of a single point, and we know that a particle possesses acceleration only when a force act on it. Here, the force can act for two reasons.

- One reason is that the particle is in external field. For example, the particle is inside the gravity and the gravity is pulling it down or up or in some direction.
- The second reason is that the other particles exert force and push or pull the particle in different directions.

The two types of forces that act on every particle are divided into external and internal forces. Therefore, the acceleration of each particle is due to the internal and external forces, but the internal force gets cancelled (because action and reactions are equal and opposite) and only external force remains.

$$M\vec{a}_{cm} = \sum \vec{F}_n = \sum (\vec{F}_{ext} + \vec{F}_{int})$$

$$\sum \vec{F}_{ext} = M\vec{a}_{cm}$$

In conclusion, the center of mass moves in such a way to concentrate the whole mass on it.

A center of mass is a point but it's not necessarily for a mass to be actually there. It's a common misconception that there's actually something. It is only appearing that all the masses concentrated there but when Newton's Law is applied, this imaginary point moves just like a mass that is concentrated at one point.

The center of mass for a combination of objects is the average center of mass location of objects. The center of mass can be outside the body; it does not have to be inside for all the structures as can be seen from figure 12.6.

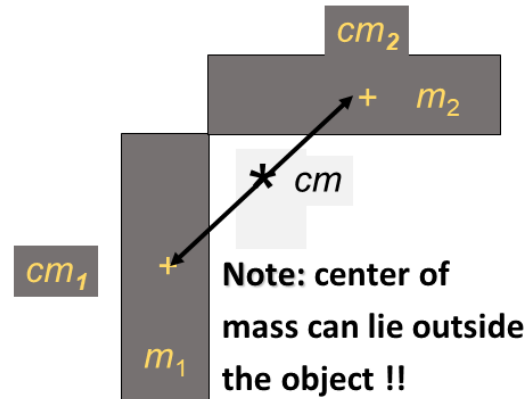


Figure 12.6. The center of mass represents the average location, and it can be located outside the physical boundaries of the objects.

Rotational Energy of Rigid Bodies:

Where there is movement there will be energy called kinetic energy. Motion is not only in one direction it can also be rotational motion. Consider a rigid body rotating about a fixed axis as shown in figure 12.7.

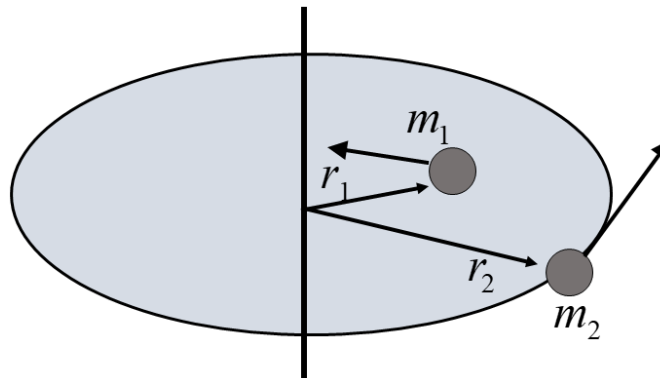


Figure 12.7. Rigid body rotation about a fixed axis.

Now let's know what a rigid body is. A body which has no elasticity, in which all its particles move together, such as wheel. If you calculate the total kinetic energy of a rigid body, then it is the sum of all the kinetic energies of each body.

Total kinetic energy is:

$$K = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \frac{1}{2}m_3v_3^2 + \dots$$

$$\therefore v = r\omega$$

For a rigid body or an inelastic body, all the particles are moving together, which means that each particle has the same angular velocity ' ω ' as the other particles, when we add all the K.E we get,

$$K = \frac{1}{2}m_1r_1^2\omega^2 + \frac{1}{2}m_2r_2^2\omega^2 + \frac{1}{2}m_3r_3^2\omega^2 + \dots$$

$$K = \frac{1}{2}\left(\sum m_i r_i^2\right)\omega^2$$

Where $\sum m_i r_i^2$ is called as 'moment of inertia I'.

Rotational inertia:

$$K = \frac{1}{2}\left(\sum m_i r_i^2\right)\omega^2$$

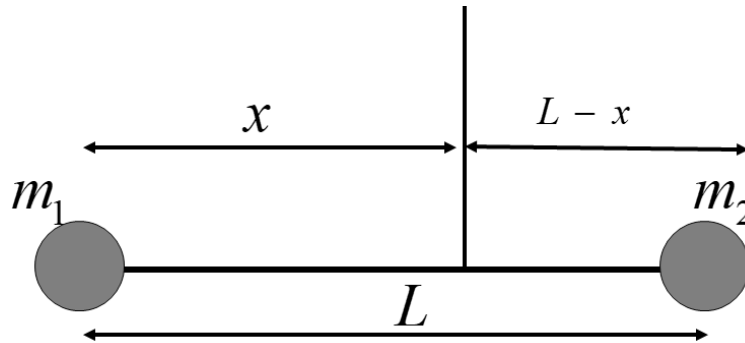
$$K = \frac{1}{2}I\omega^2 \quad \therefore I = \sum m_i r_i^2$$

This implies that we will take the mass of each body and use the square of the distance to calculate each body's moment of inertia. Then, we will add up all of the body's moments of inertia to get the total moment of inertia. When a body moves on straight line than its KE is:

$$K = \frac{1}{2}Mv^2 \quad \therefore v \text{ is the linear velocity.}$$

Problem:

Two particles m_1 and m_2 are connected by a light rigid rod of length L . neglect the mass of rod, find the rotational inertia I of this system about an axis perpendicular to the rod and at a distance x from m_1 .



Solution:

$$I = I_1 + I_2$$

$$I = m_1 x^2 + m_2 (L - x)^2$$

For what x , is I the largest?

$$\frac{dI}{dx} = \frac{d}{dx} (m_1 x^2 + m_2 (L - x)^2)$$

$$\frac{dI}{dx} = m_1 (2x) + m_2 2(L - x)(-1)$$

For a maxima, $\frac{dI}{dx} = 0$

$$0 = 2m_1 x - 2m_2 (L - x)$$

$$0 = 2m_1 x - 2m_2 L + 2m_2 x$$

$$(2m_1 + 2m_2)x = 2m_2 L$$

$$x = \frac{2m_2 L}{2(m_1 + m_2)} = \frac{m_2 L}{(m_1 + m_2)}$$

Conclusion: At this distance, the system of particle possess the highest value of moment of inertia. If $m_1 = m_2$ then $x = L/2$ for maxima.

Problem:

Three particles of masses m_1 (2.3 kg), m_2 (3.2 kg) and m_3 (1.5 kg) are at the vertices of a triangle.

Part-I: Find the rotational inertia about axes perpendicular to the xy plane and passing through each of the particles.

Part-II: What is the moment of inertia about the center of mass?

Solution:

Moment of inertia about each axis of rotation =?

When passing through particle 1:

$$I = I_1 + I_2 + I_3 = mr_1^2 + mr_2^2 + mr_3^2$$

$$I = 2.3kg(0) + 3.2kg(3)^2 + 1.5kg(4)^2 = 52.8kg$$

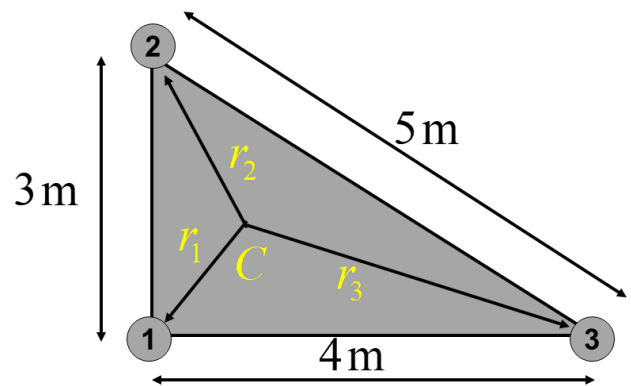
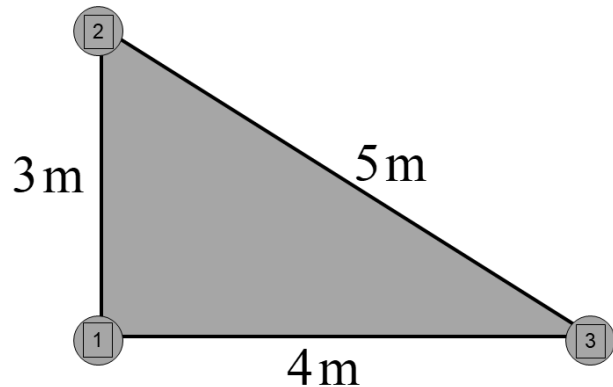
When passing through particle 2:

$$I = 2.3kg(3)^2 + 3.2kg(0) + 1.5kg(5)^2 = 58.2kg$$

When passing through particle 3:

$$I = 2.3kg(4)^2 + 3.2kg(5)^2 + 1.5kg(0) = 116.8kg$$

Part-II: What is the moment of inertia about the center of mass?



Lets find the center of mass first.

Since, masses are in xy-plane so,

$$\vec{r}_{cm} = x_{cm} \hat{i} + y_{cm} \hat{j}$$

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3} = \frac{m_1(0) + m_2(0) + m_3(4)}{2.3 + 3.2 + 1.5} = \frac{6}{7} = 0.86m$$

$$y_{cm} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3}{m_1 + m_2 + m_3} = \frac{m_1(0) + m_2(3) + m_3(0)}{2.3 + 3.2 + 1.5} = \frac{9.6}{7} = 1.37m$$

Using pythagoras theorem,

$$r_1^2 = x_{cm}^2 + y_{cm}^2 = (0.86)^2 + (1.37)^2 = 2.62m^2$$

$$r_2^2 = x_{cm}^2 + (y_2 - y_{cm})^2 = (0.86)^2 + (3 - 1.37)^2 = 3.40m^2$$

$$r_3^2 = (x_3 - x_{cm})^2 + y_{cm}^2 = (4 - 0.86)^2 + (1.37)^2 = 11.74m^2$$

$$I_{cm} = \sum m_i r_i^2 = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2$$

$$I_{cm} = 2.3 * 2.62 + 3.2 * 3.40 + 1.5 * 11.74$$

$$I_{cm} = 34.5 \text{ kg m}^2$$

This problem is for three masses, you may solve it for infinite masses. We use integration if we have infinite masses. If we divide a body into small parts, if mass of a small parts is dm . Take the distance r from the center and then take square of distance and then multiply with dm and then sum up over all the parts of the body that make up. This is called integration, and the total moment of inertia is the integration or the integral of the entire body of r square times dm .

For solid bodies:

$$I = \int r^2 dm$$

Hoop about cylinder axes:

$$I = \int r^2 dm \quad \because r = R = \text{Fix}$$

$$I = R^2 \int dm$$

$$I = MR^2$$

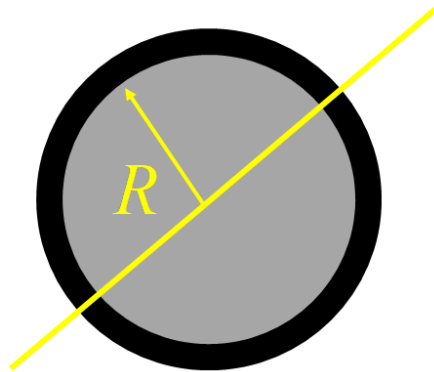


Figure 12.8. Hoop rotation about cylinder axis.

Solid plate about cylinder axes:

$$I = \int r^2 dm$$

$$\therefore \rho_0 = \frac{M}{\text{Area}} = \frac{M}{\pi r^2}$$

$$\text{if } \rho_0 = \frac{dm}{dA} = \frac{dm}{2\pi r dr}$$

$$dm = 2\pi r dr \rho_0$$

$$I = \int_0^R r^2 \cdot 2\pi r dr \rho_0$$

$$I = \int_0^R 2\pi r^3 dr \rho_0 = 2\pi \rho_0 \int_0^R r^3 dr$$

$$I = 2\pi \rho_0 \left. \frac{r^4}{4} \right|_0^R$$

$$I = \frac{1}{2} (\pi R^4 \rho_0) = \frac{1}{2} (\pi R^4) \frac{M}{\pi R^2}$$

$$I = \frac{1}{2} MR^2$$

Solid sphere about diameter:

$$I = \frac{2}{5} MR^2$$

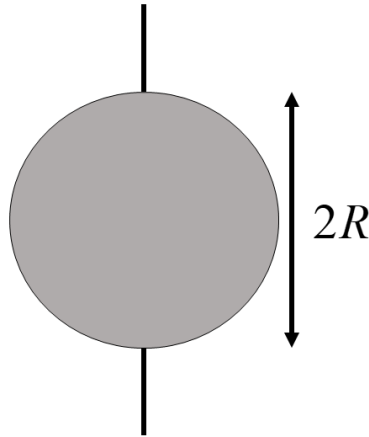


Figure 12.9. Solid sphere rotation about diameter.

For a hollow sphere, the mass is concentrated towards the outside, and you would get the result that a hollow sphere of mass m will have a greater moment of inertia than a sphere of mass m .

The moment of inertia measures how much energy is generated inside an object when you spin it. For example as shown in figure 12.10 (a) and (b).

(a) Solid cylinder or disk about cylinder axis

$$I = \frac{1}{2} MR^2$$

(b) Solid cylinder or disk about central diameter:

$$I = \frac{1}{4} MR^2 + \frac{1}{12} ML^2$$

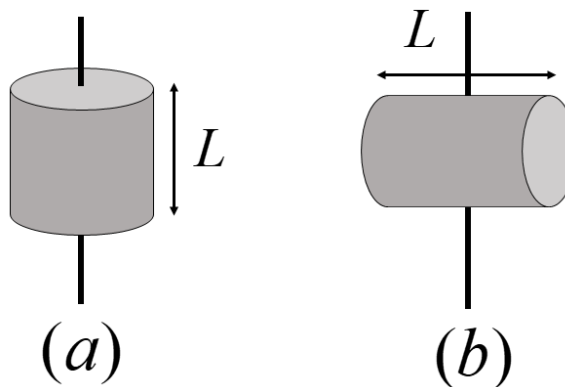


Figure 12.10. (a) solid cylinder or disk about cylinder axis, (b) solid cylinder or disk about central diameter .

Rectangular plate about central axis:

$$I = \frac{1}{12} M (a^2 + b^2)$$

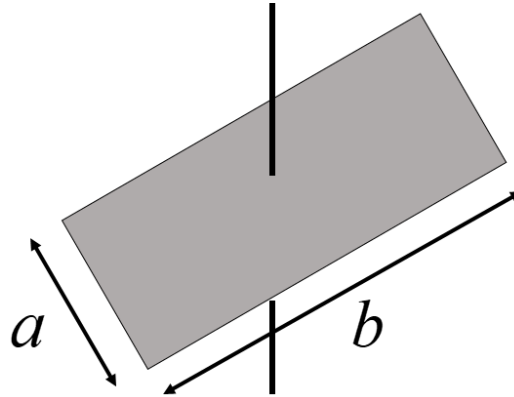


Figure 12.11. Rectangular plate about central axis.

So in conclusion, if an object rotates, its kinetic energy becomes equal to the $\frac{1}{2} I \omega^2$, but the question arises here is that, why the object rotates? Of course, force helps to rotate the object, but it depends on where the force is applied. For example, we have wrenches of various sizes. Because of torque, when we use a small-length wrench, it is difficult to tighten the nut; however, when we use a large-length wrench, the nut is easily tightened. A wrench with a long length has higher torque than one with a little length.

Now we define **torque** mathematically,

$$\vec{\tau} = \vec{r} \times \vec{F}$$

Where 'r' is the distance at which force acts. If we take its magnitude, we have

$$\tau = rF \sin \theta$$

And θ is the angle between r and F as shown in figure 12.12.

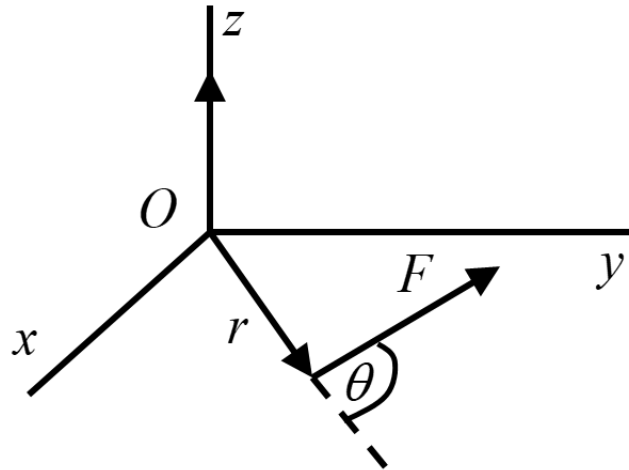


Figure 12.12. Illustration of torque, where 'r' is the distance at which force (F) acts, and θ is the angle between r and F.

Work can be done wherever a force occurs and when it acts and moves a distance S, the force into a distance is equal to work as shown in figure 12.13. So $F \cdot ds$ is the amount of work done.

$$dW = \vec{F} \cdot d\vec{s}$$

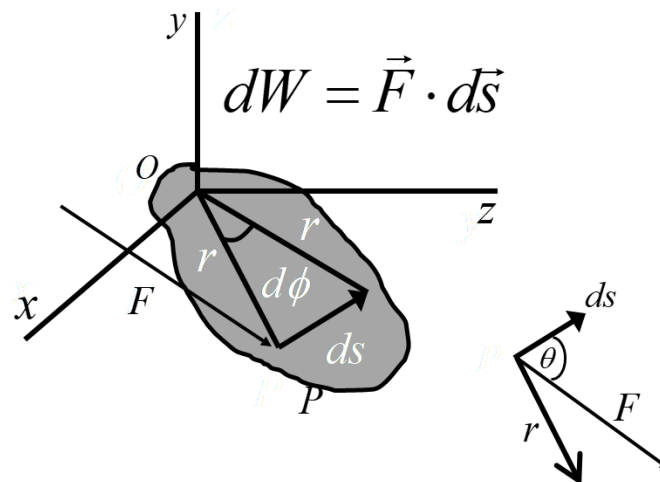


Figure 12.13. Work (W) is the product of force (F) and displacement (ds), where the force acting over a distance results in the performance of work.

So suppose there are several bodies and forces acting on them. Now the work done is,

$$\begin{aligned}
 dW &= \vec{F} \cdot d\vec{s} = F \cos \theta ds \\
 &= (F \cos \theta)(r d\phi) \\
 &= \tau d\phi
 \end{aligned}$$

For all particles,

$$\begin{aligned}
 dW_{net} &= (F_1 \cos \theta_1) r_1 d\phi + (F_2 \cos \theta_2) r_2 d\phi + \dots + (F_n \cos \theta_n) r_n d\phi \\
 &= (\tau_1 + \tau_2 + \dots + \tau_n) d\phi
 \end{aligned}$$

See, there's only one angle here, and that's because all the particles move together because it's a rigid body.

$$W_{net} = \left(\sum \tau_{ext} \right) d\phi = \left(\sum \tau_{ext} \right) \omega dt$$

When we differentiate the K.E with respect to ω ,

$$\begin{aligned}
 K &= \left(\frac{1}{2} I \omega^2 \right) \\
 \frac{dK}{d\omega} &= \frac{d}{d\omega} \left(\frac{1}{2} I \omega^2 \right) = \frac{1}{2} I (2\omega) = I \omega \\
 dK &= I \omega d\omega \quad \therefore \alpha = \frac{d\omega}{dt} \Rightarrow d\omega = \alpha dt \\
 dK &= (I \alpha) \omega dt
 \end{aligned}$$

Remember that α is the angular acceleration.

$$dW_{net} = dK$$

Work has been done by you in rotating it so that work is converted into kinetic energy. Then the result is the total external torque and is equal to moment of inertia multiplied by angular acceleration,

$$\sum \tau_{ext} = I \alpha$$

This equation is analogous to the Newton's law $F = ma$.

Before continuing we should know where translational and rotational motions are similar.

‘x’ is displacement at the one side and ϕ is angular displacement on the other side. As x increases with the time and we take the derivative dx/dt , likewise ϕ increases with time and we take the derivative from there we get the angular speed.

Translational motion

$$x, M$$

$$v = \frac{dx}{dt}$$

$$a = \frac{dv}{dt}$$

Angular motion

$$\phi, I$$

$$\omega = \frac{d\phi}{dt}$$

$$\alpha = \frac{d\omega}{dt}$$

In term of dynamics, the moment of inertia has exactly the same role for rotational motion as that of mass in translational motion.

Translational

$$F = Ma$$

$$W = \int F dx$$

$$K = \frac{1}{2} Mv^2$$

Rotational motion

$$\tau = I\alpha$$

$$W = \int \tau d\phi$$

$$K = \frac{1}{2} I\omega^2$$

Combined Rotational and Translational Motion:

In order to understand this type of motion, let's take the example of a car. Take the wheel of a car, put a mark on the wheel, and put a mark on its rim. The car goes in a straight line, but at the same time, its wheel rotates. So, this is an example of combined rotational and translational motion.

Now we study it in detail, there is a body with two vectors on it. One vector is of center of mass and the second is at point p, point P is any random point on a body and a vector which goes from the center of mass to P is called r_i' as shown in figure 12.14.

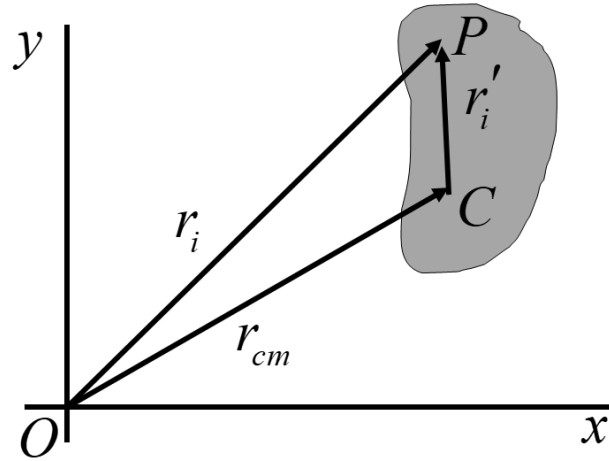


Figure 12.14. Body with center of mass vector and position vector at point P. The vector from the center of mass to a random point p on the body is defined as the position vector r'_i .

If we take out the total kinetic energy of it, then it will have two parts, one part is the kinetic energy of the center of the mass and the other part is due to its rotation.

$$K = \frac{1}{2} M v_{cm}^2 + \frac{1}{2} I_{cm} \omega^2$$

Remember, we didn't take a regular shape here. The body is rolling along its path in addition to traveling in a straight line. We now aim to find out if it is indeed the case that this body possesses two different kinds of kinetic energy. Let's prove it.

First of all, we will extract the kinetic energies of all its different particles, then we divide the velocity of every particle into two parts. One is the part of center of mass and other is due to rotation.

$$\begin{aligned}
K &= \sum \frac{1}{2} m_i v_i^2 = \sum \frac{1}{2} m_i \vec{v}_i \cdot \vec{v}_i \\
&= \sum \frac{1}{2} m_i (\vec{v}_{cm} + \vec{v}'_i) \cdot (\vec{v}_{cm} + \vec{v}'_i) \\
&= \sum \frac{1}{2} m_i (v_{cm}^2 + 2\vec{v}_{cm} \cdot \vec{v}'_i + v_i'^2) \\
&= \sum m_i \vec{v}_{cm} \cdot \vec{v}'_i = \vec{v}_{cm} \cdot \sum m_i \vec{v}'_i \\
&= \sum \vec{p}'_i = \sum m_i \vec{v}'_i = M \vec{v}'_{cm} \\
\vec{v}'_{cm} &= 0
\end{aligned}$$

$\vec{v}'_{cm} = 0$ in the center of mass frame.

$$\begin{aligned}
K &= \sum \frac{1}{2} m_i v_{cm}^2 + \sum \frac{1}{2} m_i v'^2 \\
&= \frac{1}{2} M v_{cm}^2 + \sum \frac{1}{2} m_i r_i'^2 \omega^2 \\
K &= \frac{1}{2} M v_{cm}^2 + \frac{1}{2} I_{cm} \omega^2
\end{aligned}$$

Here, we emphasized that it is a rigid body that moves along. However, we cannot apply this to a non-rigid body in such a way that, for example, rotating a bucket of water will cause the water inside to rotate at one speed while the water attached to the bucket rotates at a different speed, indicating that the bucket is not a rigid body.

Rolling without slipping:

Take a rigid body with a center of the mass speed (v_{cm}) moving toward the right side and this is just translational motion. And on the other hand it is just rotational motion that is rotating along the fix center and its speed is $R\omega$ on their edges as shown in figure 12.15.

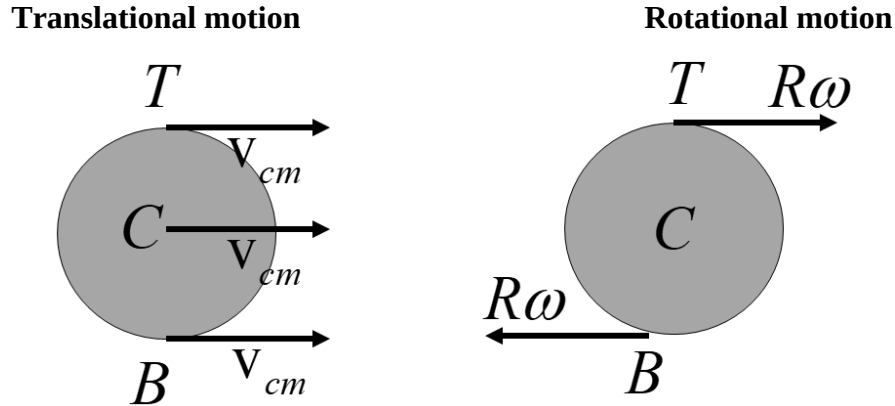


Figure 12.15. The body exhibits translational motion with center of mass speed (v_{cm}) to the right and rotational motion about a fixed center with speed $R\omega$.

It is possible that it is rolling and rotating together, so it is a combination of translational plus rotational motion. Here point B is attached to the ground so it is at rest. It is moving at the speed of V_{cm} , and the top point is moving at two times the velocity of center of mass as shown in figure 12.16.

Translational + Rotational motion

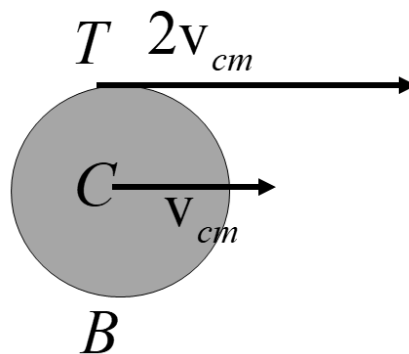


Figure 12.16. Rigid body with combined translational (v_{cm}) and rotational ($R\omega$) motion about a fixed axis.

If we find the total energy then we see the result of rolling without slipping is,

$$v_{cm} = R\omega$$

$$K = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I_{cm}\left(\frac{v_{cm}^2}{R^2}\right)$$

$$K = \frac{1}{2}MR^2\omega^2 + \frac{1}{2}I_{cm}\omega^2$$

We didn't say that this is a sphere or a hoop or the other shape we only discuss it as a rigid body.

Example:

In the absence of friction, energy is conserved, meaning that the same amount of energy at the beginning remains at the end and since we know that energy is divided into two parts; kinetics energy of center of mass plus rotational motion so from this we can solve many types of questions.

Now we take the example of a body rolling down a slope as shown in figure 12.17. For example a hoop which has the moment of inertia $\frac{1}{2}mR^2$. Now we want to know how fast it will be when it leaves the slope.

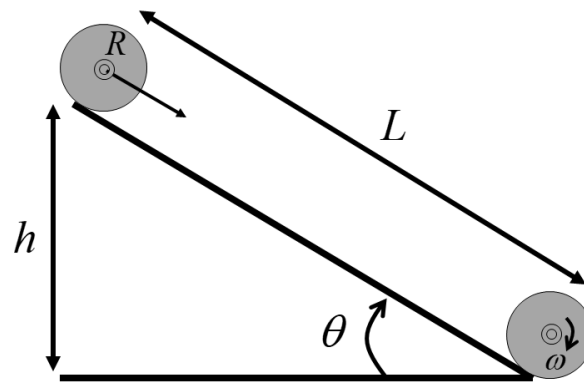


Figure 12.17. Rolling Motion of a Hoop on a Slope.

Solution:

Here we use the conservation of energy, in the start it has potential energy but later it loses the potential energy and only kinetic energy remains. When the body leaves the slope its K.E is,

At starting point, object possess gravitational P.E = Mgh

As it moves down, it possess rotational and translational K.E, which is:

$$K = \frac{1}{2} M v_{cm}^2 + \frac{1}{2} I_{cm} \omega^2$$

$$Mgh = \frac{1}{2} M v_{cm}^2 + \frac{1}{2} \left(\frac{1}{2} MR^2 \right) \left(\frac{v_{cm}}{R} \right)^2 \quad \therefore v_{cm} = r\omega$$

$$gh = \frac{v_{cm}^2}{2} + \frac{v_{cm}^2}{4} = \frac{3v_{cm}^2}{4}$$

$$v_{cm} = \sqrt{\frac{4}{3} gh}$$

Summary: Kinetic energy consists of two parts, one related to the center of mass and one related to its rotation. Center of mass and moment of inertia are two properties associated with a body and which define the body. What we've discussed is very important for mechanical engineers, people who design machines that have rotating objects inside, whether they're aircraft engineers or automobile engineers or others. We will go far with these concepts, which are essential in every other subject that we have learnt.

Lecture 13

Angular momentum

In physics, momentum is a fundamental concept. Momentum is determined by multiplying the velocity of a particle by its mass. Its significance lies in that when a force acts on a particle, its momentum changes. Conversely, in the absence of force, the momentum remains constant. When dealing with multiple particles, the total momentum equals the sum of the momenta of all individual particles. It is crucial to note that momentum is a vector quantity.

Today's lecture focuses on angular momentum, which is equally fundamental. This lecture will explore how angular momentum shares properties with regular momentum and possesses additional characteristics. This distinguishes it from a general definition of angular momentum.

Angular momentum of a single particle:

Angular momentum describes the rotational motion of an object. It is a vector quantity defined as the cross product of the object's position vector and its linear momentum vector, for a chosen origin, as shown in figure 13.1.

Mathematically,

$$\vec{L} = \vec{r} \times \vec{p}$$

There are different ways to write angular momentum.

$$L = r p \sin \theta$$

$$L = (r \sin \theta) p = r_{\perp} p$$

$$L = r(p \sin \theta) = r p_{\perp}$$

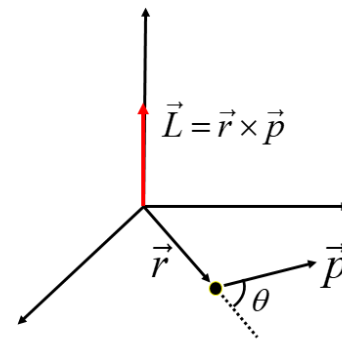


Figure13.1: Relation between angular and linear momentum.

Angular momentum of projectile:

A projectile is launched, following the trajectory of a parabola, originating from a specified point. Our objective is to determine its angular momentum for point O, as shown in figure 13.2.

It is crucial to clearly define point O, as the calculation of angular momentum depends on the reference point from which the projectile is thrown.

Let's find angular momentum $\vec{L}$ about origin O after some time t:

$$x = (v_0 \cos \theta)t \quad \text{eq(1)}$$

$$y = (v_0 \sin \theta)t - \frac{1}{2}gt^2 \quad \text{eq(2)}$$

$$v_x = v_0 \cos \theta \quad \text{eq(3)}$$

$$v_y = v_0 \sin \theta - gt \quad \text{eq(4)}$$

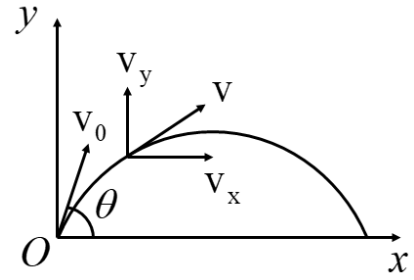


Figure13.2: Velocity components of a projectile.

$$\vec{L} = \vec{r} \times \vec{p} = (x\hat{i} + y\hat{j}) \times (v_x\hat{i} + v_y\hat{j})m$$

$$\vec{L} = m(xv_x\hat{i} \times \hat{i} + xv_y\hat{i} \times \hat{j} + yv_x\hat{j} \times \hat{i} + yv_y\hat{j} \times \hat{j})$$

$$\vec{L} = m(xv_x(0) + xv_y(\hat{k}) + yv_x(-\hat{k}) + yv_y(0)) \quad \because \hat{i} \times \hat{i} = \hat{k} \times \hat{k} = \hat{j} \times \hat{j} = 0$$

$$\vec{L} = m(xv_y - yv_x)\hat{k} \quad \because \hat{i} \times \hat{j} = \hat{k}, \hat{j} \times \hat{i} = -\hat{k}$$

putting values from eq(1-4), we get

$$\vec{L} = m \left((tv_0 \cos \theta)(v_0 \sin \theta - gt) - \left\{ (tv_0 \sin \theta - \frac{1}{2}gt^2)v_0 \cos \theta \right\} \right) \hat{k}$$

$$\vec{L} = m \left((tv_0 \cos \theta v_0 \sin \theta - gt^2 v_0 \cos \theta) - \left\{ tv_0 \sin \theta v_0 \cos \theta - \frac{1}{2}gt^2 v_0 \cos \theta \right\} \right) \hat{k}$$

$$\vec{L} = m \left(tv_0 \cos \theta v_0 \sin \theta - gt^2 v_0 \cos \theta - tv_0 \sin \theta v_0 \cos \theta + \frac{1}{2}gt^2 v_0 \cos \theta \right) \hat{k}$$

$$\vec{L} = m \left(\frac{1}{2}gt^2 v_0 \cos \theta - gt^2 v_0 \cos \theta \right) \hat{k}$$

$$\vec{L} = mgt^2 v_0 \cos \theta \left(\frac{1}{2} - 1 \right) \hat{k}$$

$$\vec{L} = mgt^2 v_0 \cos \theta \left(-\frac{1}{2} \right) \hat{k}$$

$$\vec{L} = -\frac{m}{2}gt^2 v_0 \cos \theta \hat{k}$$

Here we are seeing that the angular momentum is proportional to time. It changes with t-square and is increasing in the negative k-direction because it has a minus angular momentum.

Torque (reminder):

As we know torque is given as,

$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$\tau = rF \sin \theta$$

Relation between torque and angular momentum

$$\vec{L} = \vec{r} \times \vec{p}$$

$$\vec{L} + \Delta\vec{L} = (\vec{r} + \Delta\vec{r}) \times (\vec{p} + \Delta\vec{p})$$

$$\vec{L} + \Delta\vec{L} = \vec{r} \times \vec{p} + \vec{r} \times \Delta\vec{p} + \Delta\vec{r} \times \vec{p} + \Delta\vec{r} \times \Delta\vec{p}$$

$$\vec{L} + \Delta\vec{L} = \vec{L} + \vec{r} \times \Delta\vec{p} + \Delta\vec{r} \times \vec{p}$$

$\therefore \Delta\vec{r} \times \Delta\vec{p} = \text{ignored, very small term}$

$$\Rightarrow \Delta\vec{L} = \vec{r} \times \Delta\vec{p} + \Delta\vec{r} \times \vec{p}$$

$$\frac{\Delta\vec{L}}{\Delta t} = \frac{\vec{r} \times \Delta\vec{p} + \Delta\vec{r} \times \vec{p}}{\Delta t} = \frac{\vec{r} \times \Delta\vec{p}}{\Delta t} + \frac{\Delta\vec{r} \times \vec{p}}{\Delta t}$$

$$= \vec{r} \times \frac{\Delta\vec{p}}{\Delta t} + \frac{\Delta\vec{r}}{\Delta t} \times \vec{p}$$

Take limit as $\Delta t \rightarrow 0$:

$$\therefore \lim_{\Delta t \rightarrow 0} \frac{\Delta\vec{L}}{\Delta t} = \frac{d\vec{L}}{dt}$$

$$\Rightarrow \frac{d\vec{L}}{dt} = \vec{r} \times \frac{d\vec{p}}{dt} + \frac{d\vec{r}}{dt} \times \vec{p}$$

But $\frac{d\vec{r}}{dt}$ is $\vec{v}$ and $\vec{p} = m\vec{v}$!

$$\therefore \frac{d\vec{r}}{dt} \times \vec{p} = \vec{v} \times m\vec{v} = m(\vec{v} \times \vec{v}) = 0$$

And we are left with only,

$$\frac{d\vec{L}}{dt} = \vec{r} \times \frac{d\vec{p}}{dt}$$

Now use Newton's second law:

$$\vec{F} = \frac{d\vec{p}}{dt}$$

$$\Rightarrow \frac{d\vec{L}}{dt} = \vec{r} \times \vec{F}$$

$$\therefore \frac{d\vec{L}}{dt} = \vec{\tau}$$

$$\boxed{\frac{d\vec{L}}{dt} = \sum \vec{\tau}}$$

The net torque acting on a particle is equal to the time rate of change of its angular momentum.

The Spinning Top:

As we observed from experiment, the torque changes the direction but not the magnitude of the angular momentum and the motion is called precession, as shown in figure 13.3.

Let's do it mathematically, starting from torque.

$$\vec{\tau} = \vec{r} \times \vec{F}$$

where $\vec{F} = m\vec{g}$, so

$$\tau = Mgr \sin \theta \quad \text{eq(1)}$$

$\vec{\tau}$ is perpendicular to $\vec{L}$ as shown in figure 13.4

$\therefore$ it cannot change the magnitude of $\vec{L}$!!

we know from figure 13.5

$$\Delta\vec{L} = \vec{\tau} \Delta t$$

using $\theta = \frac{s}{r}$, we get

$$\Delta\phi = \frac{\Delta L}{L \sin \theta}$$

or,

$$\Delta\phi = \frac{\tau \Delta t}{L \sin \theta} \quad \text{eq(2)}$$

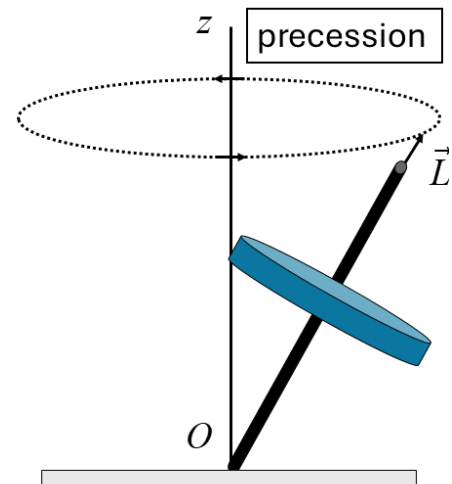


Figure 13.3: The precession about the z-axis.

Precession speed ω_p is:

$$\omega_p = \frac{\Delta\phi}{\Delta t}$$

using $\Delta\phi$ from eq(2)

$$= \frac{\tau \Delta t}{L \sin \theta} \frac{1}{\Delta t}$$

$$= \frac{\tau}{L \sin \theta}$$

using τ from eq(1)

$$= \frac{Mgr \sin \theta}{L \sin \theta}$$

$$\omega_p = \frac{Mgr}{L}$$

Here, precession is proportional to $1/L$. So, as angular momentum decreases with time the precession increase.

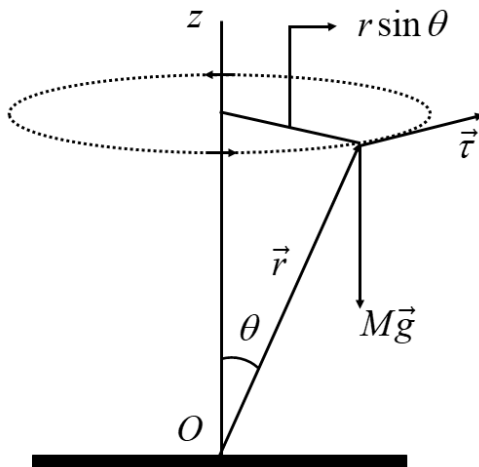


Figure 13.4: Torque, weight, and component of radius is being drawn.

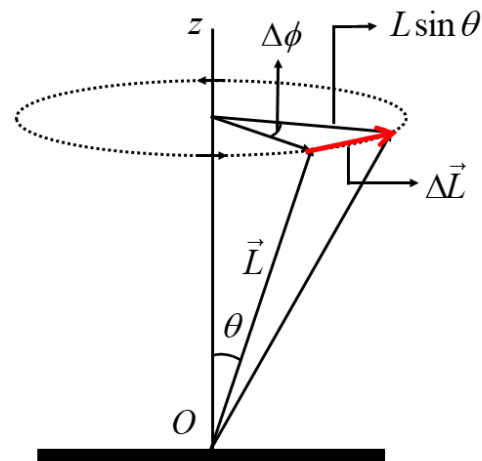


Figure 13.5: The incremental change in angular momentum and angular displacement is presented.

Example:

A mass m is released from a distance b along x -axis. A

Position vector is r from origin O , as shown in the

figure 13.6. Calculate torque and angular momentum.

Solution:

Let's calculate torque,

$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$\tau = rF \sin \theta$$

putting $r \sin \theta = b$ and $F = mg$ from figure 13.6

we get

$$\tau = mgb$$

Right hand rule shows that $\vec{\tau}$ is directed inwards, as shown in figure 13.6.

Let's calculate angular momentum,

$$\vec{L} = \vec{r} \times \vec{p}$$

$$L = rp \sin \theta$$

putting $r \sin \theta = b$ and $p = mv = mgt$

from figure 13.7, we get

$$L = mgbt$$

$\vec{L}$ is directed inwards, as shown in figure 13.7.

We know angular momentum and torque are connected, so let's calculate torque from angular momentum and check our expression.

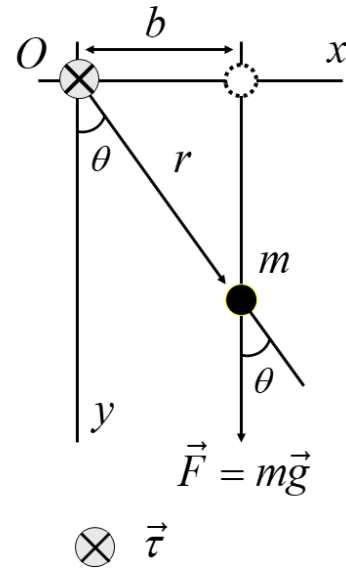


Figure 13.6: Direction of torque is into the paper.

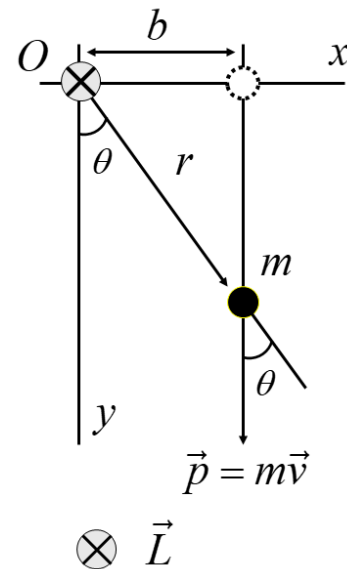


Figure 13.7: The direction of angular momentum is into the paper.

$$\begin{aligned}\frac{d\vec{L}}{dt} &= \frac{d}{dt}(mgb\hat{k}) \\ &= mgb\hat{k} \\ \tau &= mgb\hat{k}\end{aligned}$$

As we expected, it is the same.

Angular momentum for a system of particles:

Suppose we have large number of particles in a system, as shown in figure 13.8, to calculate its total angular momentum we do vector addition of angular momentum for individual particles.

$$\begin{aligned}\vec{L} &= \vec{L}_1 + \vec{L}_2 + \cdots + \vec{L}_N = \sum_{n=1}^N \vec{L}_n \\ \frac{d\vec{L}}{dt} &= \frac{d\vec{L}_1}{dt} + \frac{d\vec{L}_2}{dt} + \cdots + \frac{d\vec{L}_N}{dt} = \sum_{n=1}^N \frac{d\vec{L}_n}{dt} \\ \text{Since } \frac{d\vec{L}_n}{dt} &= \vec{\tau}_n \\ \frac{d\vec{L}}{dt} &= \sum_{n=1}^N \vec{\tau}_n\end{aligned}$$

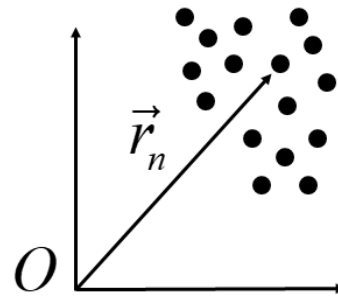


Figure 13.8: A general position vector for a system of particle.

Thus, the time rate of change of the total angular momentum of a system of particles equals the net torque acting on the system.

There are two sources of the torque acting on the system:

- 1) The torque exerted on the particles of the system by internal forces between the particles.
- 2) The torque exerted on the particles of the system by external forces.

$$\sum \vec{\tau} = \sum \vec{\tau}_{\text{int}} + \sum \vec{\tau}_{\text{ext}}$$

If the forces between two particles, as shown in figure 13.8 not only are equal and opposite but are also directed along the line joining the two particles, then the total internal torque is zero.

$$\sum \vec{\tau}_{\text{int}} = 0$$

$$\begin{aligned}\sum \vec{\tau}_{\text{int}} &= \vec{\tau}_1 + \vec{\tau}_2 \\ &= \vec{r}_1 \times \vec{F}_{12} + \vec{r}_2 \times \vec{F}_{21}\end{aligned}$$

but

$$\vec{F}_{12} = -\vec{F}_{21} = F \hat{r}_{12}$$

$$\begin{aligned}\therefore \sum \vec{\tau}_{\text{int}} &= (\vec{r}_1 - \vec{r}_2) \times \vec{F}_{12} = \vec{r}_{12} \times (F \hat{r}_{12}) \\ &= F (\vec{r}_{12} \times \hat{r}_{12}) = 0\end{aligned}$$

Hence

$$\sum \vec{\tau} = \sum \vec{\tau}_{\text{ext}} = \frac{d\vec{L}}{dt}$$

The net external torque acting on a system of particles is equal to the time rate of change of the total angular momentum of the system.

$$\sum \vec{\tau}_{\text{ext}} = \frac{d\vec{L}}{dt}$$

$$\sum \vec{F}_{\text{ext}} = \frac{d\vec{L}}{dt}$$

Conservation of Angular Momentum:

If no net external torque acts on the system, then the angular momentum of the system does not change with time.

$$\frac{d\vec{L}}{dt} = 0 \Rightarrow \vec{L} = \text{a constant}$$

Mathematical formulism of linear motion and angular motion shows some resemblance. As clear from eq(1) now go for angular momentum L since it is origin dependent while linear momentum p is not.

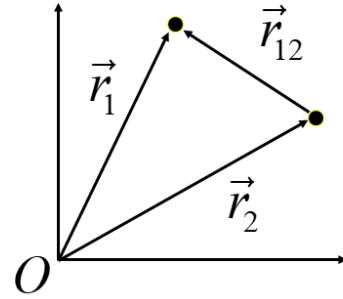


Figure 13.9: Two particles at position vector $\vec{r}_1$ and $\vec{r}_2$ respectively.

$$\vec{F} = \frac{d\vec{p}}{dt} \Leftrightarrow \vec{\tau} = \frac{d\vec{L}}{dt} \quad \text{eq(1)}$$

$$\vec{p} = m\vec{v} \Leftrightarrow \vec{L} = ??$$

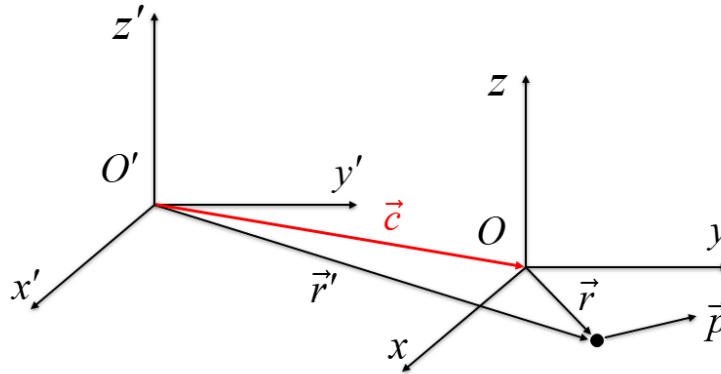


Figure 13.10: Angular momentum for two different origins is different.

$\vec{L}$ depends on the choice of the origin, from figure 13.10 we get

$$\vec{L}' = \vec{r}' \times \vec{p} = (\vec{c} + \vec{r}) \times \vec{p} = \vec{c} \times \vec{p} + \vec{L}$$

Rotation of Rigid Bodies:

A rigid body is an object whose shape doesn't deform under the influence of external forces. In other words, the distances between points on the object remain constant. Consider a rigid body that rotates about z-axis, as shown in figure 13.11. As reference line AP rotates through an angle all the points on it move with the same angular speed.

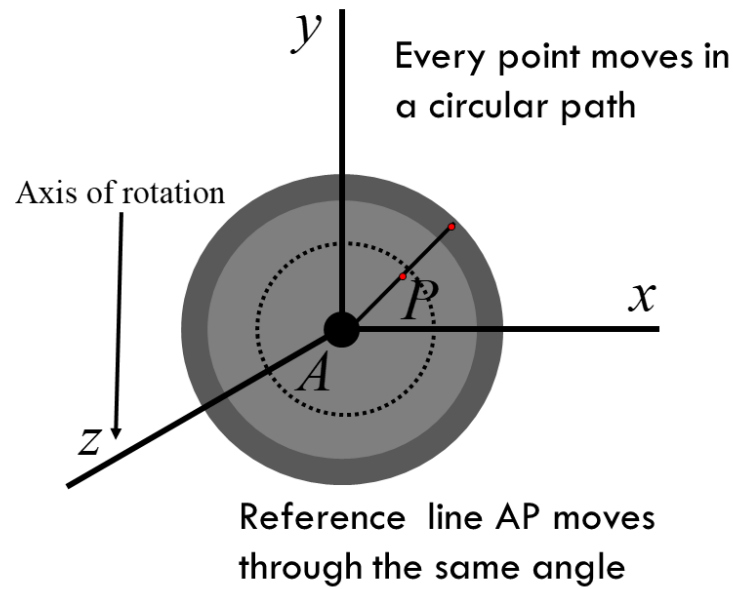


Figure 13.11: Rigid body rotates about z-axis.

Kinematics of a rigid body can be described by the motion of point P. There are two ways to understand one is by observing the motion of the point P as shown in figure 13.12 and the other is by observing the motion of cross-sectional slice of the rigid body, as shown in figure 13.13.

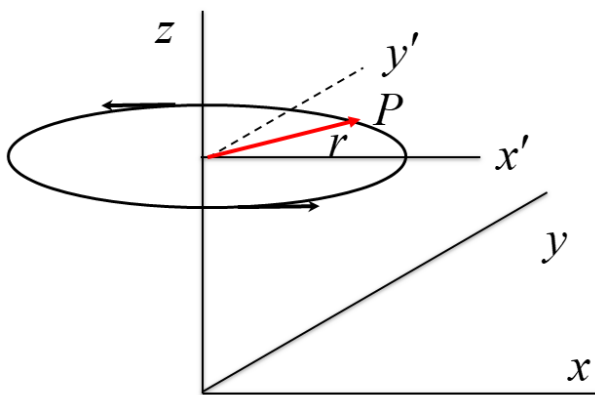


Figure 13.12: Motion of point P with translated coordinates.

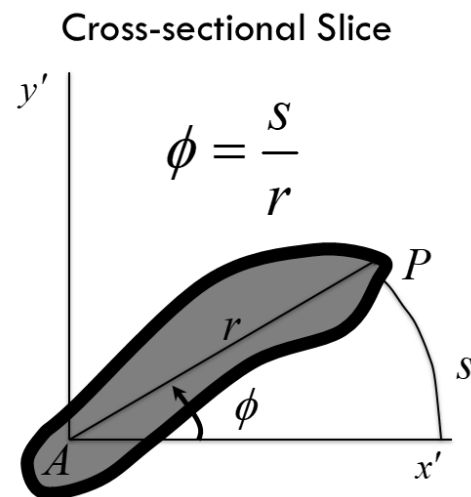


Figure 13.13: Motion of a cross-section of rigid body.

Linear and angular velocity:

Let's look at linear and angular velocity of point P. Their vector representation along with components of position vector $\mathbf{r}$ is shown in figure 13.14.

$$\vec{v} = \vec{\omega} \times \vec{r} \quad \text{eq(1)}$$

$$v = \omega r \sin \theta$$

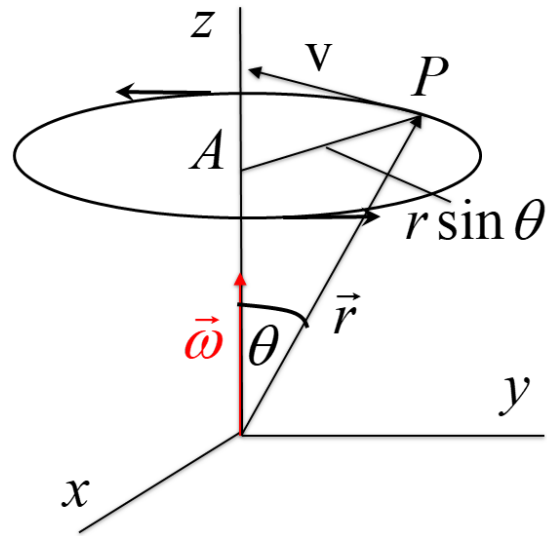


Figure 13.14: Linear and angular velocities of point P.

Linear and angular acceleration:

As we know acceleration is rate change of velocity so using eq(1) we can get,

$$\begin{aligned} \vec{a} &= \frac{d\vec{v}}{dt} = \frac{d}{dt}(\vec{\omega} \times \vec{r}) \\ &= \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \frac{d\vec{r}}{dt} \\ &= \vec{\alpha} \times \vec{r} + \vec{\omega} \times \vec{v} \quad \because \vec{\alpha} = \frac{d\vec{\omega}}{dt} \text{ and } \vec{v} = \frac{d\vec{r}}{dt} \\ \vec{a} &= \vec{a}_T + \vec{a}_R \end{aligned}$$

Where $\vec{a}_T$ and $\vec{a}_R$ are tangential and radial (towards the point A) components of acceleration, respectively, as shown in the figure 13.15.

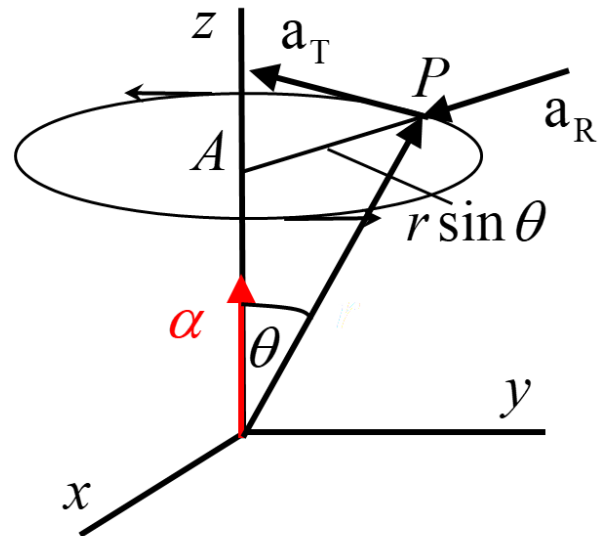


Figure 13.15: linear and angular acceleration of point P.

Now let's discuss a special case in which there are two equal masses. In this way, the center of mass will be in between the two. Axis of rotation passes through the middle, as shown in figure 13.16. Here, the direction of angular momentum is along the z-axis same as of angular momentum.

$$\vec{L} = (2mr^2)\vec{\omega}$$

$$\vec{L} = I\vec{\omega}$$

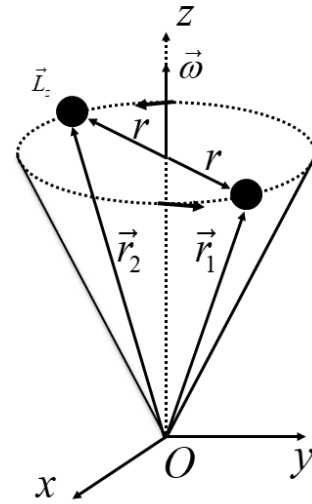


Figure 13.16: Two equal masses rotating about the z-axis.

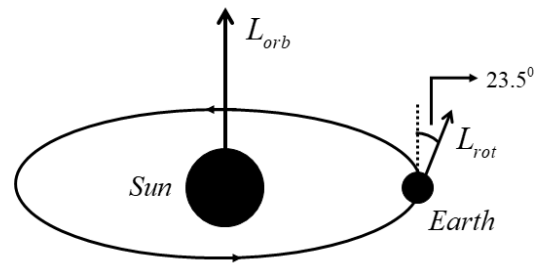
Problem # 1:

Which is greater?

- The angular momentum of the Earth due to rotation on its axis.
- The angular momentum of the Earth due to its orbital motion around the sun.

Solution:

L_{rot} is the angular momentum of earth due to rotational motion about its own axis and L_{orb} is the angular momentum of earth due to the orbital motion about the sun. R_E is the radius of the earth and R_{orb} is the distance between the sun and the earth (clearly $R_{orb} \gg R_E$).



$$L_{rot} = I\omega = \left(\frac{2}{5}MR_E^2\right)\omega \quad \text{eq(1)} \quad \because I = \frac{2}{5}MR_E^2 \text{ for solid sphere}$$

$$L_{orb} = R_{orb}p = R_{orb}Mv = R_{orb}M(R_{orb}\omega) = MR_{orb}^2\omega \quad \text{eq(2)}$$

By dividing eq(2) by eq(1), we get

$$\frac{L_{orb}}{L_{rot}} = \frac{MR_{orb}^2\omega}{\left(\frac{2}{5}MR_E^2\right)\omega}$$

$$\frac{L_{orb}}{L_{rot}} = \frac{5}{2} \left(\frac{R_{orb}}{R_E}\right)^2$$

we know, $R_{orb} \gg R_E$ or $\frac{R_{orb}}{R_E} \gg 1$ so,

$$\frac{L_{orb}}{L_{rot}} = \frac{5}{2} \left(\frac{R_{orb}}{R_E}\right)^2 \gg 1 \Rightarrow L_{orb} \gg L_{rot}$$

The angular momentum of the earth due to its orbital motion around the sun is much greater than the angular momentum of the earth due to rotation about its axis because of the fact that the distance between the sun and the earth is much larger than the radius of the earth.

Problem # 2:

A mass m is tied to a pulley of mass M with a massless rope, as shown. As mass m moves down, the pulley starts rotating. What is angular acceleration?

Solution:

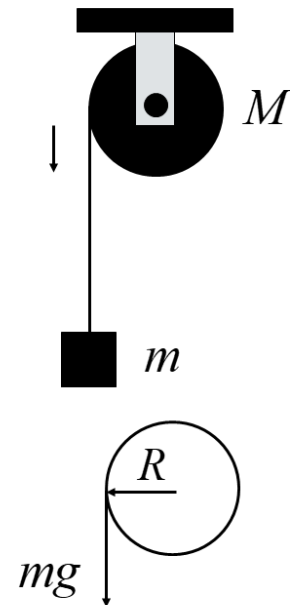
Total angular momentum of system consists of two sections, one is due to pulley and other is due to mass m . Origin is at center of the pulley so,

$$L = I\omega + mvR$$

We also know,

$$\tau = \frac{dL}{dt} \quad \because \tau = (mg)R$$

We get,



$$(mg)R = \frac{d}{dt}(I\omega + mvR)$$

$$(mg)R = I\left(\frac{d\omega}{dt}\right) + mR\left(\frac{dv}{dt}\right)$$

$$(mg)R = I\alpha + mRa \quad \therefore \alpha = \frac{d\omega}{dt} \text{ and } a = \frac{dv}{dt}$$

$$a = \alpha R \text{ or } \frac{a}{R} = \alpha \quad \therefore \text{relation between linear and angular acceleration}$$

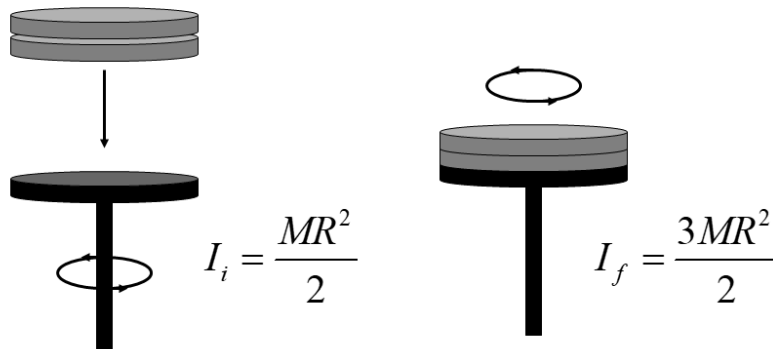
$$I = \frac{1}{2}MR^2 \quad \therefore \text{moment of inertia of pulley}$$

$$\Rightarrow mgR = \left(\frac{1}{2}MR^2\right)(a/R) + mRa$$

$$a = \frac{2mg}{M + 2m}$$

Problem # 3:

A disc which comes rotating and gets attached to another disc below. This upper disc which is twice as heavy as the lower disc and twice as big, when it gets attached to the lower disc, then these three discs start rotating together. What is the angular velocity?



Solution:

Here, angular momentum is conserved because there is no external torque. So,

$$I_i \omega_i = I_f \omega_f$$

$$\Rightarrow \omega_f = \omega_i \left(\frac{I_i}{I_f} \right)$$

$\therefore$ For one disc $I_i = I = \frac{1}{2}MR^2$ and for three discs $I_f = I = \frac{3}{2}MR^2$

$$\omega_f = \omega_i \left(\frac{MR^2}{2} \times \frac{2}{3MR^2} \right)$$

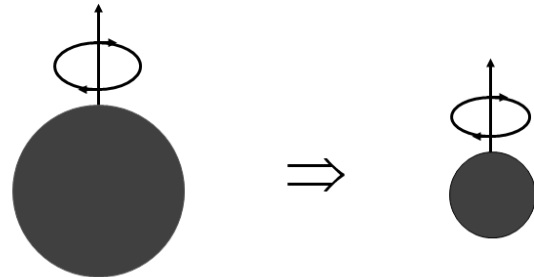
$$\omega_f = \frac{1}{3} \omega_i$$

So, the final angular velocity is one third of the initial angular velocity.

Additional Problems:

Problem # 1:

If the radius of the earth suddenly shrinks to half its present value, while the mass of the Earth remains unchanged, what will be the duration of one day where earth assumed to be a perfect sphere?



Solution:

Applying the law of conservation of angular momentum

$$I_i \omega_i = I_f \omega_f$$

$\therefore$ Moment of inertia of a solid sphere $I = \frac{2}{5}MR^2$

$$\therefore \omega = \frac{\text{angular displacement}}{\text{time}} = \frac{2\pi}{T}$$

$$\left(\frac{2}{5}MR_i^2 \right) \left(\frac{2\pi}{T_i} \right) = \left(\frac{2}{5}MR_f^2 \right) \left(\frac{2\pi}{T_f} \right)$$

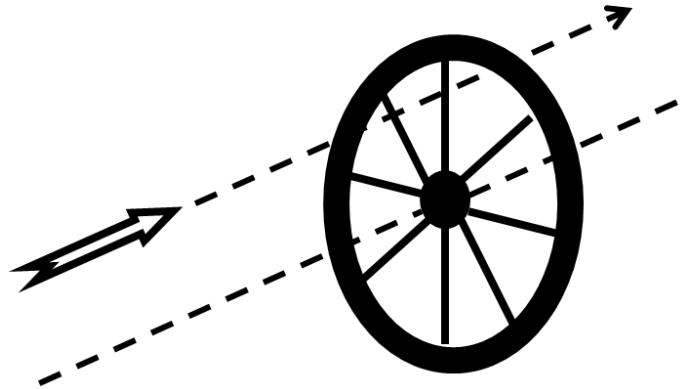
$$\frac{R_i^2}{T_i} = \frac{R_f^2}{T_f} \quad \text{or} \quad T_f = \left(\frac{R_f}{R_i} \right)^2 T_i = \left(\frac{1}{2} \right)^2 (24 \text{ hours})$$

$$T_f = \left(\frac{1}{4} \right) (24 \text{ hours}) = 6 \text{ hours}$$

So, duration of one day will be 6 hours.

Problem # 2:

A wheel has eight spokes and a radius of 30 cm. It is mounted on a fixed axle and is spinning at 2.5 rev/s. You want to shoot a 24 cm arrow parallel to this axle and through the wheel without hitting any of the spokes. Assume that the arrow and the spokes are very thin. (a) What minimum speed must the arrow have? (b) Does it



matter where between the axle and the rim of the wheel you aim? If so, where is the best location?

Solution:

$$\text{minimum speed} = \frac{\text{length of the arrow}}{\text{time to pass one spoke}}$$

$$s = \text{distance traveled by one spoke} = \frac{2\pi r}{8}$$

$$\text{time to pass one spoke} = \frac{\text{distance traveled by one spoke}}{\text{speed of spoke}}$$

$$\text{time to pass one spoke} = \frac{s}{v} = \frac{2\pi r}{8r\omega}$$

$$\text{So minimum speed} = \frac{\ell \times 8r\omega}{2\pi r} = 4.8 \text{ m/s}$$

Does not matter where we aim!!

Problem # 3:

A disk of mass $M = 2.5 \text{ kg}$ and radius $R = 20 \text{ cm}$ is mounted on a fixed horizontal axle. A block of mass $m = 1.2 \text{ kg}$ hangs from a light cord that is wrapped around the rim of the disk. Find the acceleration of the falling block, the tension in the cord, and the angular acceleration of the disk.

Solution:

$$\sum F = mg - T = ma$$

$$ma = mg - T \text{ eq(1)}$$

$$\because \tau = r F \text{ also } \tau = I\alpha$$

$$\sum \tau = TR \text{ eq(2)}$$

$$\sum \tau = \frac{1}{2}MR^2 \left(\frac{a}{R} \right) \text{ eq(3) } \because a = R\alpha$$

from eq(2) and eq(3) we get

$$TR = \frac{1}{2}MR^2 \left(\frac{a}{R} \right)$$

$$T = \frac{1}{2}Ma \text{ eq(4)}$$

substituting eq(4) into eq(1), we get

$$ma = mg - \frac{1}{2}Ma$$

$$ma + \frac{1}{2}Ma = mg$$

$$a \frac{M + 2m}{2} = mg$$

$$a = g \frac{2m}{M + 2m} \text{ eq(5)}$$

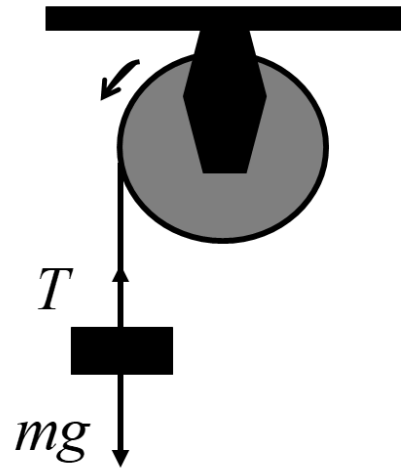
$$a = 4.8 \text{ m/s}^2$$

putting eq(5) into eq(4), we get

$$T = mg \frac{M}{M + 2m}$$

$$T = 6.0 \text{ N}$$

$$\alpha = \frac{a}{R} = 3.8 \text{ rev/s}^2$$



Physics-PHY101-Lecture

14 Equilibrium of rigid bodies:

Countless examples of stillness and motionlessness are found in our environment, for example, a ball or a chair lying on the floor or a ship sailing at a steady speed or the car is moving at a slow speed, these are some examples which are related to the ground but if you look from the sky then you will also see all the solar system is at a state of peace i.e. it is in equilibrium, so this lecture is about equilibrium of rigid/solid body.

During the last few lectures, we had talked to you in detail about momentum and angular momentum and before that we had also discussed centre of mass and moment of inertia. A body which is composed of many smaller objects, no matter how complex the interactions between them may be, but there is a point which is called the centre of mass and as this centre of mass moves, it carries momentum, depends on how much force acts on it.

“The rate of change of this momentum is equal to the net external force acting on the body”.

Therefore, if no external force acts on this body, then this is the state of equilibrium. This is a necessary condition for equilibrium, as if the total external force has to vanish. It certainly does not mean that all the forces vanish. If forces are in different directions and they cancel each other out, only then you will say that the net external force is zero.

For equilibrium, it is not enough that we say that the net external force should be zero.

Conditions for Equilibrium:

A rigid body is in mechanical equilibrium if both the linear momentum $\vec{P}$ and angular momentum $\vec{L}$ have a constant value.

$$\text{i.e., } \frac{d\vec{P}}{dt} = 0 \quad \text{and} \quad \frac{d\vec{L}}{dt} = 0$$

$$\vec{P} = 0 \quad \text{and} \quad \vec{L} = 0 \Rightarrow \text{static equilibrium}$$

Examples of Equilibrium:

Here are some practical examples of equilibrium.

1) *How does a lever work?*

Solution:

Torques balance about an axis through the fulcrum, as shown in the figure 14.1:

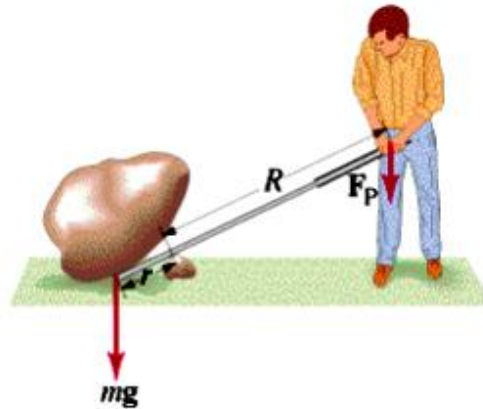


Figure 14.1: Torques by applied force and by force of gravity balance about an axis through the fulcrum.

Torque produced by person = Torque produced by stone

$$\vec{R} \times \vec{F}_p = \vec{r} \times \vec{F}_g$$

$$F_p \cdot R = mg \cdot r$$

$$\because F_g = mg$$

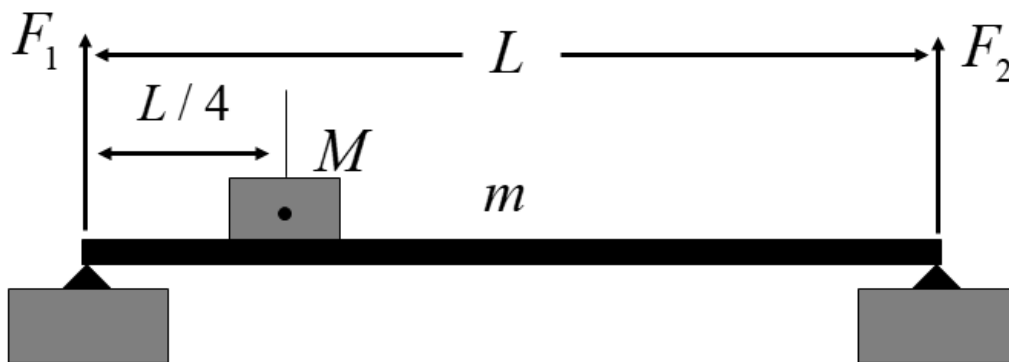
$$\because \theta = 90 \text{ so } \sin \theta = 1$$

Solve for the applied force (F_p):

$$F_p = mg \frac{r}{R}$$

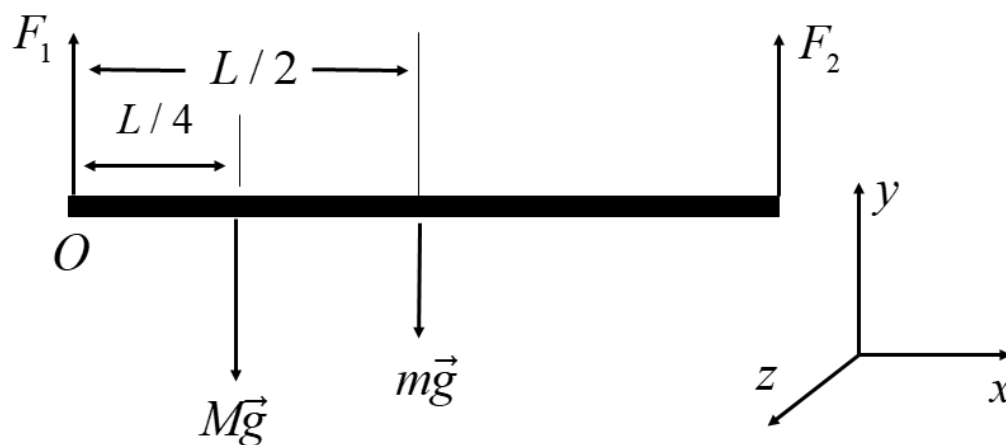
F_p (force applied by person) can be rather small depending on r and R . In the given figure R is bigger than r so a small F_p is required to lift the stone.

2) Consider a uniform rod of mass m (lies at center of mass) and length L . A mass M is placed on it as shown in figure. Find forces F_1 and F_2 ?



Solution:

For equilibrium, the sum of all forces and sum of all torque must be equal to zero.



Sum of all forces must be equal to zero

$$\sum F_y = F_1 + F_2 - Mg - mg = 0$$

$$\Rightarrow F_1 + F_2 - Mg - mg = 0$$

$$\Rightarrow F_1 + F_2 = Mg + mg \text{ or } F_1 = Mg + mg - F_2 \dots\dots (eq 14.1)$$

Sum of all torques must be equal to zero

$$\sum \tau_y = (F_1)(0) + (F_2)(L) - (Mg)(L/4) - (mg)(L/2) = 0$$

$$\Rightarrow F_2 L - Mg \frac{L}{4} - mg \frac{L}{2} = 0$$

$$\text{or } F_2 L = Mg \frac{L}{4} + mg \frac{L}{2} \text{ or } L(Mg \frac{1}{4} + mg \frac{1}{2})$$

$$F_2 L = L(Mg \frac{1}{4} + mg \frac{1}{2})$$

Cancelling L on both sides

$$F_2 = (Mg \frac{1}{4} + mg \frac{1}{2})$$

$$F_2 = \frac{Mg + 2mg}{4} = \frac{(M + 2m)g}{4}$$

$$F_2 = \frac{(M + 2m)g}{4}$$

Now put this value of F_2 in above eq 14.1,

$$\text{we get } F_1 = \frac{(3M + 2m)g}{4}$$

For a body in equilibrium, the choice of origin for calculating torques is unimportant. Torque must vanish everywhere irrespective of position of O.

Proof:

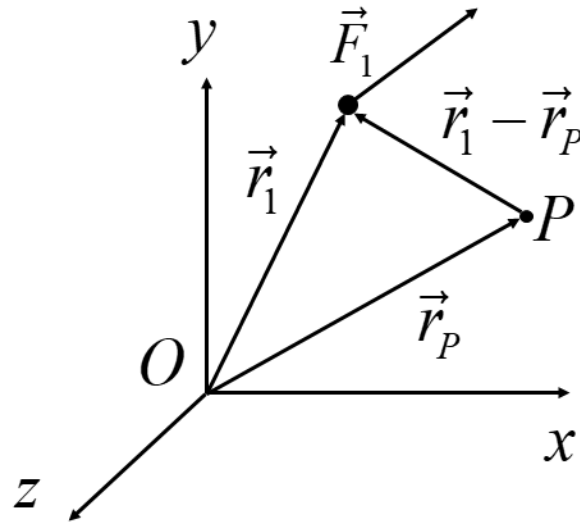


Figure 14.2: Torque about a general point P is being described.

Total torque is the sum of torques

$$\begin{aligned}\vec{\tau}_O &= \vec{\tau}_1 + \vec{\tau}_2 + \cdots + \vec{\tau}_N \\ &= \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 + \cdots + \vec{r}_N \times \vec{F}_N\end{aligned}$$

Torque about point P as shown in figure 14.2, written as

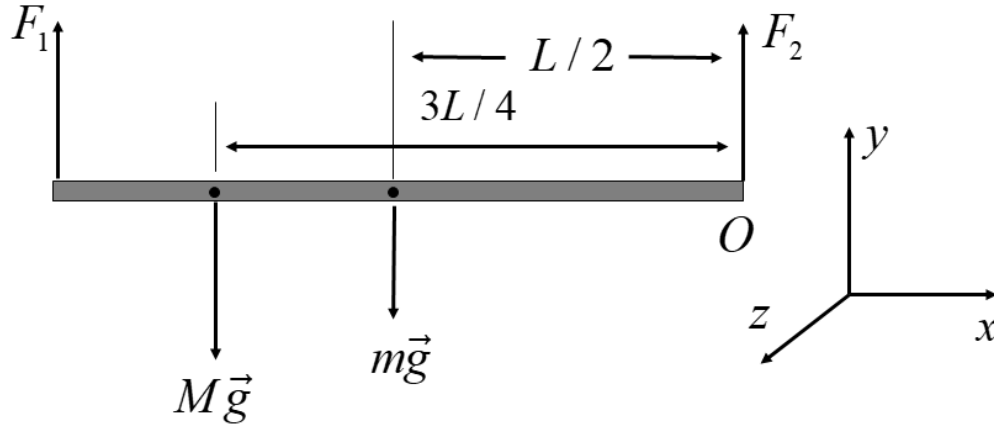
$$\begin{aligned}\vec{\tau}_P &= (\vec{r}_1 - \vec{r}_P) \times \vec{F}_1 + (\vec{r}_2 - \vec{r}_P) \times \vec{F}_2 + \cdots + (\vec{r}_N - \vec{r}_P) \times \vec{F}_N \\ &= [\vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 + \cdots + \vec{r}_N \times \vec{F}_N] - [\vec{r}_P \times \vec{F}_1 + \vec{r}_P \times \vec{F}_2 + \cdots + \vec{r}_P \times \vec{F}_N] \\ &= \vec{\tau}_O - [\vec{r}_P \times (\vec{F}_1 + \vec{F}_2 + \cdots + \vec{F}_N)] \\ &= \vec{\tau}_O - [\vec{r}_P \times (\sum \vec{F}_{ext})]\end{aligned}$$

but $\sum \vec{F}_{ext} = 0$, for a body in translational equilibrium

$$\therefore \vec{\tau}_P = \vec{\tau}_O$$

Hence the torque about any two points has the same value when the body is in translational equilibrium.

Now, origin O is at the right side (as shown in figure) lets recalculate forces to see the effect of different origin.



$$\sum F_y = F_1 + F_2 - Mg - mg = 0$$

$$\Rightarrow F_1 + F_2 - Mg - mg = 0$$

$$\Rightarrow F_1 + F_2 = Mg + mg \text{ or } F_2 = Mg + mg - F_1 \text{ (eq ii)}$$

$$\sum \tau_y = -(F_1)(L) + (F_2)(0) + (Mg)(3L/4) + (mg)(L/2) = 0$$

$$\Rightarrow -F_1 L + Mg \frac{3L}{4} + mg \frac{L}{2} = 0$$

$$\text{or } F_1 L = Mg \frac{3L}{4} + mg \frac{L}{2} \text{ or } L(Mg \frac{3}{4} + mg \frac{1}{2})$$

$$F_1 L = L(Mg \frac{3}{4} + mg \frac{1}{2})$$

Cancelling L on both sides

$$F_1 = (Mg \frac{3}{4} + mg \frac{1}{2})$$

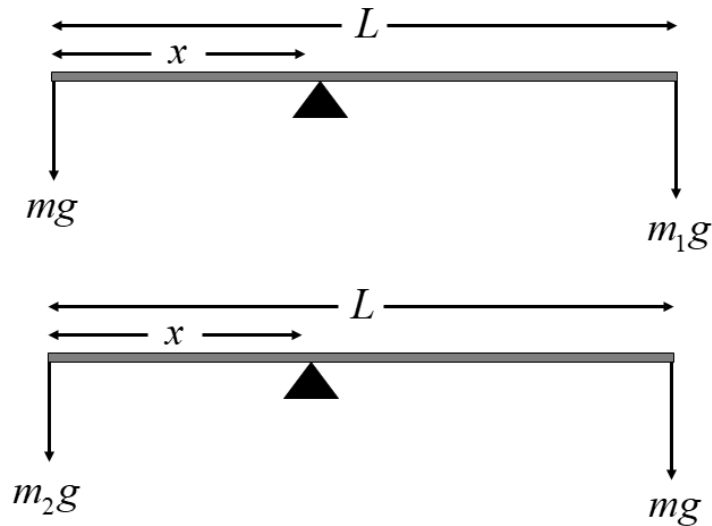
$$F_1 = \frac{3Mg + 2mg}{4} = \frac{(3M + 2m)g}{4}$$

$$F_1 = \frac{(3M + 2m)g}{4} \text{ Now put this value of } F_1 \text{ in eq ii above}$$

$$\text{we get } F_2 = \frac{(M + 2m)g}{4}$$

Conclusion: As we expected, change of origin does not affect the forces.

- 3) A massless rod of length L is placed on a fulcrum at a distance x from the left side. An unknown mass m is placed first on the left side and then on the right side of the rod (as shown in figure). If rod is balanced (by m_1 and m_2) in both situations, then calculate the value of m?

Solution:

$$m = ?$$

Taking the torques about the knife edge in the two cases (as shown in figure), we have,

$$mgx = m_1g(L - x)$$

$$m_2gx = mg(L - x)$$

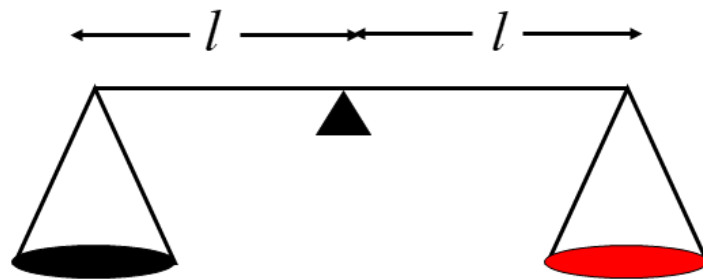
by dividing these equations we get

$$\frac{mgx}{m_2gx} = \frac{m_1g(L - x)}{mg(L - x)}$$

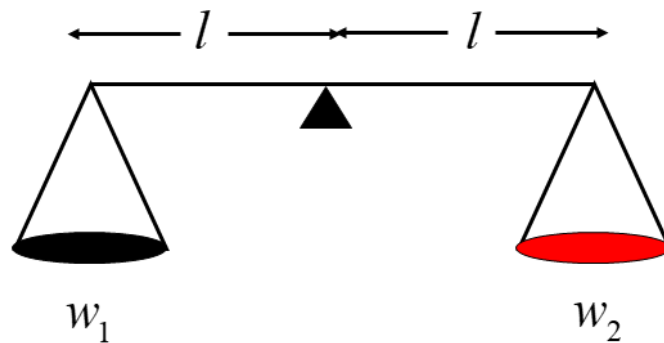
$$\Rightarrow \frac{m}{m_2} = \frac{m_1}{m} \text{ or } m = \sqrt{m_1m_2}$$

So, we can find out the unknown mass with the help of two known masses using this methodology.

- 4) A false balance (weight of both pans are not equal) has equal arms. An object weights x when placed in one pan and y when placed in the other pan. What is the true weight of the object?



Solution:



Let the weights of the pans be w_1 and w_2 and the true weight be w .

Taking torques about the knife edge gives:

$$(w + w_1)l = (x + w_2)l$$

$$\Rightarrow (w + w_1) = (x + w_2) \quad \text{eq(1)}$$

$$(w + w_2)l = (y + w_1)l$$

$$\Rightarrow (w + w_2) = (y + w_1) \quad \text{eq(2)}$$

Adding eq(1) and eq(2)

$$(w + w_2) + (w + w_1) = (x + w_2) + (y + w_1)$$

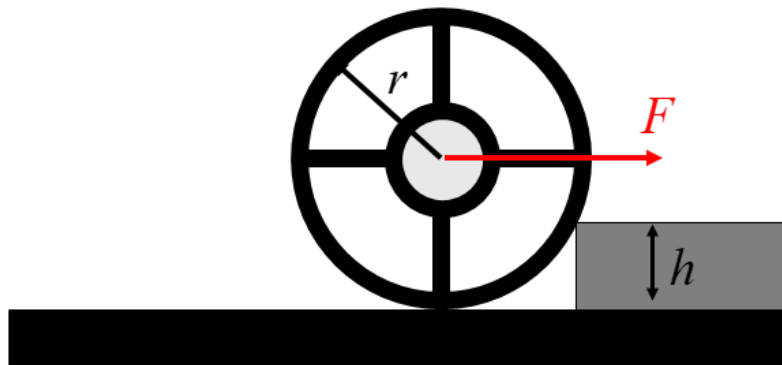
$$2w + (w_1 + w_2) = x + y + (w_1 + w_2)$$

$$2w = x + y$$

$$\Rightarrow w = \frac{x + y}{2}$$

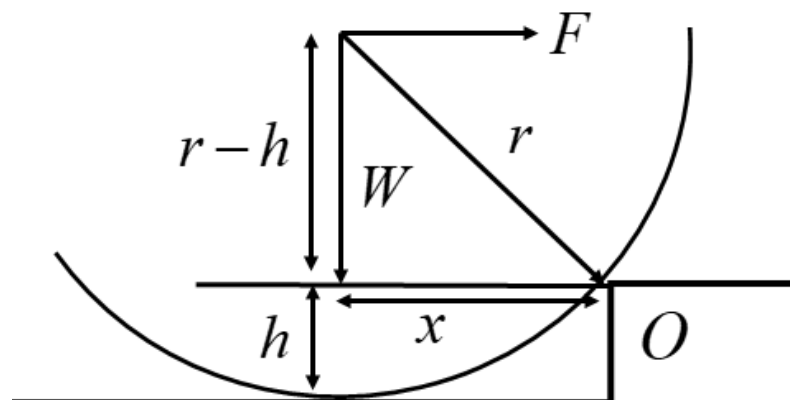
Conclusion: The actual weight is the average of both weights.

- 5) What minimum force F applied horizontally at the axle of the wheel (of the radius r) is necessary to raise the wheel over an obstacle of height h , as shown in the figure?



Solution:

The normal force vanishes as the wheel leaves the ground. The remaining two forces (applied F and weight w) participate in torque.



Take torques about O (the point of contact):

$$Wx = F(r - h)$$

$$x = \frac{F(r - h)}{W} \quad \text{eq(1)}$$

Using Pythagoras Theorem on right-angle triangle:

$$x^2 + (r - h)^2 = r^2$$

$$x^2 = r^2 - (r - h)^2$$

$$x^2 = r^2 - (r^2 + h^2 - 2rh)$$

$$x^2 = r^2 - r^2 - h^2 + 2rh$$

$$x^2 = 2rh - h^2$$

using equation (1)

$$\left\{ \frac{F(r - h)}{W} \right\}^2 = 2rh - h^2$$

$$\Rightarrow \frac{F(r - h)}{W} = \sqrt{2rh - h^2}$$

$$\Rightarrow F = W \frac{\sqrt{2rh - h^2}}{r - h}.$$

So, this F will raise the wheel over an obstacle of height h .

Review: Center of Mass (CM)

The center of mass (CM) is a theoretical point in a system of particles where, on average, the mass of the system is concentrated.

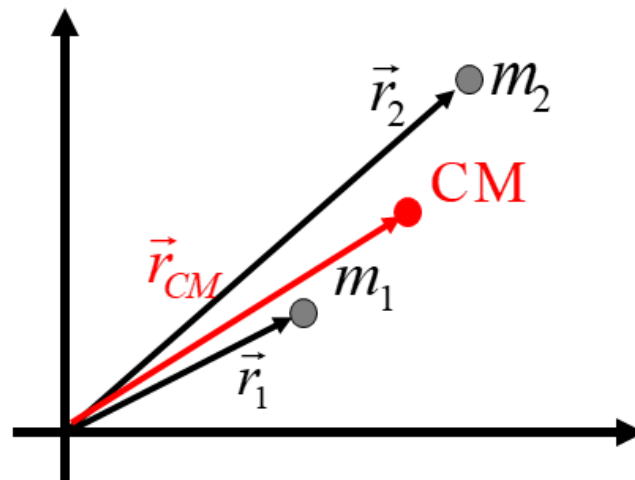


Figure 14.3: CM and $\vec{r}_{CM}$ for system of two masses.

Mathematical expression of figure 14.3 is:

$$\vec{r}_{cm} \equiv \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$$

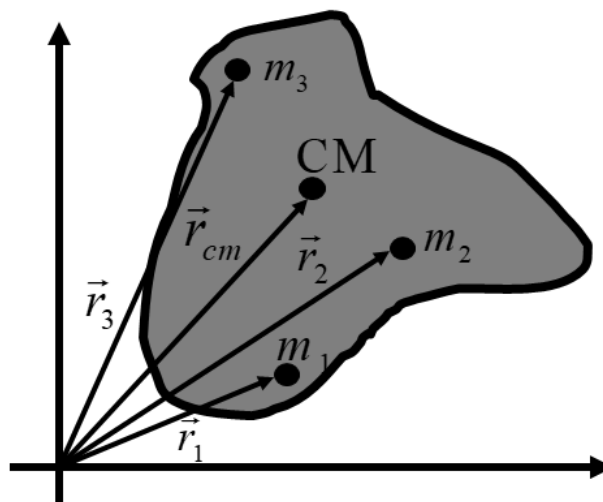


Figure 14.4: CM and $\vec{r}_{CM}$ for system of N masses is presented.

For N number of masses, as shown in figure 14.4 it can be written as

$$\vec{r}_{cm} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2 + \cdots + m_N \vec{r}_N}{m_1 + m_2 + \cdots + m_N}$$

We also know

$$\vec{F} = M \vec{a}_{cm}$$

where,

$$\vec{F} = \sum \vec{F}_{ext}$$

The specialty of the center of mass is that if we throw a body of any shape, then the trajectory of CM, the curve on which that body will move will be a parabola, its shape may be complicated, but its one point, which is the center of mass, moves on a parabola. The parabolic path (dotted line) of the center of mass (red dot) on the trajectory of the object is shown in figure 14.5.

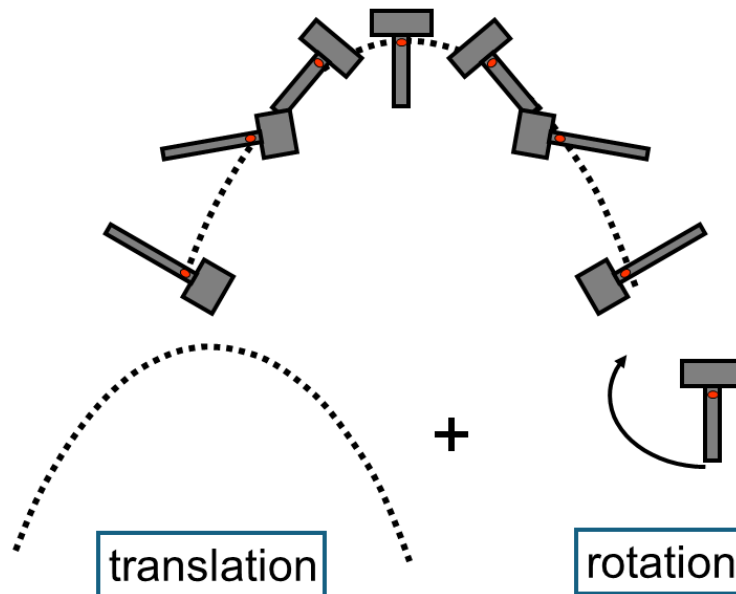


Figure 14.5: An object of arbitrary shape has translational and rotational motion while CM (red dot) has a parabolic path.

Centre of gravity:

The center of gravity is the average location of the weight of an object. As it is shown for different shapes in figure 14.6.

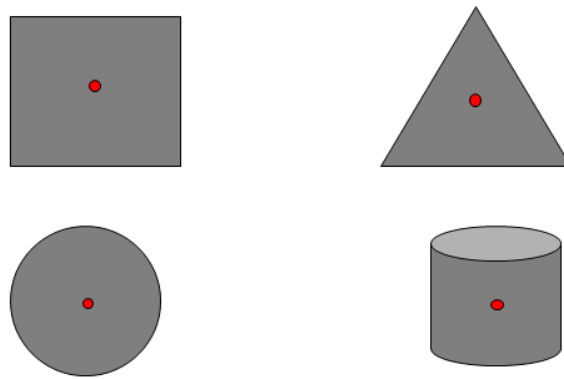


Figure 14.6: Center of gravity (as a red dot) is shown for different shapes.

Suppose the gravitational acceleration g has the same value at all points of a body. Then

- The weight is equal to mg
- And the center of gravity “coincides” with the center of mass.

Lets take look on equilibrium conditions, here

The net force on the whole = sum over all individual particles

$$\sum \vec{F} = \sum m_i \vec{g}$$

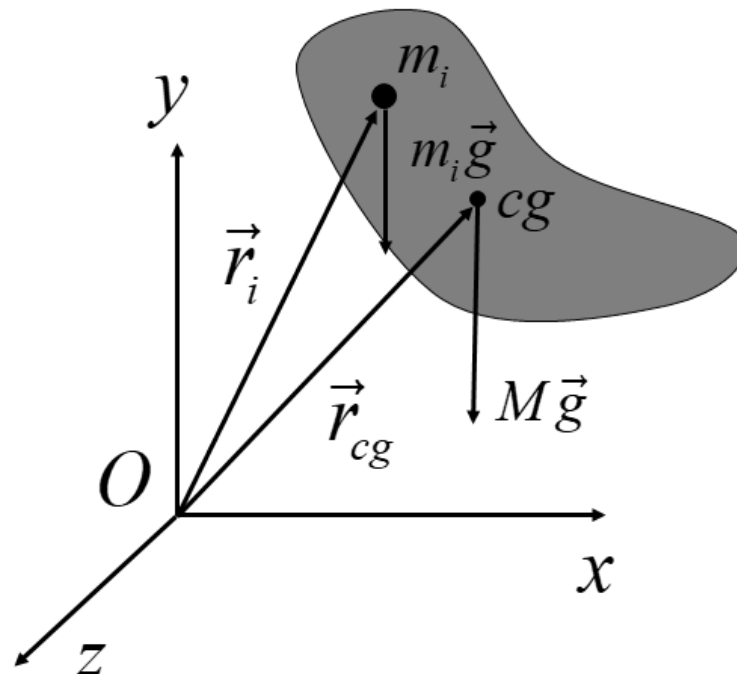


Figure 14.7: Position vector $\vec{r}_{cg}$ and cg is center of gravity for an arbitrary shape is depicted.

Since $\vec{g}$ has the same value for every particle of the body

$$\therefore \sum \vec{F} = \vec{g} \sum m_i = M \vec{g}$$

The net torque about the origin O , as shown in figure 14.7 :

$$\begin{aligned} \sum \vec{\tau} &= \sum (\vec{r}_i \times m_i \vec{g}) \\ &= \sum (m_i \vec{r}_i \times \vec{g}) \quad \because \vec{r}_{CM} = \frac{m_i \vec{r}_i}{M} \end{aligned}$$

$$\therefore \sum \vec{\tau} = M \vec{r}_{cm} \times \vec{g} = \vec{r}_{cm} \times M \vec{g}$$

Figure 14.8 shows the torque due to gravity about the centre of mass of a body is zero, and object is balanced at CM/CG.

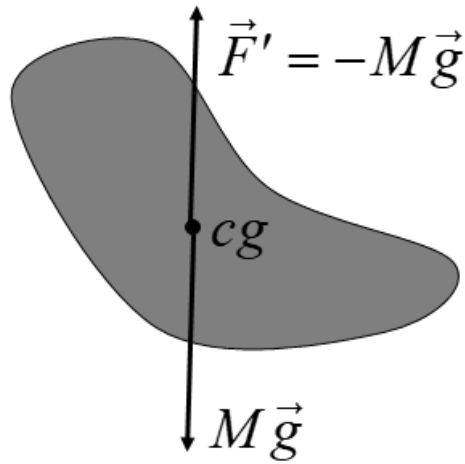


Figure 14.8: The object is balanced at CM/CG.

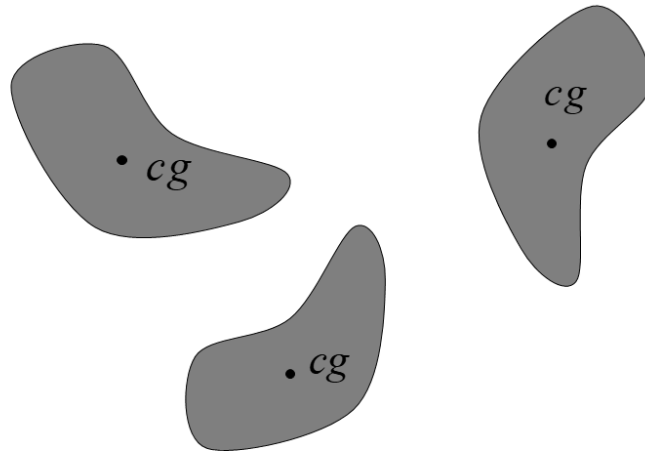


Figure 14.9: Center of gravity for different shapes is shown.

The object will be in equilibrium no matter what its orientation.

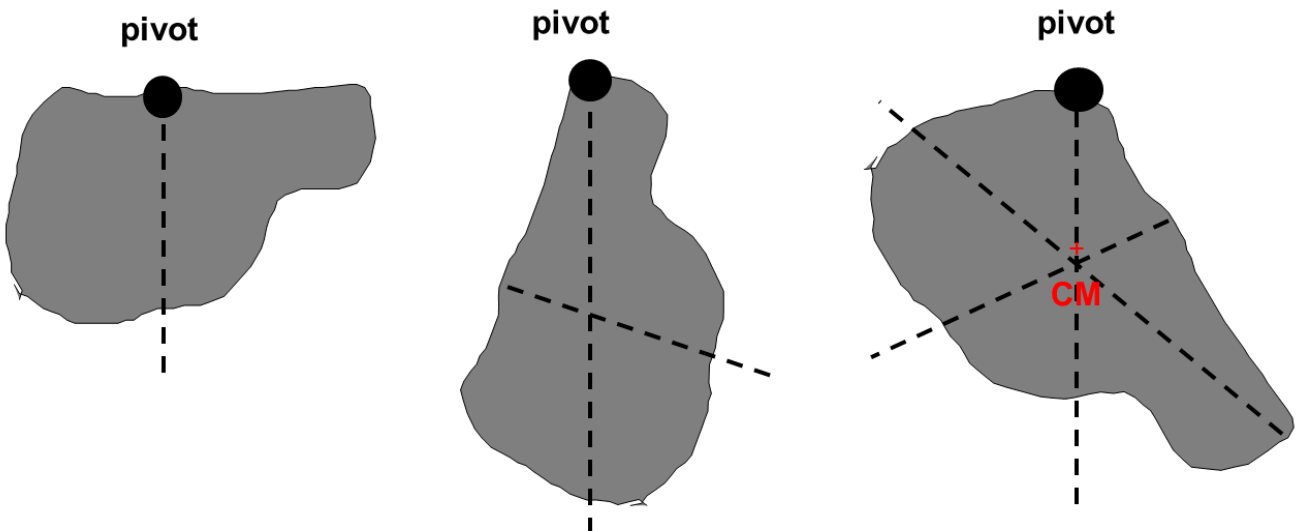


Figure 14.10: The arbitrary shaped object is being pivoted at different edges to find out CM.

We find that the center of mass is at the “center” of the object.

Difference Between Center of Mass and Center of Gravity:

If g is not constant over the body, then the center of gravity and the center of mass do not coincide, as shown in figure 14.11.

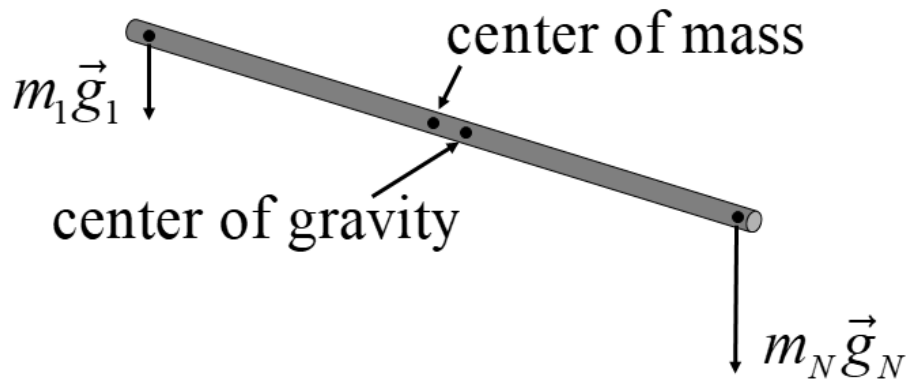


Figure 14.11: For different values of g over a body can lead to different location of CM and CG of the body.

Types of Equilibrium:

- **Stable equilibrium:** Object returns to its original position if displaced slightly, as shown in figure 14.12.

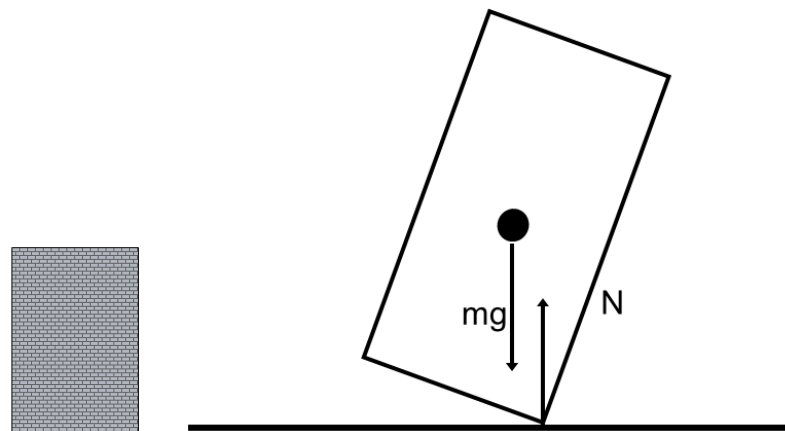


Figure 14.12: The object possesses stable equilibrium if slightly tilted.

- **Unstable equilibrium:** Object moves farther away from its original position if displaced slightly, as shown in figure 14.13.

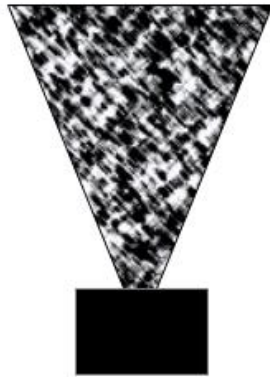


Figure 14.13: The object possesses unstable equilibrium if slightly tilted.

- **Neutral equilibrium:** Object stays in its new position if displaced slightly, as shown in figure 14.14.

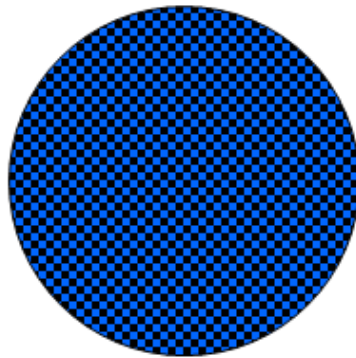


Figure 14.14: The object possesses neutral equilibrium if slightly tilted.



Figure 14.15: To achieve stable equilibrium while carrying a weight people tend to keep CG over their feet.

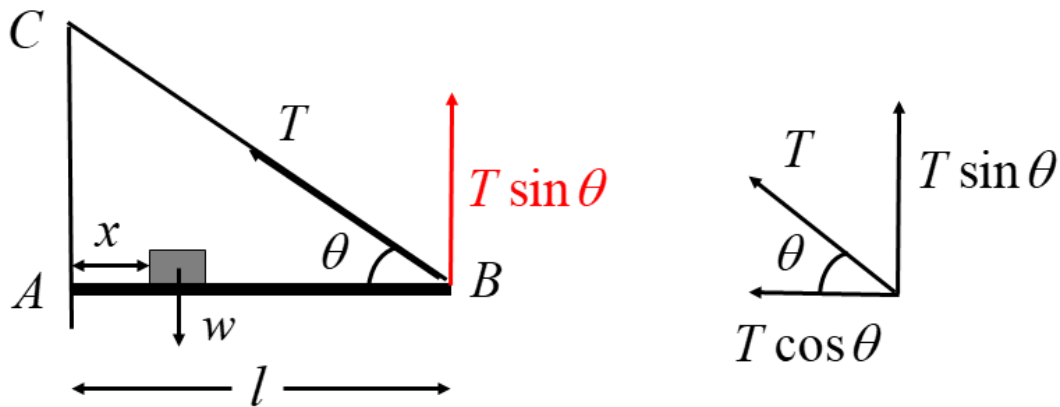
- We can observe that people try to keep the CG over their feet, to feel stable while carrying weight, as shown in figure 14.15.
- 6) A thin bar AB of negligible weight is pinned to a vertical wall at A and supported by a thin wire BC, a weight w can be moved along the bar.
 - (a) Find T as a function of x . (b) Find the horizontal and vertical components of the force exerted on the bar by the pin at A.

Solution:

Since the system is in rotational equilibrium, the net torque about A is zero.

$$\therefore w \cdot x - (T \sin \theta)l = 0$$

$$\text{or } T = \frac{w \cdot x}{l \sin \theta} \quad \text{eq(1)}$$



Let F_H and F_V be the horizontal and vertical components of the force exerted by the pin at A. Then since there is translational equilibrium we have

$$F_H = T \cos \theta$$

Using eq(1)

$$F_H = \frac{w \cdot x}{l \sin \theta} \cos \theta = \frac{wx \cot \theta}{l} \quad \therefore \cot \theta = \frac{\cos \theta}{\sin \theta}$$

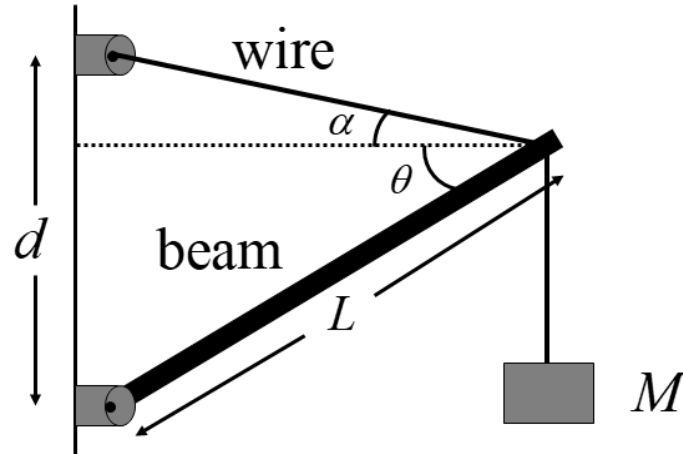
and

$$F_V = w - T \sin \theta$$

Using eq(1)

$$F_V = w - \frac{wx}{l \sin \theta} \sin \theta = w \left(1 - \frac{x}{l} \right)$$

- 7) A mass M is hanged over a pivoted beam of length L , making an angle α with the horizontal. Find the tension in the wire and the force exerted by the hinge on the beam.



Solution:

From the translational equilibrium

$$\sum F_x = F_h - T_h = 0$$

$$\sum F_y = F_v + T_v - mg - Mg = 0$$

$$T_v = T_h \tan \alpha$$

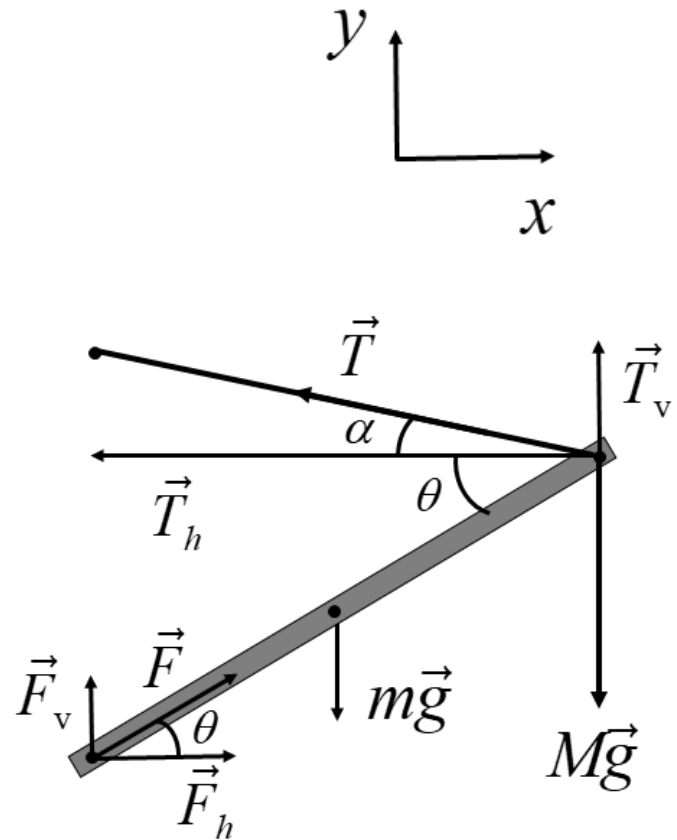
$$\tan \alpha = \frac{(d - L \sin \theta)}{L \cos \theta}$$

Applying the rotational equilibrium around the upper end of the beam

$$\sum \tau_z = F_v (L \cos \theta) + F_h (L \sin \theta) + mg \left(\frac{L}{2} \cos \theta \right) = 0 \Rightarrow F_v = F_h \tan \theta + \frac{mg}{2}$$

Solving these equations simultaneously, we get

$$F_v = \frac{(dm + L(m + 2M) \sin \theta)}{2d} g$$



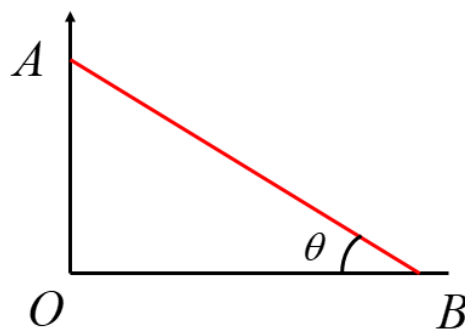
$$F_h = \frac{L(2M + m)\cos\theta}{2d}g$$

$$T_v = \frac{(m + 2M)(d - L\cos\theta)}{2d}g$$

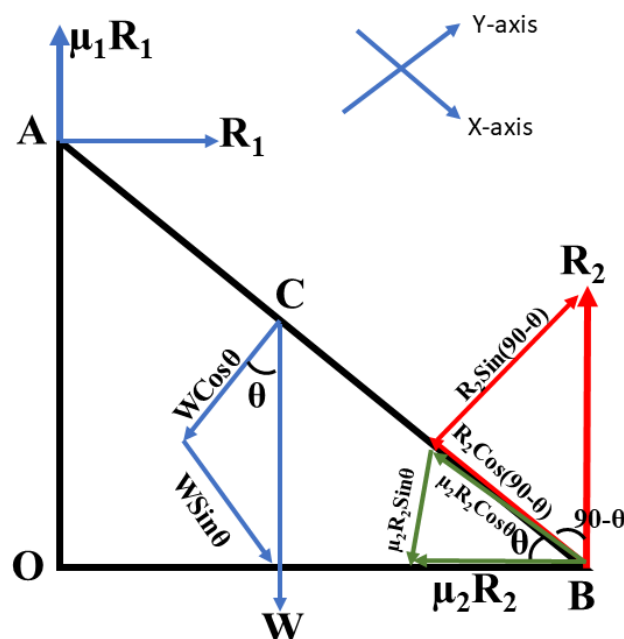
$$T_h = \frac{(m + 2M)L\cos\theta}{2d}g$$

8) Show that the least angle θ at which the rod can lean to the horizontal without slipping is

given by $\theta = \tan^{-1}\left(\frac{1 - \mu_1\mu_2}{2\mu_2}\right)$



Solution:



W = weight of the rod

R_1 and R_2 = normal force perpendicular to the surface.

According to the first condition of equilibrium,

$$\sum F_x = 0, \sum F_y = 0$$

Along x-axis:

$$R_1 - \mu_2 R_2 = 0 \Rightarrow R_1 = \mu_2 R_2 \rightarrow \text{eq(1)}$$

Along y-axis:

$$R_2 + \mu_1 R_1 - W = 0 \Rightarrow R_2 + \mu_1 R_1 = W \rightarrow \text{eq(2)}$$

substituting R_1 from eq(1) into eq(2), we get

$$R_2 + \mu_1 \mu_2 R_2 = W$$

$$R_2 (1 + \mu_1 \mu_2) = W$$

$$\Rightarrow R_2 = \frac{W}{(1 + \mu_1 \mu_2)} \quad \text{eq(3)}$$

According to the second condition of equilibrium,

$$\sum \tau = 0$$

Since A is our center of rotation which means, there is no torque contribution from R_1 and $\mu_1 R_1$ (i.e., $\tau = r \cdot F$ where, $r = 0$)

Now Considering the rotational equilibrium about A,

$$R_2 \times OB = W \times OD + \mu_2 R_2 \times OA$$

$$R_2 \times AB \cos \theta = W \times \frac{AB \cos \theta}{2} + \mu_2 R_2 \times AB \sin \theta$$

$$R_2 \cos \theta = W \frac{\cos \theta}{2} + \mu_2 R_2 \sin \theta$$

$$R_2 \cos \theta - W \frac{\cos \theta}{2} = \mu_2 R_2 \sin \theta$$

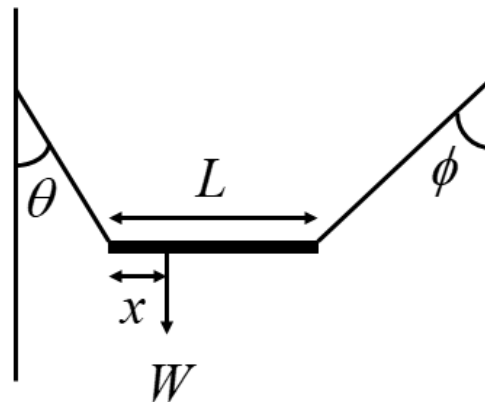
$$\text{or } \cos \theta \left(R_2 - \frac{W}{2} \right) = \mu_2 R_2 \sin \theta$$

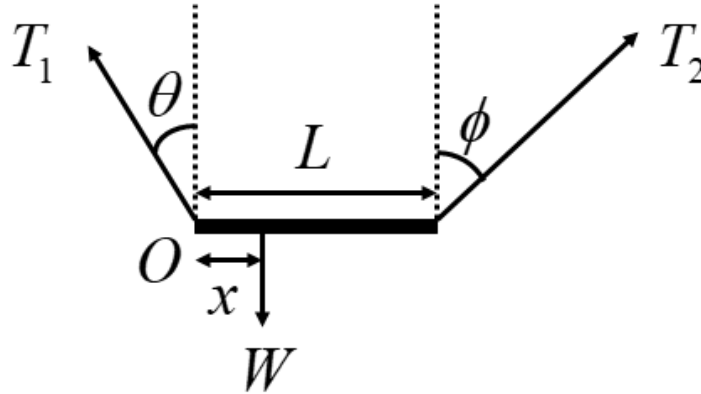
$$\Rightarrow \tan \theta = \frac{R_2 - \frac{W}{2}}{\mu_2 R_2} \quad \therefore \tan \theta = \frac{\sin \theta}{\cos \theta}$$

using the value of R_2 from eq(3), we get

$$\begin{aligned}\tan \theta &= \frac{\frac{W}{(1+\mu_1\mu_2)} - \frac{W}{2}}{\mu_2 \frac{W}{(1+\mu_1\mu_2)}} \\ \tan \theta &= \frac{\frac{2W - W(1+\mu_1\mu_2)}{2(1+\mu_1\mu_2)}}{\mu_2 \frac{W}{(1+\mu_1\mu_2)}} \\ \tan \theta &= \frac{2W - W(1+\mu_1\mu_2)}{2W\mu_2} \\ \tan \theta &= \frac{W(2-1-\mu_1\mu_2)}{2W\mu_2} \\ \Rightarrow \tan \theta &= \frac{1-\mu_1\mu_2}{2\mu_2}\end{aligned}$$

- 9) A non-uniform bar of weight W is suspended at rest in a horizontal position by two light cords. Find the distance x from the left-hand end to the center of gravity.



Solution:

Summation of horizontal components

$$T_2 \sin \phi - T_1 \sin \theta = 0$$

multiplying it with $\cos \theta$, we get

$$T_2 \sin \phi \cos \theta - T_1 \sin \theta \cos \theta = 0 \text{ eq(1)}$$

Summation of vertical components

$$T_2 \cos \phi + T_1 \cos \theta - W = 0$$

multiplying it with $\sin \theta$, we get

$$T_2 \cos \phi \sin \theta + T_1 \cos \theta \sin \theta = W \sin \theta \text{ eq(2)}$$

adding equation 1 and 2

$$T_2 \sin \phi \cos \theta - T_1 \sin \theta \cos \theta + T_2 \cos \phi \sin \theta + T_1 \cos \theta \sin \theta = W \sin \theta$$

$$T_2 \sin \phi \cos \theta + T_2 \cos \phi \sin \theta = W \sin \theta$$

$$T_2 (\sin \phi \cos \theta + \cos \phi \sin \theta) = W \sin \theta$$

$$\therefore \sin(\theta + \phi) = \sin \phi \cos \theta + \cos \phi \sin \theta$$

$$\Rightarrow T_2 = \frac{W \sin \theta}{\sin(\theta + \phi)}$$

Now for rotational equilibrium about O:

$$-Wx + (T_2 \cos \phi)L = 0$$

$$\Rightarrow x = \frac{(T_2 \cos \phi)L}{W} = \frac{W \sin \theta}{\sin(\theta + \phi)} \frac{\cos \phi L}{W}$$

$$x = \frac{L \sin \theta \cos \phi}{\sin(\theta + \phi)}$$

Equilibrium means no net force and no net torque but, in the several physics books, it is called static equilibrium when a body is stationary, for example, a ladder is attached to a wall, then it is called static equilibrium or if a box is lying on a table, it is also static. However, if a plane is flying in the air and is moving at a constant speed if it is doing so, then it is called dynamic equilibrium. However, the difference between this and that is not in the sense that if you were to move along with that ship, then it would appear static to you, hence whether something is static or dynamic depends on your frame of reference so, I have not differentiated between these two. As you have seen during this lecture how many different types of systems, we can apply the concepts of equilibrium and how much we can learn about the subject of mechanics.

Lecture#15 PHY101

Oscillations

Oscillation and vibration are fundamental physical phenomena. Oscillation is a back-and-forth movement, like stirring a pan or playing cards. It's all around us, seen in a swinging pendulum or a plucked guitar string. Even a floating object on water demonstrates oscillatory motion. This repetitive pattern continues unless acted upon by outside forces. For example, a spring with added weight will oscillate continuously. While some oscillations are visible, others need special tools like an oscilloscope to detect. Interestingly, the quartz crystal in a watch uses oscillations for accurate timekeeping.

We're exploring the idea of oscillation.

The **time period** is the amount of time it takes to reach a certain point again, often referred to as "T". It's the time it takes to complete one full cycle. The time period is measured in seconds.

Now, let's talk about **frequency** - it's closely related to the time period. Frequency is how often a cycle happens in one second. It's the inverse of the time period and is measured in hertz (Hz).

Finally, there's **amplitude** - this is the extent or size of the oscillation. Amplitude reflects how much the movement varies. A larger amplitude means a greater range of motion.

Let's understand these ideas using a **simple pendulum**. Picture a weight hanging from a string. When you move it sideways and it swings back, that's its time period - the time it takes to complete one full swing. Now, how many times does it swing back and forth in a second? That's the frequency. Imagine it takes 3 seconds to complete a full swing; that means the frequency is one-third swing per second. Finally, consider the amplitude - how far the pendulum swings. Notice how changing the amplitude changes the motion. In an ideal situation without air resistance, the pendulum would keep swinging forever. But air resistance gradually drains its energy over time, causing it to eventually stop.

Now, let's explore the concept of **equilibrium position** - the point where a body naturally stays. Any force acting on it near this position causes it to oscillate. Consider a pendulum: when displaced, gravity pulls it towards equilibrium, accelerating its motion. As it moves away, its velocity decreases. Notice, the pivotal force always directs towards equilibrium, governing the body's movement dynamics.

In the pendulum example, we observed that the force is directly proportional to the height it moves. When this force aligns horizontally, it's called a **linear restoring force**. However, if it's proportional to x^2 , it pulls in the opposite direction. Whether the force is linear depends on not over stretching the spring. A mass attached to a spring as shown in Fig 15.1 will cause the system to oscillate, as the mass experiences the force of gravity. Let's see this in action by adding a mass to the spring.

Now, observe that the system is at a balanced position when still. When released, it starts moving up and down in a repeating cycle. We call this the equilibrium point, which we'll mark as $x = 0$. As we change the mass attached to the spring, we notice the

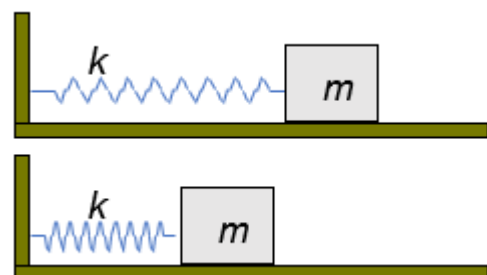


Fig 15. 1: Mass spring system

frequency of the oscillations changing. A heavier mass requires more force to stretch the spring, affecting the balance between gravity and spring force. This causes the oscillation frequency to decrease. This demonstrates how adjusting the mass can alter the oscillation frequency, an important concept to understand in oscillatory systems.

What describes oscillations?

- *period* T = time for completing one cycle
- *frequency* $f = 1 / T$
- *amplitude* A = maximum displacement from *equilibrium* position

Restoring force

- Suppose a force is always directed towards a central *equilibrium* position
- Force always acts to return the object to its equilibrium position
- The object will oscillate around the equilibrium position
- This “back-and-forth” motion around an equilibrium position is called: Periodic motion.

Periodic motion

- Simple Harmonic Motion (SHM)
- The restoring force depends on the displacement Δx

$$F_{\text{restore}} = -k \Delta x$$

Derivation of Equation of Motion for a Mass-Spring System

1. Hooke's Law

Hooke's Law describes the behaviour of an ideal spring:

$$F(x) = -kx$$

where:

- $F(x)$ is the force exerted by the spring
- k is the spring constant (a measure of the stiffness of the spring)
- x is the displacement from the equilibrium position.

Hooke's Law tells us that the force exerted by the spring is directly proportional to the displacement from its equilibrium position, with the negative sign indicating that the force is opposite to the direction of displacement.

2. Stored Energy in the Spring

When a spring is stretched or compressed, it stores potential energy. The amount of energy stored in the spring is given by:

$$U(x) = \frac{1}{2} kx^2$$

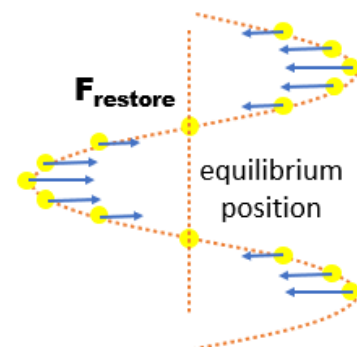


Fig 15. 2: Restoring force of a body at its equilibrium position.

where:

- $U(x)$ is the potential energy stored in the spring
- k is the spring constant, and
- x is the displacement from the equilibrium position.

This equation shows that the potential energy stored in the spring is directly proportional to the square of the displacement from its equilibrium position.

3. Equating Force and Acceleration

According to Newton's Second Law, the force acting on an object is equal to the product of its mass and acceleration:

$$F = ma$$

Substituting $F = -kx$ (from Hooke's Law), we have:

$$ma = -kx$$

This equation relates the acceleration (a) of the mass to the force (F) exerted by the spring.

4. Differential Equation of Motion

By expressing acceleration as the second derivative of displacement with respect to time, we obtain:

$$a = \frac{d^2x}{dt^2}$$

$$m \frac{d^2x}{dt^2} = -kx$$

This equation represents the differential equation of motion for the mass-spring system, where:

- m is the mass of the object,
- $\frac{d^2x}{dt^2}$ is the acceleration of the mass, and
- $-kx$ is the restoring force exerted by the spring.

5. Solving the Differential Equation

Dividing both sides by m , we get:

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

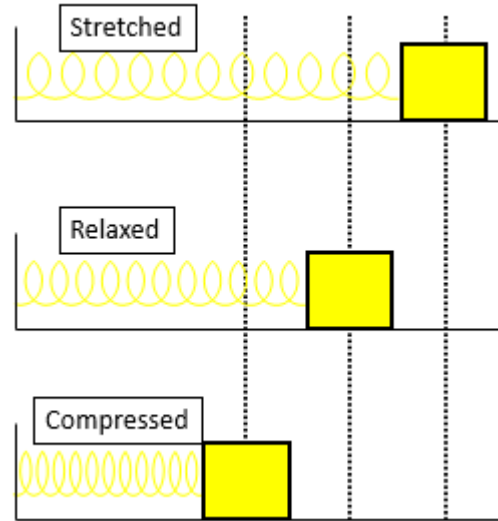


Fig 15. 3: A spring at equilibrium, neither compressed nor stretched. A block of mass m on a frictionless surface is pushed against the spring. If x is the compression in the spring, the potential energy stored in the spring is $\frac{1}{2} kx^2$. When the block is released, this energy is transferred to the block in the form of kinetic energy.

This is a second-order linear homogeneous ordinary differential equation with constant coefficients. Its solution provides the equation of motion for the mass-spring system, describing how the displacement (x) of the mass varies with time during simple harmonic motion.

Simple Harmonic Motion

1. Equation of Motion for Simple Harmonic Oscillator:

We start with the general equation of motion for a simple harmonic oscillator:

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

This equation represents the second-order linear ordinary differential equation governing the motion of the oscillator.

2. Rearrangement:

We can rearrange the equation by moving the term involving x to one side:

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

This equation shows that the acceleration of the oscillator ($\frac{d^2x}{dt^2}$) is directly proportional to its displacement (x) from the equilibrium position, with a negative sign indicating that the acceleration is opposite to the displacement.

3. Introduction of Angular Frequency:

We introduce the concept of angular frequency (ω) for the oscillator:

$$\omega^2 = \frac{k}{m}$$

The angular frequency (ω) represents the rate at which the oscillator oscillates back and forth. It is related to the stiffness of the oscillator (determined by k) and its mass (determined by m). Squaring both sides simplifies the equation.

How to calculate $\frac{d}{dt} \cos \omega t$?

$$x(t) = \cos \omega t$$

$$x(t + \Delta t) = \cos \omega(t + \Delta t)$$

Difference between $x(t + \Delta t)$ and $x(t)$

Apply Trigonometric Identity (Angle Addition Formula for Cosine):

$$\cos(A + B) - \cos(A) = -2 \sin\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right)$$

Applying this formula with $A = \omega t + \omega \Delta t$ and $B = -\omega t$

$$\cos(\omega t + \omega \Delta t) - \cos(\omega t) = -2 \sin\left(\frac{\omega t + \omega \Delta t - (-\omega t)}{2}\right) \sin\left(\frac{\omega t + \omega \Delta t - \omega t}{2}\right)$$

Simplifying the expression inside the sine functions:

$$\sin\left(\frac{\omega t + \omega \Delta t + \omega t}{2}\right) \sin\left(\frac{\omega \Delta t}{2}\right) = \sin(\omega t + \omega \Delta t / 2) \sin(\omega \Delta t / 2)$$

Approximation:

Since Δt is very small, $\omega \Delta t$ is also very small.

For small angles, $\sin(\theta) \approx \theta$

So, $\sin(\omega \Delta t / 2) \approx \omega \Delta t / 2$

Therefore, $\sin(\omega t + \omega \Delta t / 2) \sin(\omega \Delta t / 2)$ is approximated as $(\omega \Delta t / 2)^2$, and the negative sign from the trigonometric identity is retained.

Result:

Substituting the approximation into the equation:

$$x(t + \Delta t) - x(t) \approx -\sin \omega \Delta t \cdot \sin(\omega t + \omega \Delta t / 2) \approx -\omega \Delta t \cdot \sin \omega t$$

$$\therefore \frac{d}{dt} \cos \omega t = -\omega \sin \omega t$$

Remember two important results:

$$\frac{d}{dt}(\sin \omega t) = \omega \cos \omega t$$

$$\frac{d}{dt}(\cos \omega t) = -\omega \sin \omega t$$

What happens if you differentiate twice?

$$\begin{aligned} \frac{d^2}{dt^2}(\sin \omega t) &= \omega \frac{d}{dt} \cos \omega t \\ &= -\omega^2 \sin \omega t \end{aligned}$$

$$\begin{aligned} \frac{d^2}{dt^2}(\cos \omega t) &= -\omega \frac{d}{dt} \sin \omega t \\ &= -\omega^2 \cos \omega t \end{aligned}$$

Any function of the form:

$$x = a \cos \omega t + b \sin \omega t$$

$$\text{is a solution of } \frac{d^2 x}{dt^2} = -\omega^2 x$$

Physical significance of constant ω

$$x = x_m \cos \omega \left(t + \frac{2\pi}{\omega} \right)$$

$$x = x_m \cos(\omega t + 2\pi)$$

$$x = x_m \cos \omega t$$

Why is $\sin \theta \approx \theta$ for small θ ?

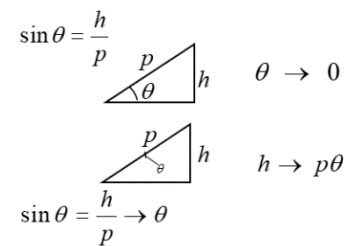


Fig 15. 4: For small angles, $\sin \theta$ is approximately equal to θ due to the linear behaviour near the origin."

That is, the function merely repeats itself after a time $2\pi / \omega$

So $2\pi / \omega$ is the period of the motion T

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{m}{k}}$$

The frequency ν of the oscillator is the number of complete vibrations per unit time:

$$\nu = \frac{1}{T} = \frac{1}{2\pi} \sqrt{\frac{k}{m}}$$

$$\omega = 2\pi\nu = \frac{2\pi}{T} = \sqrt{\frac{k}{m}}$$

ω is called the angular frequency

$$\dim[\omega] = T^{-1}$$

Unit of ω is radian/second

$$x(t) = a \cos \omega t + b \sin \omega t$$

$$x(0) = a$$

$$\begin{aligned} \frac{d}{dt} x(t) &= -\omega a \sin \omega t + \omega b \cos \omega t \\ &= \omega b \quad (\text{at } t=0) \end{aligned}$$

This solution can also be written as:

$$x(t) = x_m \cos(\omega t + \phi)$$

Physical significance of constant x_m

$$x = x_m \cos(\omega t + \phi)$$

$$\Rightarrow -x_m \leq x \leq +x_m$$

x_m is called the amplitude of the motion

The frequency of the simple harmonic motion is independent of the amplitude of the motion

Phase

We're exploring the concept of "phi" (a Greek letter) in sinusoidal waves. Phi represents a phase shift, which changes where the wave begins on the vertical axis. When we assign different constant values to phi, we see distinct changes in the waveform. For example, setting phi to

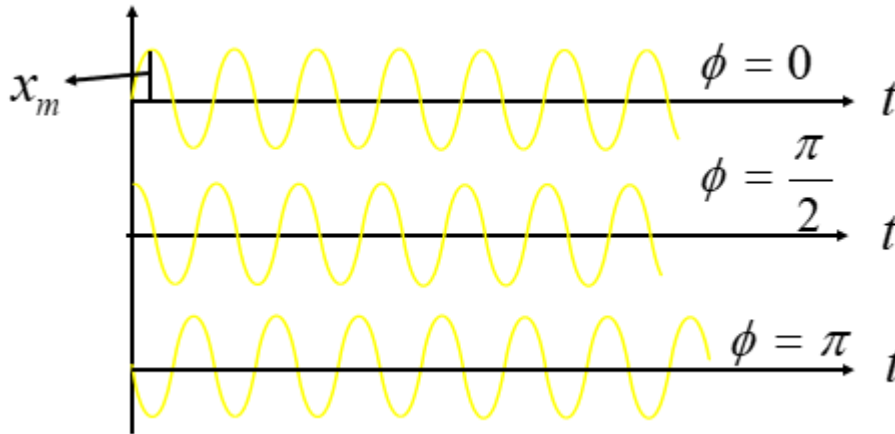


Fig 15. 5: Impact of Phase Shift (Phi) on Sinusoidal Waves: Different constant values of phi shift waveforms along the vertical axis."

zero gives us a standard sine wave with a frequency denoted by ω as shown in Fig.15.5. Increasing ϕ shifts the wave along the axis, moving its starting point from a peak to a trough. We can visually see this shift through plots. Additionally, differentiating the wave reveals its acceleration, which is linked to ω and the displacement amplitude. By exploring ϕ , we grasp its significant role in shaping sinusoidal waves and its importance in wave analysis.

$$x = x_m \sin(\omega t + \phi)$$

The quantity $\theta = \omega t + \phi$ is called the phase of the motion. The constant ϕ is called the phase constant.

Energy of simple harmonic motion

1. Expressing Displacement (x):

In simple harmonic motion, the object oscillates back and forth around a central point as shown in Fig 15.6. We express this oscillation as $x = x_m \cos(\omega t)$

Where:

x is the displacement from the central point at time t .

x_m is the amplitude, representing the maximum displacement from the central point.

ω is the angular frequency, determining the rate of oscillation.

2. Calculating Potential Energy (U):

The potential energy of the oscillator, U , arises from the restoring force exerted by the spring or restoring element. We use Hooke's Law to calculate it:

$$U = \frac{1}{2} kx^2$$

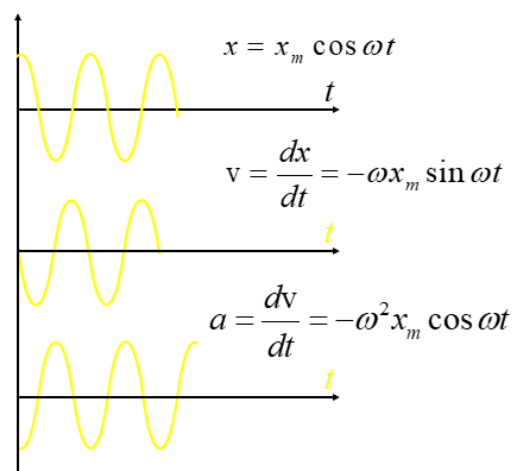


Fig 15. 6: Representation of Displacement in Simple Harmonic Motion: Explaining the components of displacement (x) in the oscillatory motion.

Substituting $x = x_m \cos(\omega t)$, we find $U = \frac{1}{2} kx_m^2 \cos^2(\omega t)$

This equation shows how the potential energy of the oscillator varies with its displacement from the central point.

3. Calculating Kinetic Energy (K):

The kinetic energy of the oscillator, K , arises from its motion. We calculate it using the formula:

$$K = \frac{1}{2} mv^2$$

Since velocity $v = \frac{dx}{dt}$, and $x = x_m \cos(\omega t)$, we find $v = -x_m \omega \sin(\omega t)$

Substituting this into the equation for kinetic energy, we find

$$K = \frac{1}{2} m\omega^2 x_m^2 \sin^2(\omega t)$$

This equation illustrates how the kinetic energy of the oscillator varies with its velocity and displacement.

4. Calculating Total Mechanical Energy (E):

The total mechanical energy (E) of the oscillator is the sum of its potential and kinetic energies:

$$E = K + U$$

Substituting the expressions for K and U , we find

$$E = \frac{1}{2} kx_m^2 \cos^2(\omega t) + \frac{1}{2} kx_m^2 \sin^2(\omega t)$$

Since $\cos^2(\omega t) + \sin^2(\omega t) = 1$, we find $E = \frac{1}{2} kx_m^2$

This equation shows that the total mechanical energy of the oscillator remains constant throughout its motion, illustrating the principle of energy conservation as shown in Fig 15.7.

$$E = \frac{1}{2} mv^2 + \frac{1}{2} kx^2 = \frac{1}{2} kx_m^2$$

$$v = \frac{dx}{dt} = \pm \sqrt{\frac{k}{m} (x_m^2 - x^2)}$$

speed is maximum at $x=0$

speed is zero at $x = \pm x_m$

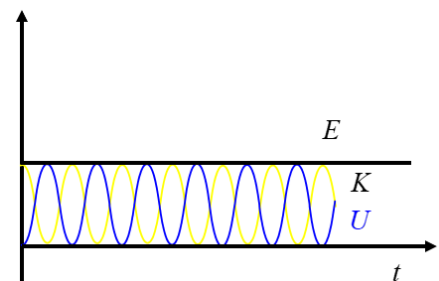


Fig 15. 7: Total mechanical energy of the oscillator remains constant.

Problem: 15.1: Two spring systems, illustrated in Fig 15.8, require determination of their respective time period.

Solution:

$$T = t_1 + t_2 + t_3$$

t_1 = period of oscillation of spring k_1

t_2 = period of oscillation of spring k_2

t_3 = time to cover the distance d

$$T = 2\pi\sqrt{\frac{m}{k_1}} + 2\pi\sqrt{\frac{m}{k_2}} + \frac{2d}{v}$$

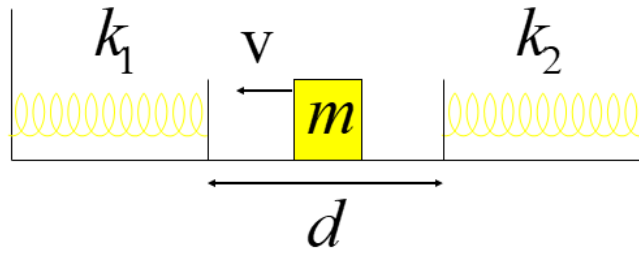


Fig 15. 8: Illustration of Two Spring System.

Springs Coupled in Series

Consider a system where two springs, each with spring constants k_1 and k_2 , are connected end to end, such that they act in series.

1. Total Displacement (y)

When a force (F) is applied to the system, it causes a total displacement (y) in the system, which can be expressed as the sum of the displacements (y_1 and y_2) caused by each spring as shown in Fig 15.9:

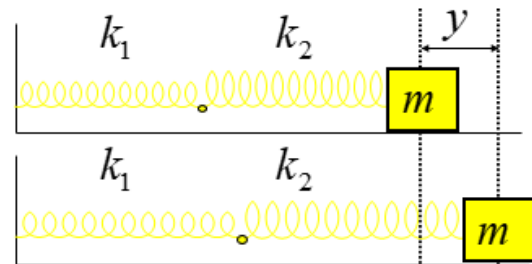


Fig 15. 9: Illustration of Springs Coupled in Series.

$$y = y_1 + y_2$$

2. Hooke's Law for Each Spring

According to Hooke's law, the force exerted by each spring (F) is proportional to its displacement (y_1 and y_2). Therefore:

$$F = -k_1 y_1 = -k_2 y_2$$

where k_1 and k_2 are the spring constants of the respective springs.

3. Displacement of Each Spring

Solving for the displacements y_1 and y_2 in terms of the force F :

$$y_1 = -\frac{F}{k_1}, y_2 = -\frac{F}{k_2}$$

4. Total Displacement and Effective Spring Constant

Substituting the expressions for y_1 and y_2 into the total displacement equation, we get:

$$y = -F \left(\frac{1}{k_1} + \frac{1}{k_2} \right) = -F \left(\frac{k_1 + k_2}{k_1 k_2} \right)$$

This expression shows the total displacement y in terms of the force F and the inverse of the sum of the reciprocals of the spring constants k_1 and k_2 .

5. Effective Spring Constant (k_{eff})

The effective spring constant (k_{eff}) of the coupled springs can be obtained by equating the force F to $-k_{eff} y$, which yields:

$$F = - \left(\frac{k_1 k_2}{k_1 + k_2} \right) y, \quad k_{eff} = \left(\frac{k_1 k_2}{k_1 + k_2} \right)$$

This equation provides a way to calculate the effective spring constant of the series-coupled springs based on their individual spring constants.

6. Period of Oscillation (T)

Finally, the period of oscillation (T) of the system can be determined using the effective spring constant (k_{eff}) and the mass (m) connected to the springs:

$$\frac{1}{k_{eff}} = \frac{1}{k_1} + \frac{1}{k_2}, \quad T = 2\pi \sqrt{\frac{m}{k_{eff}}}$$

This equation relates the period of oscillation to the effective spring constant and the mass of the system, providing insight into the dynamics of the coupled springs in series.

Springs in parallel

1. Introduction to Displacement (x):

We begin by defining x as the displacement of the mass from its equilibrium position.

2. Force Exerted by Spring 1 (F_1)

Spring 1 exerts a force (F_1) on the mass, given by Hooke's Law:

$$F_1 = -k_1 x$$

Here,

k_1 represents the spring constant of spring 1.

3. Force Exerted by Spring 2 (F_2)

Similarly, spring 2 exerts a force (F_2) on the mass, also given by Hooke's Law:

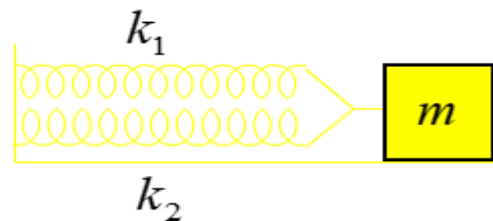


Fig 15. 10: Illustration of Springs in parallel.

$$F_2 = -k_2x$$

k_2 is the spring constant of spring 2.

4. Total Force (F):

The total force (F) exerted on the mass is the sum of the forces exerted by both springs:

$$F = F_1 + F_2 = -(k_1 + k_2)x$$

This equation shows that the total force is the sum of the forces from each spring acting in the same direction.

5. Effective Spring Constant (k_{eff})

We define an effective spring constant (k_{eff}), representing the combined stiffness of the two springs:

$$k_{eff} = k_1 + k_2$$

This effective spring constant accounts for the total restoring force acting on the mass due to both springs.

6. Angular Frequency (ω):

The angular frequency (ω) of the system is given by:

$$\omega = \sqrt{\frac{k_1 + k_2}{m}} = \sqrt{\frac{k_{eff}}{m}}$$

It represents the rate at which the mass oscillates back and forth under the influence of the combined springs.

Torsional Oscillator

Consider a system where a wire or a shaft is twisted, exerting a restoring torque (τ) proportional to the angle of twist (θ) as shown in Fig 15. The constant of proportionality (κ), known as the torsional constant, depends on the properties of the wire or shaft.

1. Expression for Torque (τ)

The torque (τ) acting on the system can be expressed as:

$$\tau = -\kappa\theta$$

Here, θ represents the angular displacement from the equilibrium position.

2. Newton's Second Law for Rotation

Applying Newton's second law for rotation, we equate the torque (τ) to the moment of inertia (I) times the angular acceleration (α):

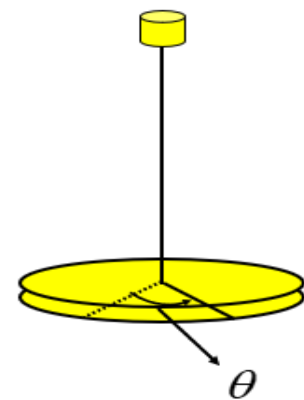


Fig 15. 11: Illustration of a Torsional System with Expression for Torque.

$$\tau = I\alpha = I \frac{d^2\theta}{dt^2}$$

This equation relates the torque to the angular acceleration, where I is the moment of inertia of the system.

3. Equating Torque and Angular Acceleration

Equating the expressions for torque and angular acceleration, we get:

$$-\kappa\theta = I \frac{d^2\theta}{dt^2}$$

4. Differential Equation for Angular Displacement

Rearranging terms, we obtain the differential equation for angular displacement:

$$\frac{d^2\theta}{dt^2} = -\left(\frac{\kappa}{I}\right)\theta$$

This equation governs the motion of the torsional oscillator, relating the angular displacement (θ) to its second derivative with respect to time.

5. Solution for Angular Displacement

Assuming harmonic motion, we propose a solution of the form:

$$\theta = \theta_m \cos \omega t$$

Here,

θ_m represents the maximum angular displacement, and ω is the angular frequency of oscillation.

6. Angular Frequency (ω)

Substituting the proposed solution into the differential equation, we find the angular frequency (ω) of the oscillation:

$$\omega = \sqrt{\frac{\kappa}{I}}$$

This equation gives the rate at which the angular displacement oscillates back and forth, determined by the torsional constant (κ) and the moment of inertia (I) of the system.

Physics-PHY101-Lecture 16

OSCILLATIONS: II

Oscillations are a fundamental concept in physics that describes the repetitive motion of a system around a central equilibrium point. In this chapter, we will explore various types of oscillatory motion and their properties. We begin with the simple pendulum, which is a classic example of a harmonic oscillator. Next, we will discuss the physical pendulum, which is a more complex system than the simple pendulum. We will also discuss the composition of two harmonic oscillators with the same period, which leads to interesting phenomena like beats and resonance. Additionally, we will look into damped oscillators, which describe systems where the amplitude of oscillations decreases over time due to damping forces. We will analyze the behavior of damped oscillators and derive their equations of motion. Finally, we will examine forced and free oscillations. **Forced oscillations** occur when an external force is applied to a system, while **free oscillations** occur without any external force. By the end of this chapter, you will have a deep understanding of the fundamental concepts of oscillatory motion and be able to analyze and describe a wide range of oscillatory systems.

Simple Pendulum:

A simple pendulum is an ideal body consisting of a body of mass m and suspended by a light in-extensible string. The pendulum oscillates periodically under the action of gravity when it is pulled to one side of its equilibrium point and then released.

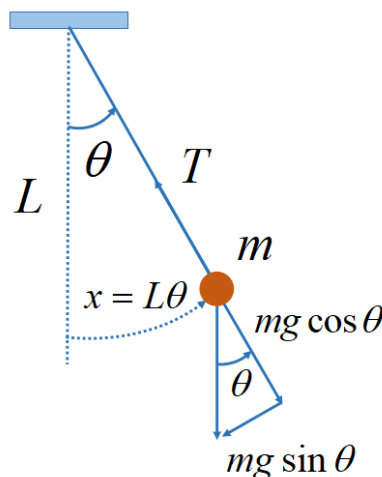


Figure 16. 1: The simple pendulum. A pendulum of mass m with a length L and making an angle θ with the vertical. We choose the x axis to be in the tangential direction and the y axis to be in the radial direction at this particular time.

The forces acting on mass are the weight mg and the tension T in the string. The weight mg is resolved into a radial component of magnitude $mg \cos \theta$ and a

tangential component of magnitude $mg \sin \theta$. The centripetal acceleration required to keep the particle travelling in a circular path is provided by the radial components of the forces. The restoring force on m that pushes it towards equilibrium is represented by the tangential component. The restoring force is thus

$$F = -mg \sin \theta \quad (1)$$

The negative sign shows that F is opposite to the direction of increasing $L = x / \theta$.

It should be noted that the restoring force is proportional to $\sin \theta$ rather θ . As a result, the motion is not simply harmonic. However, for small angles, the sine function can be approximated using the small-angle approximation:

$$\sin \theta \approx \theta$$

Thus

$$x = L\theta \rightarrow \theta = \frac{x}{L}$$

Putting the value in eq. (1) we get

$$F = -mg \frac{x}{L}$$

where

$$a = -g \frac{x}{L} \quad (2)$$

$$F = -\left(\frac{mg}{L}\right)x$$

By Newton's second law, $F = ma$ where $a = \frac{d^2x}{dt^2}$. Hence

$$F = m \frac{d^2x}{dt^2}$$

$$\text{Also, } F = -\left(\frac{mg}{L}\right)x$$

On comparing, we get,

$$\frac{d^2x}{dt^2} = -\left(\frac{g}{L}\right)x$$

$$\frac{d^2x}{dt^2} + \left(\frac{g}{L}\right)x = 0$$

The general solution to this differential equation is of the form:

$$x(t) = A \cos(\omega t) + B \sin(\omega t)$$

Where A and B are constants, and ω is the angular frequency, related to g and L by the equation $\omega = \sqrt{L/g}$. For the specific, where the pendulum is released from its

maximum displacement (x_m), the initial conditions are $x(0) = x_m$ and $\frac{dx}{dt}(0) = 0$. Solving for A and B based on these conditions yields:

$$x(0) = A\cos(0) + B\sin(0) \rightarrow x_m = A$$

So, $A = x_m$ and $B = 0$. Therefore, the solution for the given initial condition is:

$$x = x_m \cos \omega t$$

In SHM acceleration is given by

$$a = -\omega^2 x \quad (3)$$

After comparing eq. 2 and 3, we will get the expression of angular frequency defined as,

$$\omega^2 = \frac{g}{L}$$

$$\omega = \sqrt{\frac{g}{L}}$$

The period of the oscillation of physical pendulum is given by,

$$T = \frac{2\pi}{\omega}.$$

Substituting ω into the expression, we get

$$T = 2\pi \sqrt{\frac{m}{mg/L}} = 2\pi \sqrt{\frac{L}{g}}$$

Physical Pendulum

A physical pendulum is a rigid body suspended from a pivot point that oscillates back and forth in a vertical plane under the influence of gravity. Unlike a simple pendulum, which consist of a mass suspended from weightless string, a physical pendulum has a non-negligible mass and size. The motion of a physical pendulum can be described by using the principle of rotational motion and the conservation of energy.

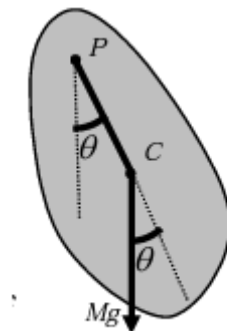


Figure 16. 2: Physical pendulum consisting of a rigid body of mass M.

Let us consider a physical pendulum consisting of a rigid body of mass M and length L , suspended from a pivot point P .

The pivot point P is fixed, and the pendulum is free to oscillate about it. Let the pendulum make a small angular displacement θ from its equilibrium position. The torque acting on the pendulum on the pendulum due to gravity is given by,

$$\tau = -Mgd \sin \theta \quad (4)$$

Where g is the acceleration due to gravity and L is the distance between the pivot point P and the center of mass of the pendulum. The negative sign indicates that the torque acts in the opposite direction to the displacement.

By Newton's second law for rotational motion, the torque is defined as,

$$\tau = I \alpha \quad (5)$$

where I is the moment of inertia of the pendulum and α is the angular acceleration, $\alpha = \frac{d^2\theta}{dt^2}$.

Using the small angle approximation $\sin \theta \approx \theta$, and the definition of angular acceleration, we can write the above equation as,

$$I \frac{d^2\theta}{dt^2} = -Mgd \sin \theta,$$

Dividing both sides by I and rearranging, we get

$$\frac{d^2\theta}{dt^2} + \left(\frac{Mgd}{I}\right) \theta = 0. \quad (6)$$

The solution to this differential equation is a sinusoidal function of the form,

$$\theta(t) = A \cos(\omega t + \varphi),$$

where A is the amplitude of oscillation, ω is the angular frequency, and φ is the phase angle. To find ω , take the second derivative of $\theta(t)$.

$$\theta''(t) = -A\omega^2 \cos(\omega t + \varphi)$$

Substituting this into the differential equation, we get

$$-A\omega^2 \cos(\omega t + \varphi) + \left(\frac{Mgd}{I}\right) A \cos(\omega t + \varphi) = 0$$

Dividing both sides by $A \cos(\omega t + \varphi)$ and after simplification we get

$$\omega^2 = \left(\frac{Mgd}{I}\right)$$

After substituting this into the expression for the period, we get

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{I}{Mgd}}$$

Simple Harmonic Motion and Uniform Circular Motion

Simple harmonic motion (SHM) and uniform circular motion (UCM) are both types of periodic motion, but they differ in their nature and characteristics. Although SHM and UCM have different characteristics, they are related in some ways.

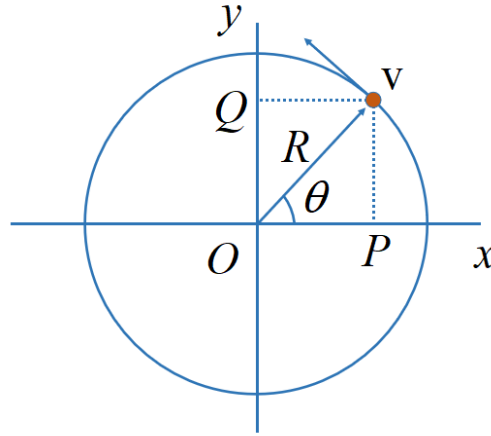


Figure 16. 3: Particle P executing uniform circular motion.

For example, the projection of UCM onto any diameter of the circle is SHM, with the amplitude of the SHM equal to the radius of the circle and the period of the SHM equal to the period of the UCM. Similarly, the motion of a particle in SHM can be considered as the projection of UCM onto a straight line. Furthermore, the equation for SHM can be derived from the equation for UCM by considering the projection of UCM onto a diameter of the circle.

Figure shows a particle P in uniform circular motion where $\mathbf{v}$ is the linear velocity and R is radius of circle.

At a time t , the vector R which locates point P relative to the origin O , makes an angle θ with the x axis and the x component of vector R is,

$$x = R \cos \theta$$

Differentiating above equation two times w.r.t time we will get

$$\begin{aligned} \frac{dx}{dt} &= R(\sin \theta) \frac{d\theta}{dt} \\ \frac{d^2x}{dt^2} &= -R \cos \theta \frac{d^2\theta}{dt^2} \\ \frac{d^2x}{dt^2} &= -x \frac{d^2\theta}{dt^2} \end{aligned}$$

As we knew that the left side of above equation is the definition of acceleration and right side is the definition of angular displacement so above equation simplified as,

$$a = -\omega^2 x$$

Which is the expression of the body executing simple harmonic motion. Therefore, we can conclude that uniform circular motion can be considered a form of simple harmonic motion. When referring to "circular motion," the acceleration connected to this concept is referred to as centripetal acceleration, which is defined as:

$$\vec{a} = -\frac{v^2}{R}\hat{r} = -R\omega^2\hat{r}$$

where $v = R\omega$. The acceleration along x direction is

$$a_x = -R\omega^2 \cos \theta$$

By substituting $R\cos\theta = x$ (i. e. $\cos \theta = \frac{x}{R}$) in above equation we will get

$$a_x = -\omega^2 x$$

which shows that point P executes simple harmonic motion.

Similarly, the y component of acceleration is

$$a_y = -R\omega^2 \sin \theta = -\omega^2 y \quad \therefore y = R\sin \theta.$$

Composition of Two Simple Harmonic Oscillator along Same Line

When two simple harmonic motions with the same period are added together, the resulting motion is also a simple harmonic motion with the same period. The amplitude and phase of the resulting motion depend on the amplitudes and phases of the two individual motions.

Let's consider two simple harmonic motions with the same period T,

$$x_1(t) = A_1 \sin \omega t,$$

$$x_2(t) = A_2 \sin(\omega t + \varphi),$$

where A_1 and A_2 are the amplitudes and φ is any fixed angle and ω is the angular frequency. The sum of two motions is,

$$x(t) = x_1(t) + x_2(t),$$

$$x(t) = A_1 \sin \omega t + A_2 \sin(\omega t + \varphi).$$

To simplify this expression, we can use the identity

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

Using this identity we get,

$$x(t) = A_1 \sin \omega t + A_2 \sin \omega t \cos \varphi + A_2 \cos \omega t \sin \varphi,$$

$$x(t) = \sin \omega t (A_1 + A_2 \cos \varphi) + \cos \omega t (A_2 \sin \varphi).$$

Let $A_1 + A_2 \cos \varphi = R \cos \theta$ and $A_2 \sin \varphi = R \sin \theta$. After using some simple trigonometry we can write x in the form, $x = R \sin(\omega t + \theta)$. Thus, the resultant

motion is also simple harmonic motion along the same line and has the same time period. Its **amplitude is R** which is defined as:

$$\begin{aligned}
 R &= \sqrt{(R_x^2 + R_y^2)} \\
 R &= \sqrt{(A_1 + A_2 \cos \phi)^2 + (A_2 \sin \phi)^2} \\
 R &= \sqrt{(A_1^2 + A_2^2 \cos^2 \phi + 2A_1A_2 \cos \phi + A_2^2 \sin^2 \phi)} \\
 R &= \sqrt{(A_1^2 + A_2^2(\cos^2 \phi + \sin^2 \phi) + 2A_1A_2 \cos \phi)} \\
 R &= \sqrt{(A_1^2 + A_2^2(1) + 2A_1A_2 \cos \phi)} \\
 R &= \sqrt{(A_1^2 + A_2^2 + 2A_1A_2 \cos \phi)}
 \end{aligned}$$

And its direction is,

$$\tan \theta = \frac{A_2 \sin \phi}{A_1 + A_2 \cos \phi}.$$

Special Cases

1. Constructive Interference

If $\phi = 0$, the waves are in phase, resulting in a rise in the total amplitude to a certain degree from the combination of wave amplitudes.

$$R = \sqrt{A_1^2 + A_2^2 + 2A_1A_2} = \sqrt{(A_1 + A_2)^2} = (A_1 + A_2),$$

And $\tan \theta = 0$,

which means that if displacements of two harmonic oscillators is equal then the resultant displacement is sum of two individual displacements.

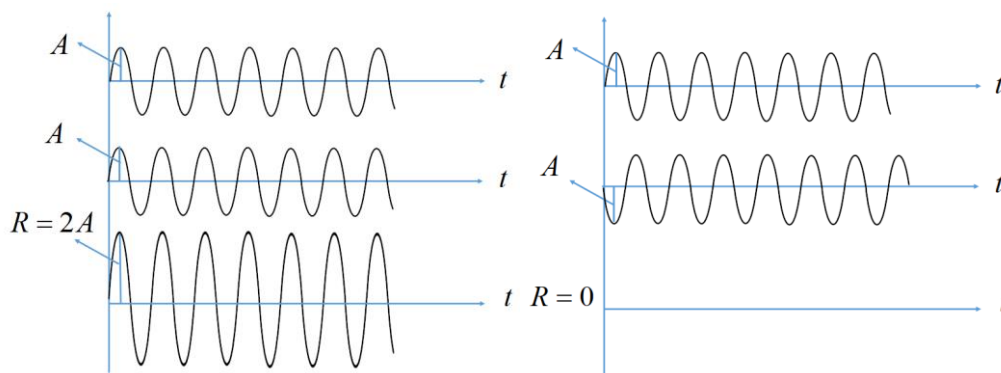


Figure 16. 4: Figure illustrates both constructive and destructive interference: the left side shows waves synchronizing in phase, resulting in reinforcement (constructive), while the right side displays waves being out of phase, causing them to nullify each other (destructive).

2. Destructive Interference

If $\phi = \pi$, the waves are out of phase, their combined amplitudes result in a reduction of the overall amplitude to a certain extent resultant is given by

$$R = \sqrt{A_1^2 + A_2^2 - 2A_1A_2} = \sqrt{(A_1 - A_2)^2} = (A_1 - A_2),$$

and $\tan\theta = 0$,

and this is the case of destructive interference. The resultant will be zero, if the amplitude of both waves are equal.

Composition of Two Simple Harmonic Oscillator at Right angles to each other

Until this point, we have only discussed the one-dimensional simple harmonic oscillator, which implies motion along a single direction. However, a particle can also move in two or three dimensions. Therefore, let's now turn our attention to the two-dimensional simple harmonic oscillator.

Let's consider two simple harmonic motions with different amplitudes i.e.

$$x = A\sin\omega t \Rightarrow \frac{x}{A} = \sin\omega t \quad (7)$$

and

$$y = B\sin(\omega t + \phi) \Rightarrow \frac{y}{B} = \sin(\omega t + \phi) \quad (8)$$

Using eq. 7 in the trigonometric relation $\cos\theta = \sqrt{1 - \sin^2\theta}$, we can write

$$\cos\omega t = \sqrt{1 - \frac{x^2}{A^2}}. \quad (9)$$

Similarly, by using another trigonometric identity $\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ in eq. 8 we can write

$$\frac{y}{B} = \sin\omega t\cos\phi + \cos\omega t\sin\phi \quad (10)$$

Substituting the values of $\cos\omega t$ and $\sin\omega t$ from eq. 9 and from eq. 7 into eq. 10 we get

$$\frac{y}{B} = \frac{x}{A}\cos\phi + \sqrt{1 - \frac{x^2}{A^2}}\sin\phi$$

$$\frac{y}{B} - \frac{x}{A}\cos\phi = \sqrt{1 - \frac{x^2}{A^2}}\sin\phi$$

Squaring both sides

$$\therefore (a - b)^2 = a^2 + b^2 - 2ab$$

$$\left(\frac{y}{B} - \frac{x}{A}\cos\phi\right)^2 = \left(\sqrt{1 - \frac{x^2}{A^2}}\sin\phi\right)^2$$

$$\frac{y^2}{B^2} + \frac{x^2}{A^2} \cos^2 \phi - 2 \frac{y}{B} \frac{x}{A} \cos \phi = \left(1 - \frac{x^2}{A^2}\right) \sin^2 \phi$$

$$\therefore \cos^2 \phi = 1 - \sin^2 \phi$$

$$\frac{y^2}{B^2} + \frac{x^2}{A^2} (1 - \sin^2 \phi) - 2 \frac{y}{B} \frac{x}{A} \cos \phi = \left(1 - \frac{x^2}{A^2}\right) \sin^2 \phi$$

$$\frac{y^2}{B^2} + \frac{x^2}{A^2} - \frac{x^2}{A^2} \sin^2 \phi - 2 \frac{y}{B} \frac{x}{A} \cos \phi = \left(1 - \frac{x^2}{A^2}\right) \sin^2 \phi$$

$$\frac{y^2}{B^2} + \frac{x^2}{A^2} - 2 \frac{y}{B} \frac{x}{A} \cos \phi = \left(1 - \frac{x^2}{A^2}\right) \sin^2 \phi + \frac{x^2}{A^2} \sin^2 \phi$$

$$\frac{y^2}{B^2} + \frac{x^2}{A^2} - 2 \frac{y}{B} \frac{x}{A} \cos \phi = \left(1 - \frac{x^2}{A^2} + \frac{x^2}{A^2}\right) \sin^2 \phi$$

Thus, combined displacement of both harmonic motions is given by the relation

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} - 2 \frac{x}{A} \frac{y}{B} \cos \phi = \sin^2 \phi \quad (11)$$

Special Cases

Case I

If $\phi = 0$, eq (11) becomes

$$\therefore \sin 0^\circ = 0, \cos 0^\circ = 1$$

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} - 2 \frac{x}{A} \frac{y}{B} = 0$$

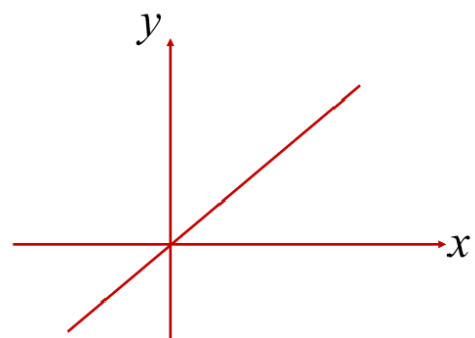
It is a special condition in which the ellipse degenerates (i.e. width of one axis becomes zero).

$$\left(\frac{x}{A} - \frac{y}{B}\right)^2 = 0$$

$$\frac{x}{A} - \frac{y}{B} = 0$$

Rearranging the above equation

$$y = \left(\frac{B}{A}\right) x$$



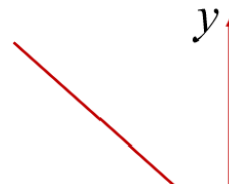
This is the equation of a straight line ($y = mx + c$)

with slope $c = \frac{B}{A}$. Thus, the resultant motion is a S. H. M. along a straight line passing through the origin.

Case II

If $\phi = \pi$, then eq. 11 becomes

$$\therefore \sin 180^\circ = 0, \cos 180^\circ = -1$$



$$\frac{x^2}{A^2} + \frac{y^2}{B^2} + 2\frac{xy}{AB} = 0$$

$$\left(\frac{x}{A} + \frac{y}{B}\right)^2 = 0$$

$$\frac{x}{A} + \frac{y}{B} = 0$$

$$y = -\left(\frac{B}{A}\right)x$$

This is also the equation of straight line with negative slope $c = -\frac{B}{A}$.

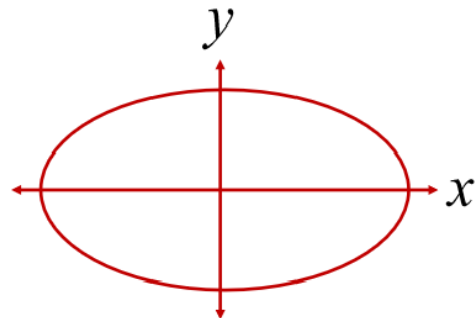
Case III

If $\phi = \frac{\pi}{2}$ (for $A > B$), we get

$$\therefore \sin 90^\circ = 1, \cos 90^\circ = 0$$

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1,$$

which is the equation of an ellipse.



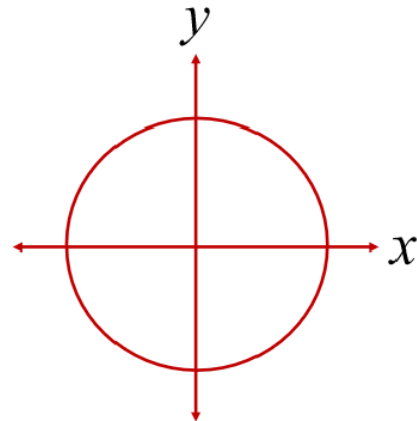
Case IV

If $\phi = \frac{\pi}{2}$ (for $A = B$), we get

$$\therefore \sin 90^\circ = 1, \cos 90^\circ = 0$$

$$x^2 + y^2 = A^2$$

which is a circle with radius A.



Lissajous Figures

If two oscillations of different frequencies at right angles are combined, the resulting motion is more complicated. It is not even periodic unless the two frequencies are in the ratio of integers. This resulting curve is called Lissajous figures.

$$\frac{\omega_x}{\omega_y} = \text{integers} \Rightarrow \text{periodic motion}$$

Such a curve can be represented by the following pair of equations.

$$x = A \sin \omega_x t$$

$$y = B \sin(\omega_y t + \phi)$$

Damped Harmonic Motion

Damped harmonic motion refers to the motion of a system that is subjected to both a restoring force and a damping force. The restoring force tends to return the system to its equilibrium position, while the damping force tends to reduce the amplitude of the oscillation over time. This reduction of amplitude is due to the damping force. There are many cases of damping force including friction, air resistance, and internal forces.

Let us consider a simple model of damped oscillator in which the block slides on a friction-less surface.

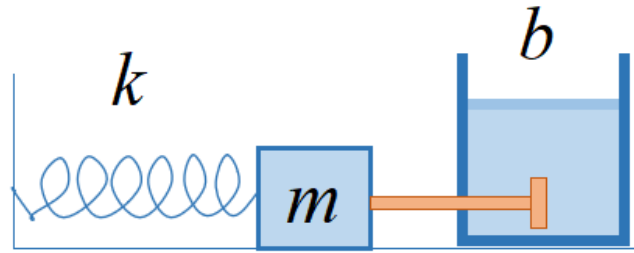


Figure 16. 5: A representation of damped harmonic oscillator.

The damping force is represented in terms of a (mass-less) vane that moves in a viscous fluid. This damping force due to fluid is $-bv_x$, where b is a positive constant called the damping constant that depends on the properties of the fluid and the size and shape of the object. With $\sum F_x = -kx - bv_x$, Newtons second law gives,

$$-kx - bv_x = ma_x$$

with $v_x = \frac{dx}{dt}$ and $a_x = \frac{d^2x}{dt^2}$ the above equation becomes,

$$m \frac{d^2x}{dt^2} + b \frac{dx}{dt} + kx = 0. \quad (12)$$

The solution of above equation is,

$$x(t) = x_m e^{\frac{-bt}{2m}} \cos(\omega' t + \varphi) \quad (13)$$

Use the above equation to calculate $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$. Then by putting the values of x , $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$ in eq. 12 to calculate the value of ω as

$$\omega' = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}.$$

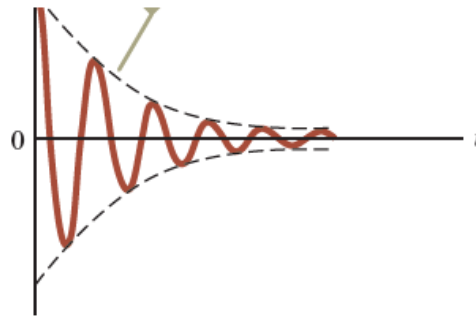


Figure 16. 6: Graph of displacement versus time showing decrease in amplitude with time for a damped oscillator.

The above solution is same as we obtain in the case of simple and physical pendulum except the extra term $e^{\frac{-bt}{2m}}$ which is due to damping force. This solution assumes that the damping constant is small so that the quantity under the square root in eq. 16.40 cannot be negative.

Figure shows the position as a function of time for an object oscillating in the presence of a retarding force. When the retarding force is small, the oscillatory character of the motion is preserved but the amplitude decreases exponentially in time, with the result that the motion ultimately becomes undetectable. Any system that behaves in this way is known as a **damped oscillator**. The dashed black lines in Figure, which define the *envelope* of the oscillatory curve, represent the exponential factor in Equation 13. This envelope shows that the amplitude decays exponentially with time.

When the magnitude of the retarding force is small such that $\frac{b}{2m} < \frac{k}{m}$, the system is said to be under-damped.

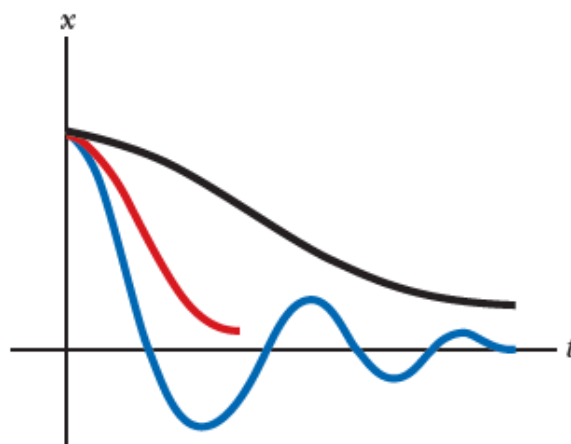


Figure 16. 7: Displacement versus time graphs of an underdamped oscillator (blue line), a critically damped oscillator (red), and an overdamped oscillator (black).

As the value of b increases, the amplitude of the oscillations decreases more and more rapidly. When b reaches a critical value b_c such that $b_c/2m = \frac{k}{m}$, the system does not oscillate and is said to be **critically damped**. In this case, the system, once released from rest at some non-equilibrium position, approaches but does not pass through the equilibrium position. The graph of position versus time for this case is the red curve in Figure.

If the medium is so viscous that the retarding force is large compared with the restoring force—that is, if $\frac{b}{2m} > \frac{k}{m}$ the system is over-damped. Again, the displaced system, when free to move, does not oscillate but rather simply returns to its equilibrium position. As the damping increases, the time interval required for the system to approach equilibrium also increases as indicated by the black curve in Figure.

Forced Oscillations and Resonance

An oscillating system undergoes **forced oscillations** whenever it is pushed periodically by an external force. The frequency at which a system would oscillate if neither a driving nor a damping force are present is known as the **natural frequency**.

Resonance is the phenomenon that occurs when the driving frequency ω is equal to the natural frequency ω_o and the forced oscillations reach their maximum displacement amplitude, when the damping is negligible.

$$\omega = \omega_o$$

Where

$$\omega_o = \sqrt{\frac{k}{m}}$$

The corresponding frequency is called the **resonant angular frequency**.

We take the driving force to be $F_o \cos \omega t$. Newtons second law gives,

$$\begin{aligned} \sum F_x &= -kx + F_o \cos \omega t \\ -kx + F_o \cos \omega t &= ma_x \end{aligned}$$

with $v_x = \frac{dx}{dt}$ and $a_x = \frac{d^2x}{dt^2}$ the above equation becomes,

$$m \frac{d^2x}{dt^2} + kx = F_o \quad (14)$$

The solution of above equation is,

$$x(t) = \frac{F_o}{m(\omega_o^2 - \omega^2)} \cos \omega' t \quad (15)$$

Use eq. 15 to calculate $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$. Then by putting the values of x and $\frac{d^2x}{dt^2}$ in eq. 14 we can verify the solution of eq. 14 i.e.

$$\frac{dx(t)}{dt} = \frac{(-\omega)F_o}{m(\omega_o^2 - \omega^2)} \sin \omega t \quad (16)$$

$$\frac{d^2x(t)}{dt^2} = -\frac{(\omega^2)F_o}{m(\omega_o^2 - \omega^2)} \cos \omega t \quad (17)$$

After substituting eq. 16 and eq. 17 in eq. 14 we get

$$m \left(\frac{(-\omega^2)F_o}{m(\omega_o^2 - \omega^2)} \cos \omega t \right) + k \frac{F_o}{m(\omega_o^2 - \omega^2)} \cos \omega t = F_o \cos \omega t$$

$$\frac{(-\omega^2)mF_o + kF_o}{m(\omega_o^2 - \omega^2)} \cos \omega t = F_o \cos \omega t \quad (18)$$

Note that the amplitude becomes infinitely large when ω approaches ω_o . This occurs due to the absence of a damping term in the equation. However, when damping is present, the amplitude remains large as ω approaches ω_o but remains finite. All mechanical structures such as buildings, bridges, and airplanes have one or more natural frequencies of oscillation. If the structure is subject to a driving frequency that matches one of the natural frequencies, the resulting large amplitude of oscillation can have disastrous consequences. The collapse of roadways and bridges in earthquakes is a more serious outcome.

Physics-PHY101-Lecture 17

PHYSICS OF MATERIALS

Solids, liquids, gases, and plasma are the four basic states of matter. In a **solid**, particles are packed tightly together with strong intermolecular forces, which give them a definite shape and prevent their flow. The particles in a **liquid** are more loosely packed than they are in a solid and can flow around one another, giving the liquid an arbitrary shape. As a result, the liquid will take the shape of its container. The particles in a **gas** have large spaces between them and possess high kinetic energy. A gas lacks a fixed volume or shape. Highly charged particles with a very high kinetic energy make up **plasma**. Although it is not a common state of matter on Earth, it could be the most typical state of matter across the universe. In short, stars like the sun are hot balls of plasma.

Despite the fact that a solid's shape and volume might be believed to be fixed, these characteristics can be altered by external forces. The term **deformation** is used for the shape change caused by the application of force.

Elasticity

The property by virtue of which a body tends to regain its original shape and size when external forces are removed. If a body completely recovers its original shape and size, it is called perfectly elastic. Quartz, steel, and glass are very nearly elastic.

Plasticity

If a body has no tendency to regain its original shape and size, it is called perfectly plastic. Common plastics, kneaded dough, solid honey, etc. are plastics.

Stress

Stress characterizes the strength of the forces causing the stretch, squeeze, or twist. It is defined usually as force per unit area but may have different definitions to suit different situations.

$$stress = \frac{F}{A}$$

The SI unit of stress is Nm^{-1} .

Types of Stresses

We distinguish between three types of stresses:

a) **Longitudinal stress**

If the deforming force is applied along some linear dimension of a body, the stress is called ***longitudinal stress or tensile stress or compressive stress***.

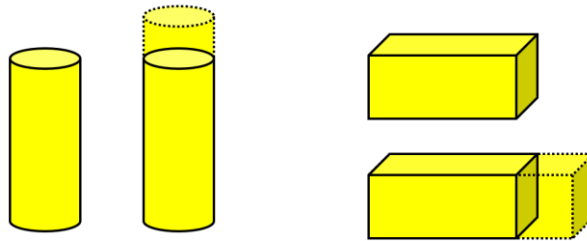


Figure 17. 1: Longitudinal stress causing deformation along the length of a material.

b) **Volume stress**

If the force acts normally and uniformly from all sides of a body, the stress is called ***volume stress***.

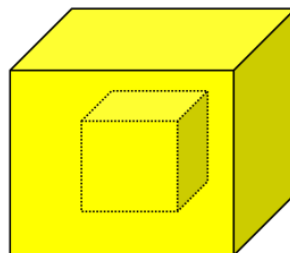


Figure 17. 2: Figure depicting volume stress, highlighting the uniform compression or expansion of a material in response to external forces acting in all directions within its volume.

c) **Shearing stress**

If the force is applied tangentially to one face of a rectangular body, keeping the other face fixed, the stress is called ***tangential or shearing stress***.

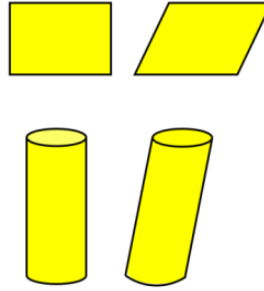


Figure 17. 3: Parallel force component that acts tangentially to a material's surface, causing deformation or slippage between adjacent layers illustrating shear stress.

Strain:

When deforming forces are applied on a body, it undergoes a change in shape or size. The fractional (or relative) change in shape or size is called the strain.

$$\text{Strain} = \frac{\text{change in dimension}}{\text{original dimension}}$$

Strain is a ratio of similar quantities, so it has no units.

Types of Strain

There are 3 different kinds of strain:

- i. **Longitudinal (linear) strain** is the ratio of the change in length (Δl) to original length (l), i.e.,

$$\text{Linear strain} = \frac{\Delta l}{l}$$

- ii. **Volume strain** is the ratio of the change in volume (ΔV) to original volume (V)

$$\text{Volume strain} = \frac{\Delta V}{V}$$

- iii. **Shearing strain** the angular deformation (θ) in radians is called shearing stress.

For a smaller angle θ ,

$$\text{Shearing strain} \equiv \theta \approx \tan \theta = \frac{\Delta x}{l}$$

The above equation shows that the sheared face moves horizontally across a distance of Δx , while the object's height is given by l .

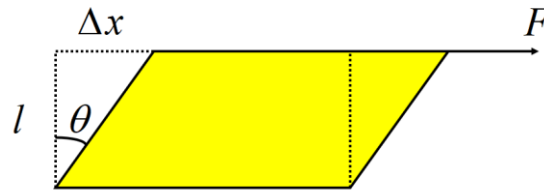


Figure 17. 4: Angular distortion or deformation experienced by a material in response to shear stress, typically represented by the change in angle between initially perpendicular lines within the material.

Hooke's Law

For small deformations, stress is proportional to strain.

$$\text{Stress} \propto \text{Strain}$$

$$\text{Stress} = E \times \text{Strain}$$

$$E = \frac{\text{stress}}{\text{strain}}$$

The constant E is called the modulus of elasticity. It can be interpreted as a measure of a material's stiffness: A material with a high elastic modulus is highly rigid and challenging to deform. Thus, the material being deformed, and the type of deformation affect the constant of proportionality. E has the same units as stress because strain is dimensionless.

There are three moduli of elasticity.

(a) Young's modulus

It is the measure of a solid's resistance to change in length. Consider a rod of original length ' l ' and cross-sectional area ' A '. Under the influence of external force F , the internal forces in the rod that resist deformation are balanced by the applied force, resulting in a change in length of the rod by the factor Δl .

Young's modulus (Y) for linear strain is given by the following relation.

$$Y = \frac{\text{longitudinal stress}}{\text{longitudinal strain}} = \frac{F/A}{\Delta l/l}$$

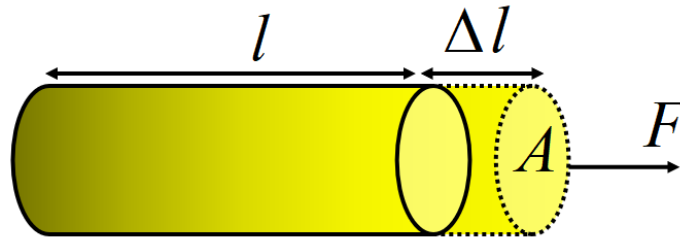
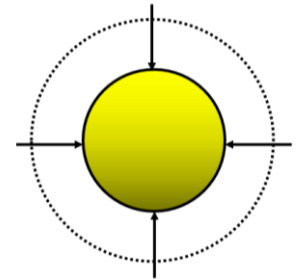


Figure 17. 5: Elastic deformation under tensile or compressive force F .

Bulk Modulus

It determines how resistant solids or liquids are to volume changes. Let a body of volume V be subjected to a uniform pressure on its entire surface, causing a change in volume ΔV and pressure ΔP . Then, bulk modulus (B) for volume strain is given by

$$B = \frac{\text{Volume Stress}}{\text{Volume Strain}} = \frac{\Delta F/A}{\Delta V/V} = -\frac{\Delta P}{\Delta V/V}$$



In the above equation, a negative sign is added to make 'B' a positive number. The need for this action arises from the fact that an increase in pressure (positive ΔP) results in a decrease in volume (negative ΔV), and vice versa. $1/B$ is called **compressibility**. A material having a small value of B can be compressed easily.

(c) Shear Modulus

It evaluates how resistant a solid's sliding planes are to motion. Let a force F produce a strain as in the figure 17.4. It is also known as modulus of rigidity. Then, shear modulus (η) for shearing strain is

$$\eta \equiv \frac{\text{Shearing stress}}{\text{Shearing strain}} = \frac{F/A}{\theta} = \frac{F}{A \tan \theta} = \frac{Fl}{A \Delta x}$$

Poisson's Ratio (σ)

When a wire is stretched, its length increases and radius decreases. The ratio of the lateral strain (the ratio of the change in diameter of a circular bar caused by longitudinal deformation to the original diameter of the material) to the longitudinal strain is called Poisson's ratio,

$$\sigma = \frac{\Delta r/r}{\Delta l/l}$$

Its value lies between 0 and 0.5.

Work done in Stretching a Wire

We can calculate the work done by stretching a wire. Obviously, we must do work against the internal restoring force (see fig 17.5). If x is the extension (i. e $\Delta l = x$) produced by the force F in a wire of length l , then

$$Y = \frac{\text{longitudinal stress}}{\text{longitudinal strain}} = \frac{F/A}{x/l}$$

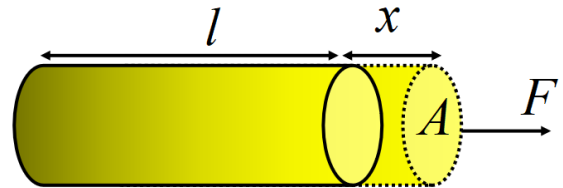
$$F = \frac{YA}{l}x \quad (1)$$

The work done in extending the wire through Δl is given by,

$$W = \int_0^{\Delta l} F dx$$

Putting the value of F from eq 1

$$W = \frac{YA}{l} \int_0^{\Delta l} x dx = \frac{YA}{l} \left[\frac{x^2}{2} \right]_0^{\Delta l}$$



Applying limits

$$W = \frac{YA}{l} \frac{(\Delta l)^2}{2}$$

Rearrange the above equation after multiplying and dividing by ' l '

$$W = \frac{1}{2} (Al) \left(\frac{Y\Delta l}{l} \right) \frac{\Delta l}{l} = \frac{1}{2} \times \text{volume} \times \text{stress} \times \text{strain}$$

Now, we can calculate the amount of energy stored in a unit volume of wire by dividing both sides by that volume.

$$\text{Work per unit volume} = \frac{1}{2} \times \text{stress} \times \text{strain}$$

We can also write this as,

$$W = \frac{1}{2} \left(\frac{YA\Delta l}{l} \right) \Delta l = \frac{1}{2} \times \text{load} \times \text{extension}$$

Behavior of wire under Stress

Figure illustrates how a sufficiently high stress might cause a material to deviate from its elastic limit. The stress-strain curve begins as a straight line in the elastic region till it reaches the **proportional limit** (maximum stress endured by the material without losing straight line proportionality between stress and strain). However, as the stress rises, the curve becomes curved.

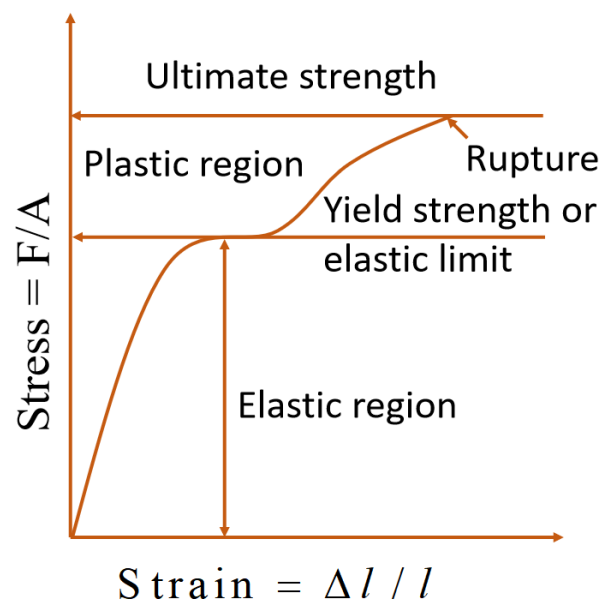


Figure 17. 6: Graph illustrating the stress-strain curve for a wire, representing the relationship between strain, and resulting stress, depicting the wire's behavior under increasing loads until it reaches its elastic limit or point of failure.

The highest stress that may be applied to a material before it undergoes permanent deformation is known as the **elastic limit** of that material. The object is permanently deformed and does not recover to its original shape when the stress has been removed when the stress surpasses the elastic limit. As a result, the object's shape changes permanently. **Ultimate strength** (the maximum stress that a substance can withstand before breaking) is exceeded as the stress is increased more.

Brittle materials have a **breaking point** that lies just above their maximum strength. After reaching their maximum strength, ductile metals like copper and gold begin to thin and stretch at lower stress levels before breaking.

Fluid Statics / Fluids at rest

A fluid is a substance that can flow and does not have a shape of its own. Thus all liquids and gases are fluids. Solids possess all the three moduli of elasticity whereas a fluid possess only the bulk modulus (B). A fluid at rest cannot sustain a tangential force. If such force is applied to a fluid, the different layers simply slide over one another. Therefore the forces acting on a fluid at rest have to be normal to the surface. This implies that the free surface of a liquid at rest, under gravity, in a container, is horizontal.

The normal force per unit area is called **pressure**

$$P = \frac{\Delta F}{\Delta A}$$

Pressure is a scalar quantity. Its **unit** is Newtons/metre², or Pascal (Pa).

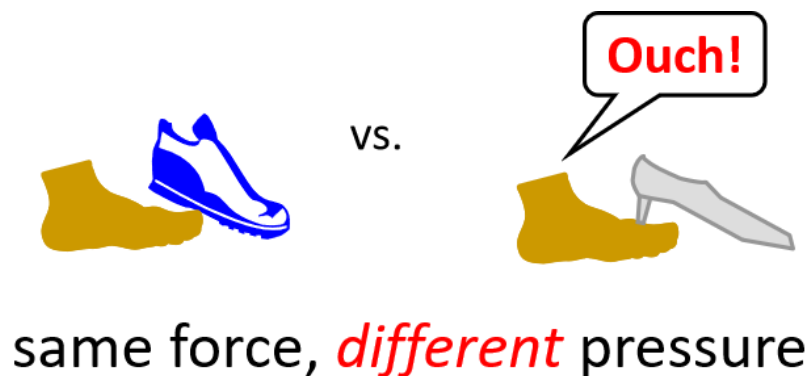


Figure 17. 7: Variation in pressure distribution across areas of contact due to differences in size or shape.

The **density** of a substance is the ratio of its mass to that of its volume.

$$\rho = \frac{\Delta m}{\Delta V}$$

Where Δm is the mass of a small piece of the material and ΔV is the volume it occupies. Density has no directional property and is a scalar quantity. Units for density are kgm^{-3} . If the density is uniform then

$$\rho = \frac{m}{V}$$

The density of water is 1000 kg/m^3 .

Variation of pressure in fluid at rest

To calculate how the pressure in a fluid changes with depth, Let's take a fluid of density ρ at rest that is exposed to the environment. Consider a small element dy in the form of a disc of fluid volume submerged within the body of the fluid. The pressure on the bottom face of the cylinder caused by the external liquid is p , while the pressure on the top face of the cylinder is caused by air pressure $p+dp$. Therefore, the upward force applied to the bottom of the cylinder by the external fluid is

$$F = pA$$

and the downward force applied to the top by the atmosphere.

$$F = (p + dp)A$$

The mass of liquid in disc is

$$dm = \rho dV = \rho A dy$$

and its weight is

$$(dm)g = \rho A dy g$$

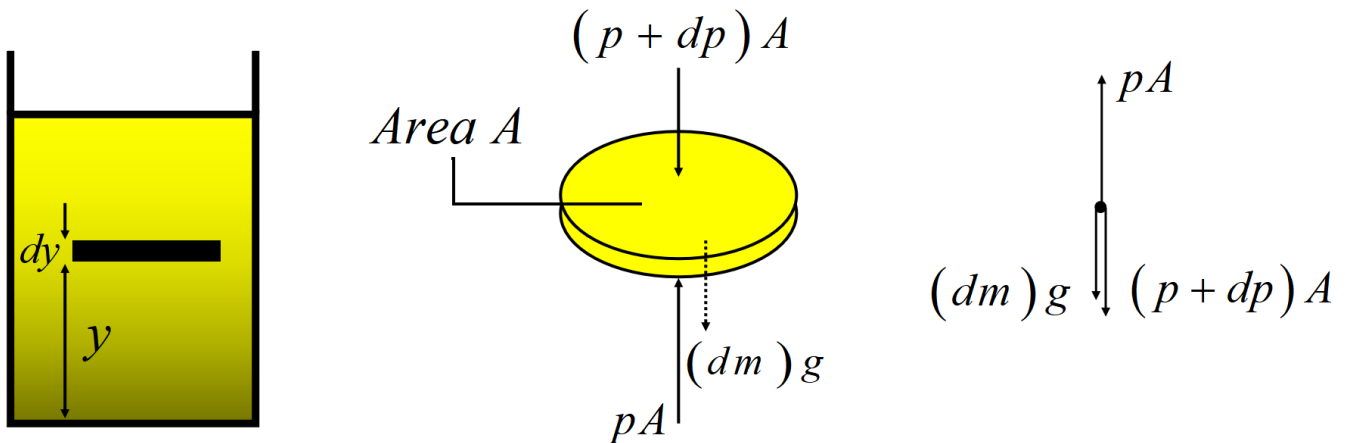


Figure 17. 8: Figure illustrating the variation of pressure on a small fluid volume segment at rest, the forces acting upon the segment and a free body diagram of the segment, highlighting how pressure changes with depth.

The resultant horizontal force is zero, because the element has no horizontal acceleration. The horizontal forces are due only to the pressure of the fluid, and by symmetry, the pressure must be the same at all points within a horizontal plane at y .

The fluid element is in equilibrium, so the resultant vertical force on it must be zero:

$$\sum F_y = 0$$

Now, let us require that the sum of the forces on the fluid element is zero

$$pA - (p + dp)A - \rho g A dy = 0$$

$$pA - pA - dpA - \rho g A dy = 0$$

$$-dpA - \rho g A dy = 0$$

$$-dpA = \rho g A dy$$

$$\frac{dp}{dy} = -\rho g$$

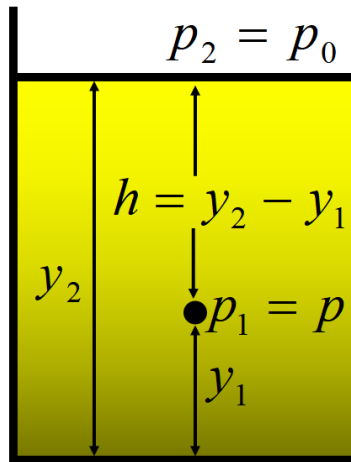


Figure 17. 9: A volume of a liquid with an exposed top surface is contained in a container. The depth h determines the pressure in the liquid at any given time.

This is the equation of hydro-static equilibrium. Note that we are taking the origin (0) at the bottom of the liquid. Therefore, as the elevation increases (dy positive), the pressure decreases (dy negative). The quantity ρg is the *weight per unit volume* of the fluid. For liquids, which are nearly in-compressible, ρ is practically constant.

$$\therefore \rho g = \text{constant}$$

$$\frac{dp}{dy} = \frac{\Delta p}{\Delta y} = \frac{p_2 - p_1}{y_2 - y_1} = -\rho g$$

$$p_2 - p_1 = -\rho g(y_2 - y_1)$$

This equation is for homogeneous liquid (constant density).

If liquid has a free surface, then this becomes the natural level from where we can measure the distances. Therefore

$$p_2 - p_1 = -\rho g(y_2 - y_1)$$

The above equation becomes

$$p_0 - p = -\rho g(y_2 - y_1)$$

But $y_2 - y_1 = h$, therefore

$$p = p_0 + \rho gh$$

This shows at the bottom of container is greater than at surface. Also the pressure is independent of the container's shape. The depth below the surface is the only factor affecting the pressure.

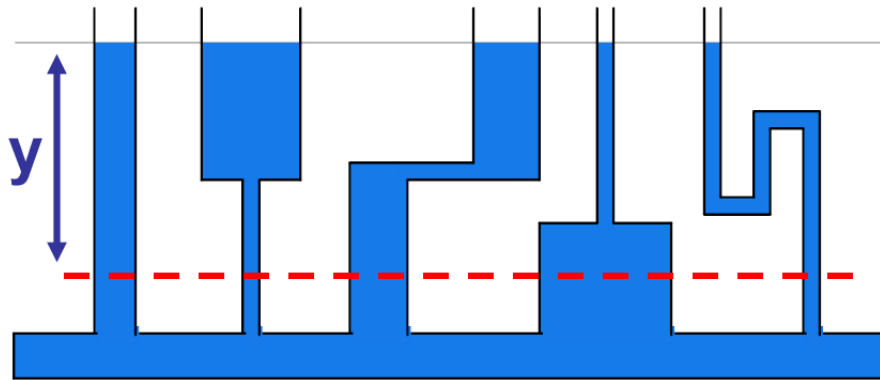


Figure 17. 10: The fluid level is the same everywhere in a connected container. As pressure only depends upon y . As the level of liquid is the same, the pressure will also be same throughout the container.

Example

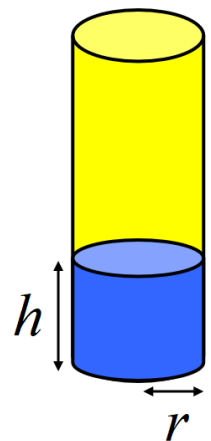
To what height should a cylindrical vessel be filled with a homogeneous liquid to make the force with which the liquid presses on the sides of the wall equal to the force exerted by the liquid on the bottom of the vessel?

Pressure at the bottom of the vessel = ρgh

Area of the bottom face = πr^2

By the definition of pressure,

$$\text{pressure} = \frac{F}{A}$$



$$\text{Force exerted by the liquid on the bottom} = (hg\rho)\pi r^2$$

As the pressure varies from top to bottom, we take an average value of pressure on the walls of the vessel.

$$\text{Mean pressure on the wall} = \frac{1}{2}(hg\rho)$$

$$\text{Area of the wall in contact with the liquid} = 2\pi rh$$

$$\text{Force exerted by the liquid on the wall} = \left(\frac{1}{2}hg\rho\right) 2\pi rh$$

Since the two forces are equal, hence

$$\left(\frac{1}{2}hg\rho\right) 2\pi rh = (hg\rho)\pi r^2$$

$$\rightarrow h = r$$

Thus the liquid should be filled up to the height equal to the radius of the cylinder.

Pascal's Principal

Pressure applied to an enclosed fluid is transmitted to every portion of the fluid and to the walls of the containing vessel.

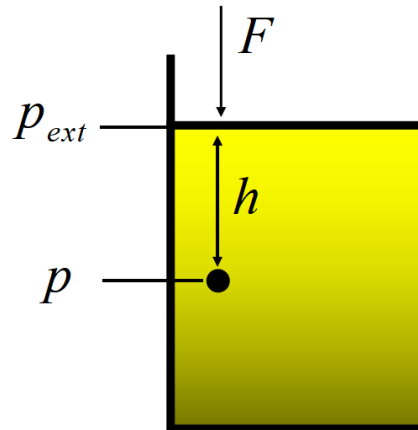


Figure 17. 11: Fluid contained in a cylinder with a moving piston. Every point P experiences pressure because of the force applied by the piston in addition to the weight of the fluid above P.

In other words, if you apply an amount Δp of external pressure to a fluid at one spot, the fluid will experience the same amount of rise in pressure throughout. For an incompressible liquid, we will demonstrate Pascal's principle. Figure depicts the fluid

within a cylinder with a piston. For example, the weight of various objects put on top of the piston can exert an external force on it. An external pressure ' p_{ext} ' is exerted on the liquid directly beneath the piston as a result of the external force. The pressure at any location 'P' at a depth 'h' below the surface can be written as follows if the liquid has a density ' ρ ':

$$p = p_{ext} + \rho gh$$

Assume that the external pressure has now been increased by an amount ' p_{ext} ', by giving the piston greater weight. The fluid's pressure changes as a result of the change in external pressure.

$$\Delta p = \Delta p_{ext} + \Delta(\rho gh)$$

Since the liquid is incompressible, the density is constant i.e. $\Delta(\rho gh) = 0$. So,

$$\Delta p = \Delta p_{ext}$$

The change in pressure at any point in the fluid is simply equal to the change in the externally applied pressure. Although we derived the above result for incompressible liquids, Pascal's principle is true for all real (compressible) fluids, gases as well as liquids.

The Hydraulic Lever

An arrangement that is frequently used to lift a heavy object, such as an automobile, is shown in Figure 17.11. It works upon Pascal's principle. A piston with area A_i is subjected to an external force F_i . The larger piston's area A_o experiences a force Mg from the object that needs to be lifted. The larger piston must experience an upward force F_o equal to the weight of the object's weight Mg in order for the system to be in equilibrium. We are trying to determine how the "output force" F_o that the system is capable of applying to the larger piston relates to the applied force F_i .

The pressure on the Liquid at the smaller piston due to our externally applied force is

$$p_i = \frac{F_i}{A_i}$$

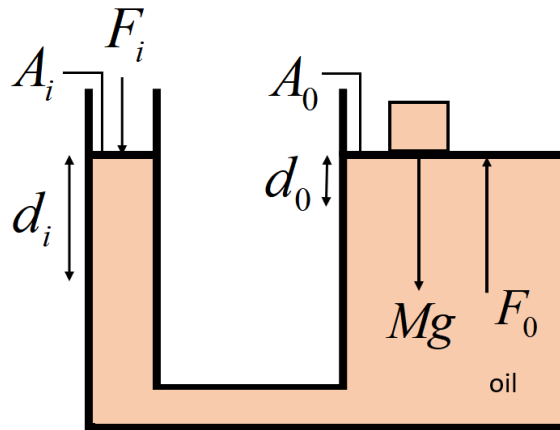


Figure 17. 12: The hydraulics lever. A force exerted on the smaller piston can result in a considerably greater force acting on the bigger piston, lifting a mass of Mg .

Pascal's principle states that this "input" pressure must equal the "output" pressure that the fluid applies to the larger piston.

$$p_0 = \frac{F_0}{A_0}$$

Therefore,

$$p_i = p_0$$

$$\frac{F_i}{A_i} = \frac{F_0}{A_0}$$

$$F_i = F_0 \left(\frac{A_i}{A_0} \right) = Mg \left(\frac{A_i}{A_0} \right) \dots \dots \dots (1)$$

$$\because A_i \ll A_0 \Rightarrow \frac{A_i}{A_0} \ll 1$$

$$\because F_i \ll F_0 = Mg$$

The downward movement of the smaller piston through the distance ' d_i ' displaces a volume of fluid (volume = area *height).

$$V = d_i A_i$$

If the fluid is incompressible, then this volume must be equal to the volume displaced by the upward motion of the larger piston

$$V = d_i A_i = d_0 A_0$$

$$d_o = d_i \left(\frac{A_i}{A_o} \right) \dots \dots \dots (2)$$

Now

$$\therefore \frac{A_i}{A_o} \ll 1 \Rightarrow d_o \ll d_i$$

Losing the ability to move anything far is the price we pay for being able to lift a heavy object. Using eq 1 and 2, we can get

$$F_i = F_o \left(\frac{A_i}{A_o} \right) = F_o \left(\frac{d_o}{d_i} \right)$$

$$F_i d_i = F_o d_o$$

Therefore, work done by the external force on the smaller piston equals the work done by the fluid on the larger fluid.

Hydraulic Brakes

Pascal's law is also used by braking systems in cars, buses, and other vehicles. Equal pressure can be transmitted throughout the liquid due to the hydraulic brakes shown in figure. The master cylinder experiences force when the brake pedal is pressed, increasing the liquid pressure inside.

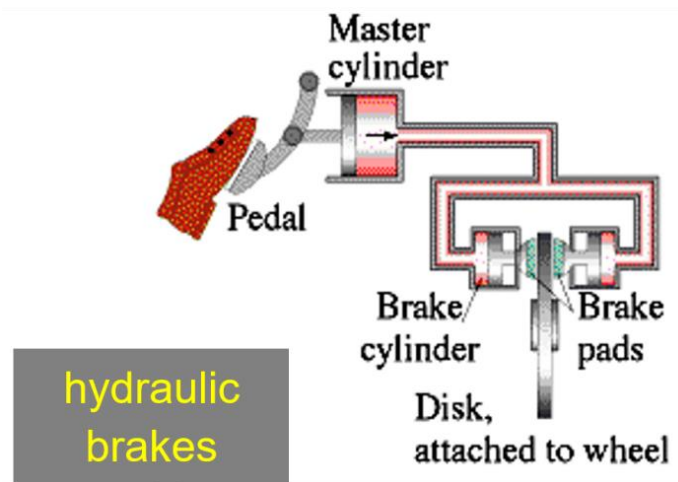


Figure 17. 13: A Schematics of hydraulic breaks

All of the pistons in other cylinders receive the same amount of liquid pressure via the liquid in the metal pipes. The pistons in the cylinders move outward, forcing the brake

pads against the braking drums as a result of the rise in liquid pressure. The wheels are stopped by the force of friction between the brake pads and brake drums.

Measuring Pressure

Gauge pressure

The pressure relative to atmospheric pressure is known as gauge pressure. Gauge pressure is positive for pressures above atmospheric pressure and negative for pressures below it.

$$\text{Gauge pressure} = \text{actual pressure} - \text{atmospheric pressure}$$

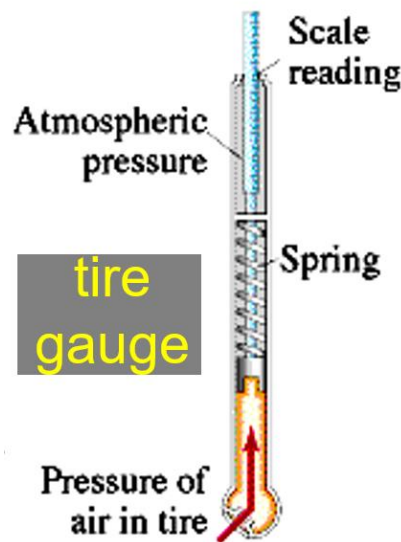


Figure 17. 14: Schematic illustrating gauge pressure, highlighting the pressure measurement relative to atmospheric pressure.

If the pressure inside a car tire is equal to the atmospheric pressure, the tire is flat. The inside pressure has to be greater than atmospheric pressure to support the car.

Manometer

The open tube manometer measures gauge pressure. The U-shaped tube often contains mercury or water. One end of the tube is exposed to the atmosphere, and the other is attached to the system whose pressure is to be determined.

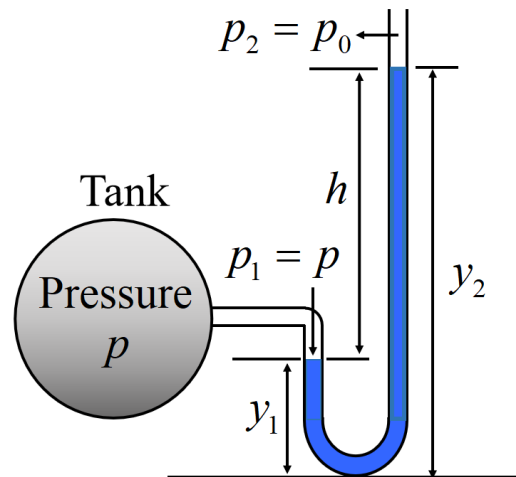


Figure 17. 15: A manometer, device used to measure fluid pressure differences, showing the column height variation between the two arms, indicating the pressure differential between the connected systems or points."

The pressure at the bottom of the tube due to the fluid in the left column is $p_0 + \rho g y_2$. And the pressure at the bottom of the tube due to the fluid in the right column is $p + \rho g y_1$. These pressures are measured at the same point so they must be equal

$$p + \rho g y_1 = p_0 + \rho g y_2$$

$$p - p_0 = +\rho g(y_2 - y_1) = \rho g h$$

Thus the gauge pressure $p - p_0$ is proportional to the difference in the height of the liquid column in the U-tube.

Mercury barometer

Evangelista Torricelli (1608–1647) created the barometer, a device used to measure pressure. The mercury barometer is made of a long glass tube that has been filled with mercury and placed upside down into a dish of mercury, as shown in Fig.

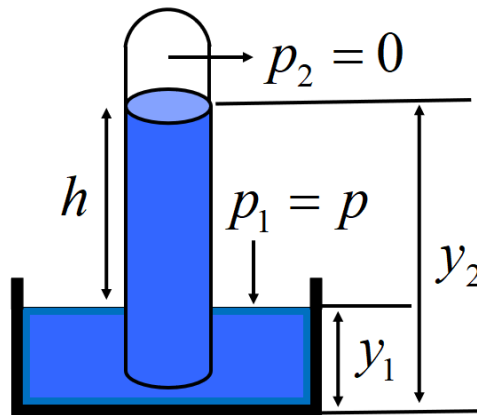


Figure 17. 16: A mercury barometer

Since only mercury vapor exists in the area above the mercury column at normal temperatures, it is effectively a vacuum that may be ignored. The pressure p_1 on the surface of mercury dish corresponds to the unmeasured pressure p .

i. e., $p_2 = 0$

$p_1 = p$ is the unknown pressure.

Therefore

$$p_2 - p_1 = -\rho g(y_2 - y_1)$$

$$\therefore h = y_2 - y_1$$

$$0 - p = -\rho g h$$

$$p = \rho g h$$

The pressure is then determined by measuring the height of the column above the dish's surface. The mercury barometer is most often used for measuring the pressure of atmosphere. The pressure of 1 atmosphere (1 atm) is equivalent to that exerted by a column of mercury of height 760 mm at 0° C under standard gravity ($g = 9.90665 \text{ m/s}^2$). The density of mercury at this temperature is $1.35955 \times 10^4 \text{ kg/m}^3$.

$$p_0 = \rho g h$$

$$1 \text{ atm} = \left(1.35955 \times 10^4 \frac{\text{kg}}{\text{m}^3}\right) \left(9.90665 \frac{\text{m}}{\text{s}^2}\right) (0.76 \text{ m})$$

$$1 \text{ atm} = 1.013 \times 10^5 \frac{\text{N}}{\text{m}^2} = 1.013 \times 10^5 \text{ Pa}$$

Sometimes, barometer values are expressed in torr, where 1 torr equals the pressure produced by a mercury column 1 mm high. Thus

$$\begin{aligned} 1 \text{ torr} &= \left(1.35955 \times 10^4 \frac{\text{kg}}{\text{m}^3}\right) \left(9.90665 \frac{\text{m}}{\text{s}^2}\right) (0.001\text{m}) \\ &= 133.326 \text{ Pa} \end{aligned}$$

Archimedes' principle

A body wholly or partially immersed in a fluid is buoyed up by a force (or upthrust) equal in Magnitude to the weight of the fluid displaced by the body.

This net upward force is called the *buoyant force* or *buoyancy*. It is possible to show that the weight of the displaced fluid is equal to the pressure differential between the object's upper and lower sides, which is the physical cause of the buoyant force.

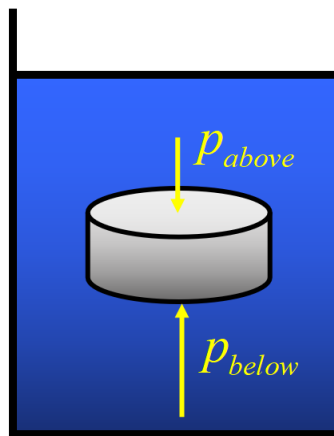


Figure 17. 17: Pressure is applied to the object's sides by the fluid around it. Because pressure increases with depth, the arrows on the underside are bigger than those on the top.

In Figure, the fluid around the object exerts pressure on all sides. The forces generated by the pressure are indicated by arrows. The arrows on the underside are larger than those on top because pressure rises with depth. The horizontal components cancel when they are all added together, but there is a net force upwards. This force, due to differences in pressure, is buoyant force B.

$$\begin{aligned} \because p_{below} &> p_{above} \\ \therefore \frac{F_{below}}{A} &> \frac{F_{above}}{A} \end{aligned}$$

$$F_{below} > F_{above}$$

$$F_b = F_{below} - F_{above} > 0$$

This net upward force F_b is called the buoyant force or buoyancy.

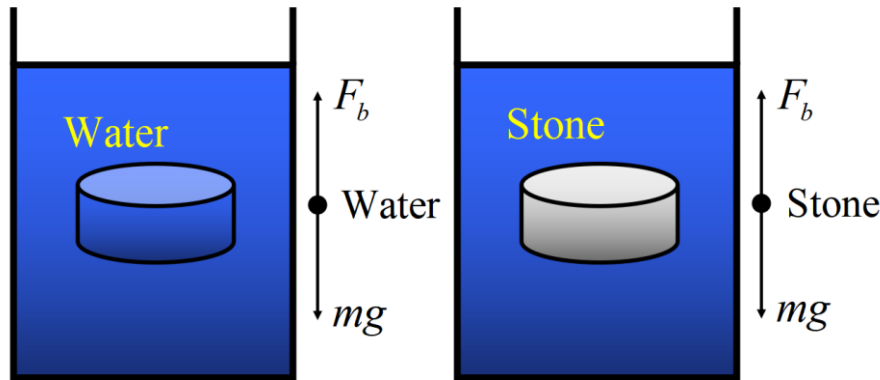


Figure 17. 18: Figure shows the submerged water segment and stone of same volume and the magnitudes of the corresponding forces (F_b and mg) acting on them.

Pressure exerted on a submerged object by the surrounding fluid certainly cannot depend on the material of which the object is made. Therefore, this force acts through the center of gravity of the displaced fluid, called the center of buoyancy. Thus a body appears to weigh less when immersed in a fluid

$$\text{Apparent Weight} = \text{True Weight} - \text{Upthrust}$$

Suppose a body of the volume V and density ρ is fully immersed in a liquid of density ρ' , Then

$$\text{Weight of the body} = W = \rho gV$$

$$\text{Weight of the liquid displaced} = W' = \rho' gV$$

Net downward force or the apparent weight is

$$W_a = (\rho - \rho')gV = \rho gV \left(1 - \frac{\rho'}{\rho}\right)$$

$$W_a = (\rho - \rho')gV = W \left(1 - \frac{\rho'}{\rho}\right)$$

The following possibilities may occur:

1. If $\rho' < \rho$, $W_a > 0$. Therefore, the body will sink to the bottom

2. If $\rho' = \rho$, $W_a = 0$. Therefore, the body will just float or remain hanging at whatever height it is left inside the liquid.
3. If $\rho' > \rho$, the upthrust will be greater than the weight of the body. Therefore, the body will move partly out of the free surface of the liquid until the upthrust becomes equal to W . The body will then float. Thus the principle of floatation is:

“For a body to float in a liquid, the weight of the liquid displaced by the immersed portion of the body must be equal to its own weight”.

Physics-PHY101-Lecture#18

Physics Of Fluids

Introduction to fluids

The field of fluid dynamics is a fascinating branch of physics and engineering. Fluid refers to a substance that flows and takes the shape of its container. In physics, fluid is a state of matter that includes both liquids and gases. Unlike solids, fluids do not have a fixed shape and can easily deform under the influence of external forces. Fluids are characterized by their ability to flow and their relatively low resistance to shear stress. This property is known as viscosity. Liquids, such as water, have a higher viscosity compared to gases like air. The behavior of fluids is governed by principles of fluid dynamics, which study the motion and properties of fluids.

Surface Tension

Liquids exhibit surface tension. Surface tension is a physical property of liquids that describes the force acting on the surface of the liquid due to the cohesive forces between its molecules. It is the reason why liquid droplets form spherical shapes and why some insects can walk on water. At the molecular level, the molecules within a liquid are attracted to each other. In the bulk of the liquid, these attractive forces are balanced in all directions. However, at the surface, the attractive forces are unbalanced, resulting in a net inward force that causes the surface to behave like a stretched elastic membrane. The molecules in the bulk of the liquid are attracted equally in all directions, but at the surface, they experience a stronger attractive force from the molecules below them than from the molecules above them. As a result, the molecules at the surface are pulled inward, creating a sort of "skin" or tension on the surface.

Surface tension can be quantified by measuring the amount of force required to break or distort the surface of a liquid. This force of contraction is at right angles to an imaginary line of unit length, tangential to the surface of a liquid, and is called its surface tension defined as:

$$\gamma = \frac{F}{L}$$

Here F is the force exerted by the "skin" of the liquid. The SI unit of surface tension is $\frac{N}{m}$.

Quantitative measurement of surface tension

In order to quantify the amount of force required to break or distort the surface of liquid can be measured with the help of the following experiment. Let w be the weight of the sliding wire and T is the force with which we pull the wire downward.

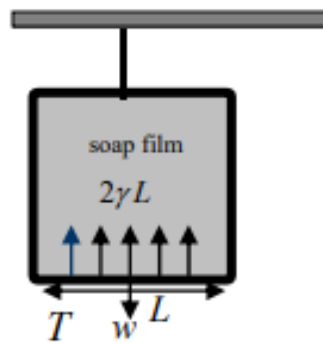


Figure 18.1: Weight of the sliding wire acting downward

From above figure it is obvious that, $T + w = F = \text{net downward force}$. Since film has both front and back surfaces, the force acts along a total length of $2L$. By using the equation of the surface tension of the film becomes,

$$\gamma = \frac{F}{2L} \Rightarrow F = 2\gamma L.$$

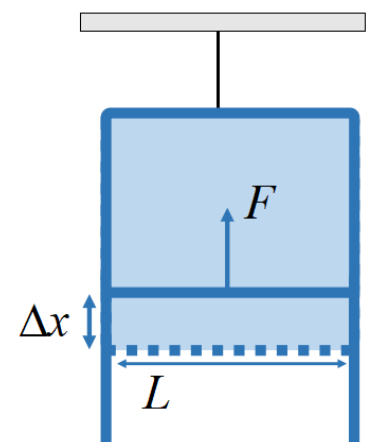
Hence the surface tension is

$$\gamma = \frac{T+w}{2L}.$$

The γ in above equation is surface tension coefficient.

Surface tension in term of energy

We can comprehend the above phenomenon in terms of energy. When the area of the film increases, the surface energy also increases proportionally. Additionally, this concept can be understood by considering the molecules within the film and the forces acting upon them, which depend on the distance



between the molecules. If the molecules within the film are larger, the corresponding forces will be greater.

Let's ask how much work is done when we stretch the skin of liquid. If we move the sliding wire through a displacement Δx , then the work done by the force of surface tension is $F\Delta x$. Here F is a conservative force, so there is potential energy defined as:

$$\Delta U = F\Delta x$$

By using, $\gamma = \frac{F}{L}$ in above equation we get

$$\Delta U = \gamma L\Delta x$$

Where L is the length of the surface layer. As $L\Delta x = \Delta A$ = change in area of the surface, so above equation becomes,

$$\gamma = \frac{\Delta U}{\Delta A}$$

So we see that surface tension is the ratio of the change in potential energy to the change in the surface area of the liquid when it is stretched.

Example: Soap bubble formation

Soap bubble formation is a fascinating phenomenon that occurs due to the surface tension of soapy water. Surface tension is a property of liquids that arises from the cohesive forces between their molecules. It is the force that causes the surface of a liquid to behave like a stretched elastic sheet.

When we blow air into a soap film or a soapy solution, the air pressure inside the film increases, stretching the film and causing it to form a bubble. This causes a pressure difference between the inside and outside of a soap bubble or a liquid drop. A soap bubble consists of two spherical surface films with a thin layer of liquid between them .

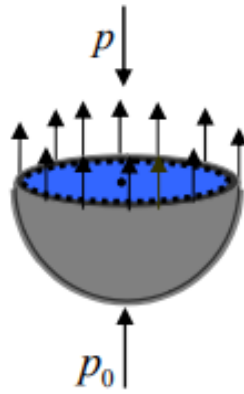


Figure 18. 2: A soap bubble consists of two spherical surface films

Let p = pressure exerted by the upper half, and p_o = external pressure. The force exerted due to surface tension is

$$F = \gamma L$$

and for the above example, the equation becomes,

$$F = 2(2\pi r\gamma)$$

where we have used $L = 2\pi r$ (circumference of soap bubble) and the “2” is for two surfaces. In equilibrium forces must be equal i.e.

Total force = Surface tension force + Force due to outside

and force can be defined as

$$F = pA = \pi r^2 p$$

So by using the expression of total force, we get

$$\pi r^2 p = 2(2\pi r\gamma) + \pi r^2 p_o$$

$$rp = 2(2\gamma) + rp_o$$

$$r(p - p_o) = 2(2\gamma)$$

$\therefore$ As, bubble has two surfaces

$$(p - p_o) = \frac{4\gamma}{r}$$

After solving the above equation for a liquid having only one surface, we get the expression of excess pressure as,

$$\text{Excess pressure} = p - p_o = \frac{2\gamma}{r}$$

Based on the equation above, it is evident that the pressure difference is inversely proportional to the radius of the bubble. This implies that it is significantly easier to form bubbles with a larger radius compared to those with smaller radii.

Fluid Motion

So far, we have only discussed fluid statics, which pertains to fluids that are not in motion. However, understanding fluid dynamics becomes more crucial when fluids are in motion. In this section, we will discuss the topic in detail due to its high significance and numerous real-life examples such as blood flow in our bodies, water flowing from a tank, and the air passing over an airplane, enabling it to fly. This is why the discussion now shifts to fluid motion.

To begin with, fluid motion can be classified as either steady or unsteady. **Steady flow** refers to a constant-speed flow of liquids, while the flow becomes unsteady when the fluid descends from a waterfall, for instance. Additionally, the fluid can either be compressible or in-compressible (constant density). Furthermore, fluid flow can also be categorized as viscous or friction-free.

Types of fluid flow

When discussing fluid dynamics, we encounter two types: laminar flow and turbulent flow. Now, let's proceed to explore the various types of fluid flow.

Laminar flow

Laminar flow is characterized by smooth and orderly movement of fluid particles in parallel layers or streamlines, with minimal mixing between adjacent layers.

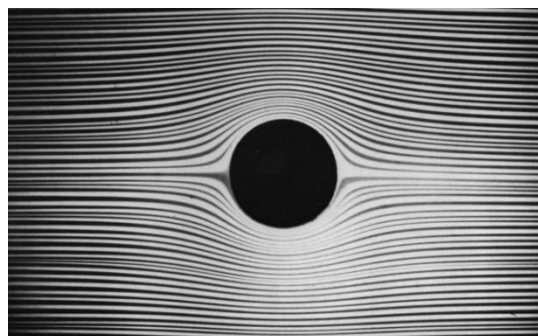


Figure 18. 3: Laminar flow of fluid

Turbulent flow

Turbulent flow, on the other hand, is characterized by irregular and chaotic motion of fluid particles, involving vigorous mixing.

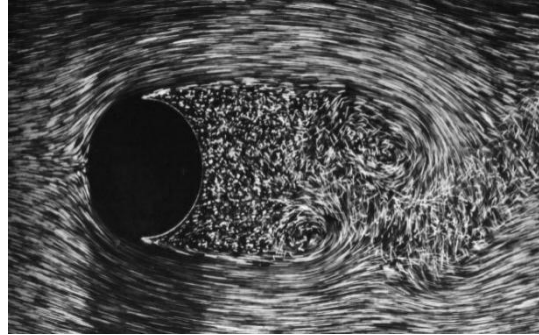


Figure 18. 4: Turbulent flow of fluid

Equation of Continuity

The equation of continuity is a fundamental principle in fluid dynamics that describes the conservation of mass in a fluid flow.

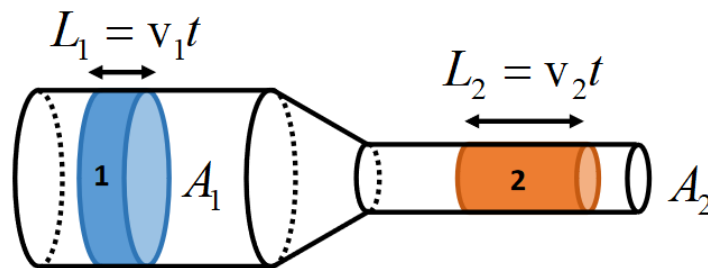


Figure 18. 5: Fluid flowing through a non-uniform pipe.

Consider water flowing in a pipe with a changing diameter. Fluid enters from left where the cross-sectional area is A_1 (volume V_1) and leaves from right where the area is A_2 (volume V_2). Let v_1 and v_2 are the speeds of fluid particles at both ends of the pipe. If we consider the volume flow rate (V/t), which is the amount of fluid passing through a section of the pipe per unit of time, and since water is incompressible, the volumes at two different points must be equal.

$$\frac{V_1}{t} = \frac{V_2}{t}$$

$$V_1 = V_2$$

The volume of water passing through a section of the pipe can be expressed as the product of cross-sectional area (A) and length (L):

$$A_1 L_1 = A_2 L_2$$

$\therefore$ displacement (L) = velocity (v) $\times$ time (t)

$$A_1 v_1 t = A_2 v_2 t$$

$$A_1 v_1 = A_2 v_2$$

This equation represents the inverse relation between the area and speed i.e., smaller the area, greater will be the speed.

This is the key equation of continuity. It states that the product of the cross-sectional area and the velocity at one point in the pipe is equal to the product at another point. This holds true as long as the flow is steady, there are no leaks or ruptures, and the fluid is incompressible. The product of cross-sectional area and velocity is equal to the derivative of volume with respect to time

$$\frac{dV}{dt} = Av = \text{const}$$

This expression emphasizes that the rate of change of volume with respect to time is constant along the pipe.

In summary, the equation of continuity states that for an in-compressible fluid, the mass flow rate remains constant along a streamline. It is based on the principle of conservation of mass, where the volume of fluid passing through a particular section of a pipe must be constant for a steady flow. Thus, as the cross-sectional area changes, the fluid velocity adjusts to ensure that the mass flow rate remains constant.

Example: Blood flow

The heart, functioning as a pump, circulates blood throughout our body, delivering oxygen and nutrients. On average, the heart pumps 60 to 90 ml of blood per heartbeat. If the heart stops, death occurs within four to five minutes.

The heart is divided into left and right sides, each with specific roles. One side pumps blood, connecting to the lungs, while the other directs blood to the body. As blood circulates, it loses oxygen and gains carbon dioxide. Like a tire valve, the heart has four valves allowing one-way blood flow, opening when blood moves and closing under pressure.

Understanding blood flow is vital for general knowledge and heart issue diagnosis. Engineers designing an artificial heart need to grasp concepts like viscous and non-viscous flow, turbulent and streamlined flow, crucial for creating a heart-replacing machine.

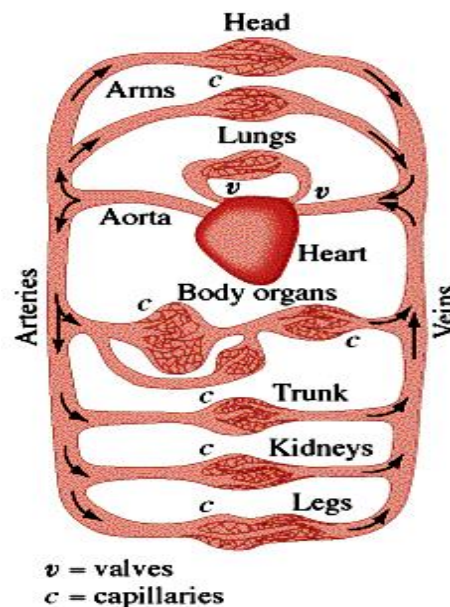


Figure 18. 6: Blood flow within a circulatory system, highlighting the movement of blood through arteries, veins, and capillaries, showcasing the direction and pathways within the vascular network.

Bernoulli's Equation

Bernoulli's Equation is the fundamental equation in fluid dynamics which tells you how fast a fluid will flow when there is also a gravitational field acting upon it. The analysis is based on applying conservation of energy,

$$\Delta K + \Delta U = W$$

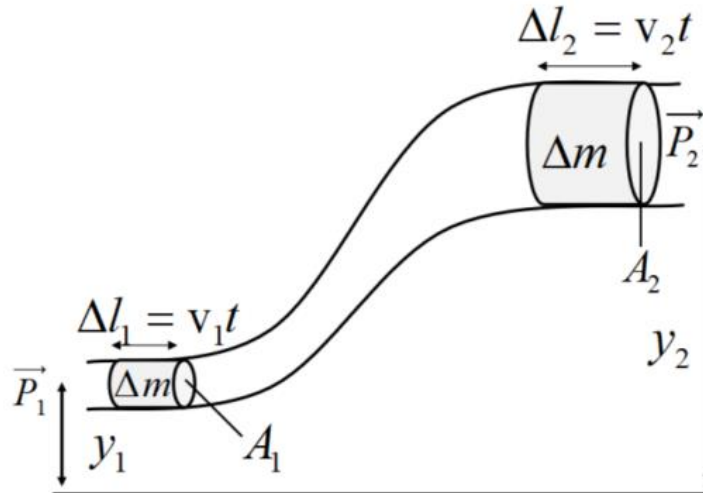


Figure 18. 7: A steady flow of fluid through the pipe in time $t+\Delta t$. Transferring the fluid element shown by the dark coloring from the tube's intake end to its output end is the overall result of the flow.

The fluid at the inlet end is subjected to a pressure p_1 (by the fluid behind it) which causes a force F_1 to push the system to the right.

$$F_1 = p_1 A_1$$

At the outlet end, a force F_2 acts on our system to the left due to pressure p_2 (by the fluid in the pipe to the right of the system)

$$F_2 = p_2 A_2$$

The fluid element Δm , moves through the pipe under the combined influence of the two pressure forces and gravity. At the inlet end, the pressure force does work in moving fluid through distance Δl ,

$$W_1 = F_1 \Delta l_1 = p_1 A_1 \Delta l_1$$

At the outlet end, the pressure force does work

$$W_2 = -F_2 \Delta l_2 = -p_2 A_2 \Delta l_2$$

The work done by gravity, as Δm moves through the vertical displacement is

$$W_g = -\Delta m g (y_2 - y_1)$$

Where y_1 and y_2 are the heights above some reference level. The negative sign in the above equations indicates that the force and displacement are in opposite directions.

Total work done on the system is

$$W = W_1 + W_2 + W_g$$

$$W = p_1 A_1 \Delta l_1 - p_2 A_2 \Delta l_2 - \Delta m g (y_2 - y_1) \quad (1)$$

In terms of the (uniform and constant) fluid density , the volume element is

$$A_1 \Delta l_1 = A_2 \Delta l_2 = \Delta V = \frac{\Delta m}{\rho},$$

Thus equation 1 becomes

$$W = (p_1 - p_2) \frac{\Delta m}{\rho} - \Delta m g (y_2 - y_1)$$

The change in kinetic energy of the fluid element Δm is

$$\Delta K = \frac{1}{2} \Delta m v_2^2 - \frac{1}{2} \Delta m v_1^2$$

Where v_1 and v_2 are the velocities at the lower and upper ends respectively. ΔU represents the potential energy due to conservative forces that act among objects within the system. Here we assume that no such forces act within the fluid, so

$$\Delta U = 0$$

Applying conservation of energy,

$$W = \Delta K$$

$$(p_1 - p_2) \frac{\Delta m}{\rho} - \Delta m g (y_2 - y_1) = \frac{1}{2} \Delta m v_2^2 - \frac{1}{2} \Delta m v_1^2$$

$$\frac{\Delta m}{\rho} [(p_1 - p_2) - \rho g (y_2 - y_1)] = \Delta m \left[\frac{1}{2} v_2^2 - \frac{1}{2} v_1^2 \right]$$

$$[(p_1 - p_2) - \rho g (y_2 - y_1)] = \rho \left[\frac{1}{2} v_2^2 - \frac{1}{2} v_1^2 \right]$$

$$p_1 - p_2 - \rho g y_2 + \rho g y_1 = \frac{1}{2} \rho v_2^2 - \frac{1}{2} \rho v_1^2$$

$$p_1 + \frac{1}{2} \rho v_1^2 + \rho g y_1 = p_2 + \frac{1}{2} \rho v_2^2 + \rho g y_2$$

Since the subscripts 1 and 2 refer to two arbitrary locations in the pipe, we can drop the subscripts and write

$$P + \frac{1}{2} \rho v^2 + \rho g y = \text{constant}$$

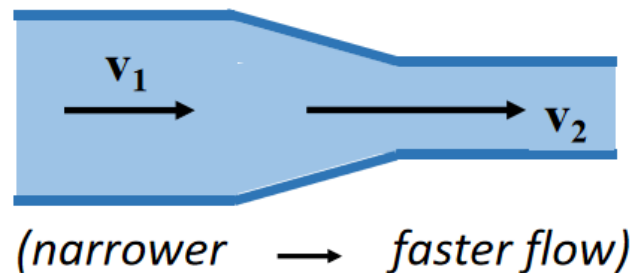
The above equation is called **Bernoulli equation** which states that “for a steady, non-viscous, in-compressible and irrotational flow fluid the sum of pressure, kinetic energy per unit volume and potential energy per unit volume is always constant”.

In summary, Bernoulli's equation indicates that the total energy of a fluid remains constant along a streamline. As the fluid moves faster (increasing its kinetic energy) or climbs to a higher elevation (increasing its potential energy), the pressure exerted by the fluid decreases. Similarly, if the fluid slows down (reducing its kinetic energy) or descends to a lower elevation (decreasing its potential energy), the pressure increases.

This principle has numerous practical applications, including understanding the lift generated by an airplane wing, the flow through pipes and nozzles, and the behavior of fluids in various engineering and natural systems.

Example 1: Let us apply **Bernoulli equation**

to water flowing in a pipe whose cross-sectional area decreases along its length as shown in figure (A_1 is area of the wide part, etc.).



The fluid is flowing horizontally, so there is no change in the height ($y_1 = y_2$) hence, the gravitational potential energies get cancelled out, and we get

$$p_1 + \frac{1}{2}\rho v_1^2 + \cancel{\rho g y_1} = p_2 + \frac{1}{2}\rho v_2^2 + \cancel{\rho g y_2}$$

$$p_1 + \frac{1}{2}\rho v_1^2 = p_2 + \frac{1}{2}\rho v_2^2$$

Now, since the liquid is in-compressible, according to equation of continuity it flows faster in the narrower part as given below

$$A_1 v_1 = A_2 v_2 \quad (2)$$

$$v_2 = \left(\frac{A_1}{A_2}\right) v_1$$

As $A_1 > A_2$, thus $v_2 > v_1$, equation 2 becomes

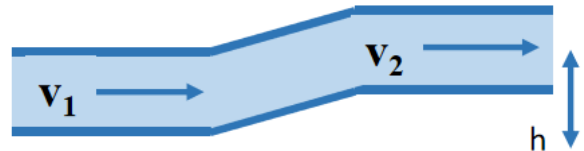
$$p_2 = p_1 - \frac{1}{2}\rho(v_2^2 - v_1^2)$$

This means that the pressure is smaller in the narrow pipe where the streamlines are closer together than in the wider pipe ($p_2 < p_1$). Hence, **where the speed of fluid is high, the pressure will be low.**

Example 2: Consider a uniform pipe ($A_1 = A_2$) in which an ideal fluid is moving with constant velocity ($v_1 = v_2$). The height of pipe changes from y_1 to y_2 ($y_2 - y_1 = h$) as shown in figure,

Applying Bernoulli's equation

$$p_1 + \frac{1}{2}\rho v_1^2 + \rho g y_1 = p_2 + \frac{1}{2}\rho v_2^2 + \rho g y_2$$



$$p_2 = p_1 - \rho g(y_2 - y_1) = p_1 - \rho g h$$

Hence, $p_2 < p_1$, greater the height lower will be the pressure and vice versa.

Applications of Bernoulli's equation

Bernoulli's equation, a fundamental principle in fluid dynamics, has wide-ranging applications that span various fields of science and engineering. From aerodynamics and hydraulics to blood flow dynamics and sports, the equation provides valuable insights into the behavior of fluids in different contexts. Understanding and applying Bernoulli's equation enables engineers, researchers, and practitioners to design efficient systems, analyze fluid flow patterns, and even explain phenomena such as lift generation and ball trajectories. In this section, we will explore some of the fascinating applications of Bernoulli's equation and its significance in understanding fluid behavior across diverse disciplines.

Dynamic lift

Dynamic lift is the force that a body experiences as a result of moving through a fluid, like an aircraft wing.

Airplanes

Dynamic lift is exactly the reason why an aircraft flies? The wing shape is curved so that when the aircraft moves through the air, the streamlines are more closely spaced above the wing than below it. The air moves faster on the top part of the wing than on the lower part. Thus, the air pressure is lower on the top compared to the bottom and

there is a net pressure upwards. The difference in pressure between the wing's upper and lower surfaces is what produces lift. This is called lift.

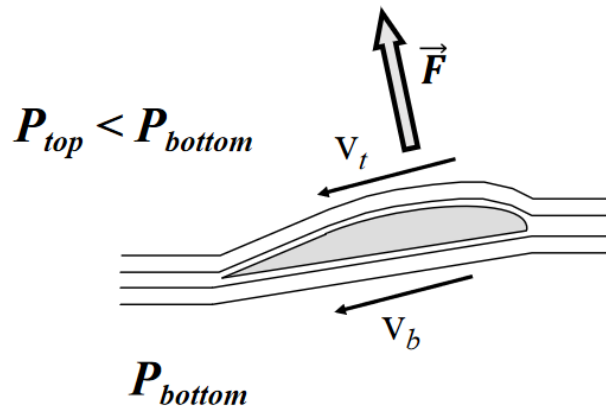


Figure 18. 8: Dynamic lift in airplanes, demonstrating the principle of airflow over the wing creating a pressure difference, resulting in upward lift force during flight.

Another explanation for the lift, however, is that the wing is made to deflect the air passing past it downward. In accordance with Newton's Third Law, this lift is therefore a reaction force $\vec{F}$ (exerted by air on the wing) to the downward force (the wing exerts on the air).

Atomizer

An atomizer is a tool that emits liquid droplets as a thin mist. In this context, the term "atomize" refers to breaking a large body into several, distinct particles. Its work is based on Bernoulli's principle. It provides a low pressure and pulls the liquid and air inside the vertical tube upward when fast horizontal air passes over it. At the end of the horizontal tube, there is a nozzle that disperses the liquid into tiny drops and mixes them with the air.

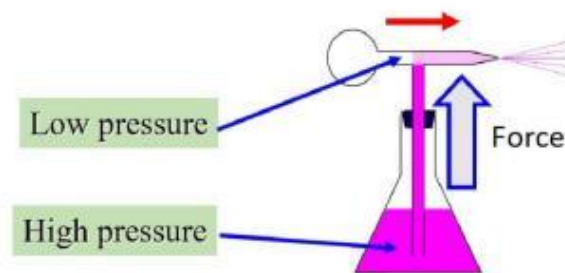


Figure 18. 9: An atomizer, a device used to break down liquids into a fine mist, illustrating the mechanism of liquid dispersion through fine nozzles."

Venturi Meter

A Venturi meter is a device used to measure the flow rate of a fluid in a pipe. It operates based on the principle of Bernoulli's equation and the Venturi effect.

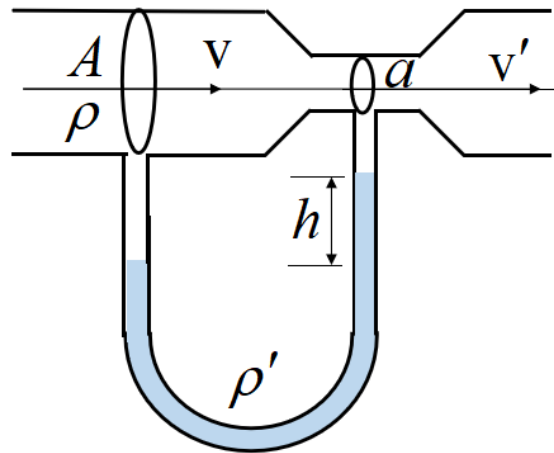


Figure 18. 10: A Venturi meter consists of a pipe with cross-sectional area A_1 is filled with a fluid of density ρ . A manometer tube is inserted at the throat, where the area is reduced to A_2 . The manometer liquid, such as mercury, have a density ρ' .

There is no difference in the heights as the pipes are horizontal. We formulate Bernoulli's equation in a more comprehensible manner.

$$p + \frac{1}{2}\rho v^2 = p' + \frac{1}{2}\rho v'^2$$

$$p - p' = \frac{1}{2}\rho v'^2 - \frac{1}{2}\rho v^2 \quad (3)$$

By using equation of continuity

$$Av = av'$$

$$v' = \frac{A}{a}v$$

By using the above equation into eq. 3 we will get

$$p - p' = \frac{1}{2}\rho \left(\frac{A}{a}v\right)^2 - \frac{1}{2}\rho v^2$$

$$p - p' = \frac{1}{2}\rho v^2 \left[\left(\frac{A}{a}\right)^2 - 1\right] \quad (4)$$

Also,

$$p - p' = h(\rho' - \rho)g \quad (5)$$

So by making the comparison of eq. 4 with eq. 5 we will get

$$h(\rho' - \rho)g = \frac{1}{2}\rho v^2 \left[\left(\frac{A}{a} \right)^2 - 1 \right]$$

$$h(\rho' - \rho)g = \frac{1}{2}\rho v^2 \left[\frac{A^2 - a^2}{a^2} \right]$$

Re arranging the above equation, we get

$$v = a \sqrt{\frac{2(\rho' - \rho)gh}{\rho(A^2 - a^2)}}$$

This equation gives the mathematical expression for the velocity of the fluid at the Venturi throat in terms of the velocities at the inlet v and the pressure difference between the two points, along with the cross-sectional areas of the Venturi meter and the density of the fluid (ρ). The physical explanation of the above equation lies in the Venturi effect, where the increase in fluid velocity in the constricted section leads to a decrease in pressure. This pressure difference can be measured and correlated to the flow rate through the Venturi meter.

The Pitot Tube

The Pitot tube is a velocity measuring device of a fluid (specifically gas), named after the inventor Henri Pitot.

Applying Bernoulli's equation, $v_1=0$, $h_1=h_2$

$$p_2 + \frac{1}{2}\rho v_2^2 = p_1$$

$$p_1 - p_2 = \frac{1}{2}\rho v_2^2 \quad (6)$$

Also, by the measurement of manometer,

$$p_1 - p_2 = \rho' gh \quad (7)$$

Comparing equation 6 and 7 we get

$$\frac{1}{2}\rho v_2^2 = \rho' gh$$

$$v_2 = \sqrt{\frac{2gh\rho'}{\rho}}$$

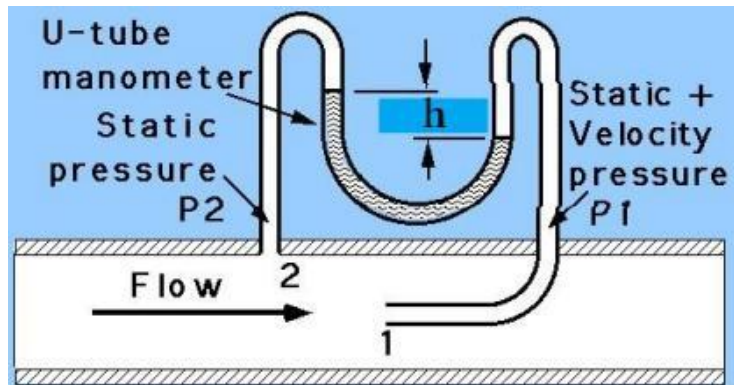


Figure 18. 11: Pitot tube, a device used in aviation to measure fluid flow velocity, illustrating its design and function in determining airspeed on aircraft.

Rotational or Irrotational Fluid Flow

Now, let's explore another perspective involving irrotational and rotational dynamics. If an element of the moving fluid does not rotate about an axis passing through the center of mass of element, the flow is said to be irrotational. If the wheel moves without rotating the motion is irrotational; otherwise it is rotational.

Previously, we covered streamline and turbulent flows, but there's another aspect to consider. When a small object rotates, introducing rotational energy into the flow, it creates a complicated situation, and the basic equation doesn't apply directly to this scenario.

Tennis Ball

Figure (a) depicts streamlines for the constant flow of air passing by a non-rotating, stationary ball at low speeds sufficient to prevent turbulence. Figure (b) displays the streamlines of air being transported by a ball that is rotating quickly. The spinning ball could not move the air in this way without viscosity and the boundary layer.

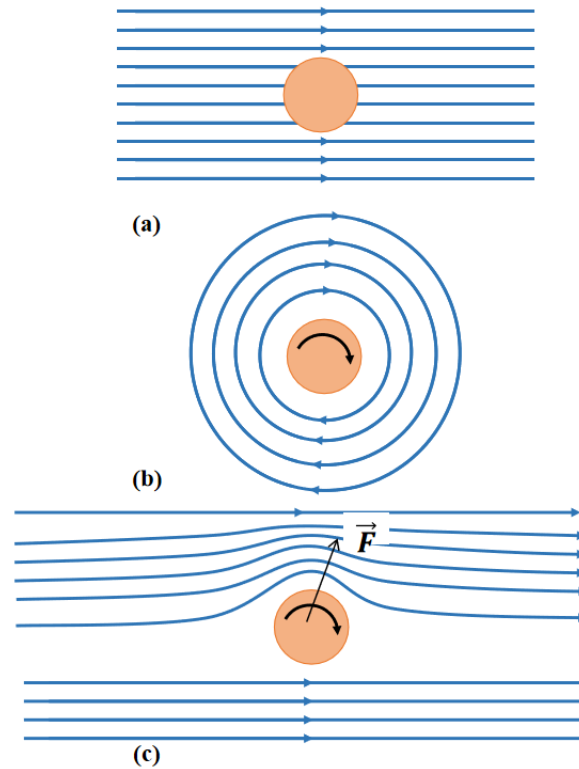


Figure 18. 12: (a) constant flow of air passing by a non-rotating, stationary ball, (b) streamlines of air being transported by a ball that is rotating quickly, (c) combined effect of circulation—caused by the ball's rotation—and steady flow—caused by the ball's translation through the air.

Figure (c) illustrates the outcome of combining circulation—caused by the ball's rotation—and steady flow—caused by the ball's translation through the air. The two velocities in the picture are added above the ball and subtracted below in the situation shown. We can observe that the space between the resulting streamlines indicates that the air velocity below the ball is lower than that above it. Since Bernoulli's equation requires that the air pressure beneath the ball be higher than the air pressure above it, the ball experiences a dynamic lift force.

Super-fluids

Most fluids exhibit viscosity, causing streamlined flow, but turbulence arises at high speeds. However, certain fluids, like liquid helium at extremely low temperatures, defy viscosity norms. At such temperatures, liquid helium undergoes quantum effects, resulting in superfluidity, where viscosity becomes negligible. This phenomenon, observed as helium spontaneously defying gravity by rising from a container's sides, has led to significant discoveries. For instance, a German team in 1988 demonstrated

that combining and cooling helium atoms induce superfluidity, eliminating viscosity and friction.

Physics-PHY101-Lecture 19

PHYSICS OF SOUND

Introduction to waves

Waves, including sound waves, are disturbances that transfer energy from one place to another without transferring matter. In the context of sound waves, the medium through which the disturbance travels are typically air, but it can also be other materials like water or solids. Sound waves are produced by vibrations. When an object vibrates, it creates compressions and rarefactions in the surrounding medium. In the case of your voice, the vibrations originate in your vocal cords. As you speak, these vibrations are transmitted to the air molecules, creating areas of high pressure (compressions) and low pressure (rarefactions). These pressure differences are subsequently transmitted as sound waves via the air. Mechanical waves, specifically water waves, are produced by tossing a stone into the water. These waves move water molecules in circular or elliptical patterns by transferring energy over the surface of the water.

For sound waves, the medium is essential. In the absence of a medium (like in outer space), sound cannot travel because there are no molecules to transmit the vibrations. It's important to note that while waves transmit energy, the particles in the medium generally oscillate around a fixed position. At the end of 2004, the biggest tsunami occurred near Indonesia. The energy from the earthquake caused water particles to move in an up-and-down motion, transmitting the energy across vast distances.

In this chapter we'll understand the principles of waves are crucial in various fields, from physics and engineering to telecommunications and environmental science.

Sound waves generated in a tube:

Moving the piston from left to right in a tube with a piston attached to one end causes a motion in the layers of air within the tube. Energy is transferred as one layer imparts energy to the next. These layers of air are constantly vibrating, and their forward movement becomes visible over time. While this is simple in one dimension, it becomes more complicated when onlookers intervene by clapping their hands, forcing the layer of air to scatter in different directions. The difficulty lies in finding and comprehending these dispersed levels.

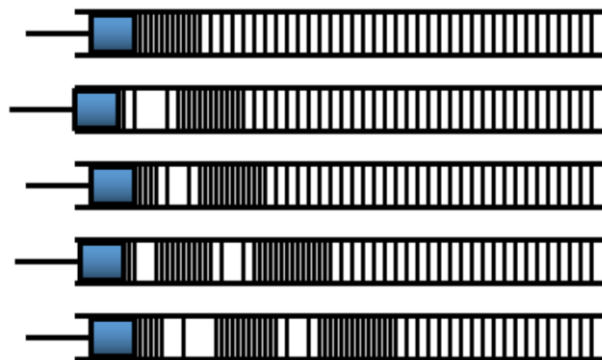
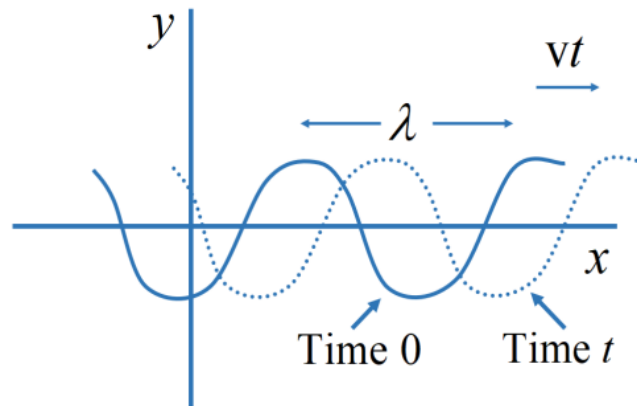


Figure 19. 1: Sound wave propagation within a tube by a moving piston, highlighting the compression and rarefaction patterns as the waves travel through the confined space. The compressible material in the tube is divided into layers with equal masses by the vertical lines.

In a previous lecture, we highlighted the significance of identifying vibrations using a variety of parameters. Firstly, the frequency of the wave determines the number of cycles per second. Secondly, the time taken to complete one cycle is referred to as the time period. Lastly, the amplitude of the vibration signifies its intensity — lower amplitude corresponds to lower sound, and in the context of earthquakes, low-amplitude seismic activity may go unnoticed.



Generation of sound

Working of a loudspeaker

A signal generator produces an electrical signal with a specific frequency and amplitude. To amplify the weak signal, we use a radio amplifier. Connecting the waves from the amplifier to a loudspeaker which consist of basic two parts: an outer diaphragm made of hard paper and a permanent magnet. The coil, when carrying current, experiences a magnetic force, leading to the diaphragm's vibration. By controlling amplitude and frequency, we can control the sound quality. High frequencies, above 15 kHz, become inaudible, demonstrating our control over amplitude and frequency.

Now by placing pallets on the diaphragm, we observe their response to changes in volume and frequency. Resonance occurs when the pallets and diaphragm frequencies align, resulting in increased movement of pallets. This experiment confirms that sound is indeed a vibration.

Intensity of sound

Sound operates on various energy levels, and here it is measured in watts. This quantifies the energy reaching us at any given time. Hence watt is equal to the joule per second (J/s). To gauge/measure the energy in space, sound intensity is measured in watts per centimeter squared.

There are various levels of sound which differ from each other, and here we employ a logarithmic scale for precision. Logarithms, such as $\log_{10}10 = 1$, $\log_{10}10^2 = 2$ and $\log_{10}10^n = n$. Hence, it helps in managing larger values to smaller ones efficiently.

Logarithmic scales are crucial for very low-intensity sounds, around 10^{-12} W/cm^2 . This is expressed in **decibels (dB) scale**, “a relative measure for comparing the intensity of different sounds with one another”.

$$I_o = \text{Threshold of hearing} = 10^{-12} \text{ W/ cm}^2$$

$$R = \text{Relative intensity of sound } I = 10 \log \frac{I}{I_o}$$

Let's now examine the intensity of various sounds by using 10^{-12} as a reference value. Following this, we will explore how sound levels are expressed on a logarithmic scale.

Waves create a displacement between one peak and another, known as wavelength. Mathematically, it is described as follows:

At time $t=0$

$$y(x, 0) = y_m \sin \frac{2\pi}{\lambda} x$$

y_m is the amplitude. At time t in x direction,

$$y(x, t) = y_m \sin \frac{2\pi}{\lambda} (x - vt)$$

Now, as the waves progress, they travel a distance vt in one second, where v is the velocity of sound. This clarifies that within one oscillation time, the wave moves forward by vt . The **period** T of a wave is the time to undergo one complete cycle of motion.

$$\lambda = vT \rightarrow T = \frac{\lambda}{v}$$

The **frequency** (ν) is the number of waves crossing a particular point every second.

$$\nu = \frac{1}{T}$$

$$y(x, t) = y_m \sin 2\pi \left(\frac{x}{\lambda} - \frac{t}{T} \right)$$

The value of y at any given position remains the same at t , $t+T$, $t+2T$,

To further describe the wave, we introduce two additional quantities: wave number (k) and angular frequency (ω).

	Sound	Intensity (W / m ²)	Relative Intensity (I / I ₀)	Sound Level (dB)
$k = \frac{2\pi}{\lambda}$	Threshold of hearing	10^{-12}	10^0	0
$\omega = \frac{2\pi}{T} = 2\pi\nu$	Rustle of leaves	10^{-11}	10^1	10
	Whisper (at 1 atm)	10^{-10}	10^2	20
	City street, no traffic	10^{-9}	10^3	30
	Conversation (at 1 atm)	10^{-6}	10^6	60
	Pressure horn (at 1 atm)	10^{-3}	10^9	90
	Ear damage	1	10^{12}	120
	Jet engine (at 1 atm)	10	10^{13}	130

In the previous discussion on oscillations, resonance was explored in detail. It was highlighted that when an oscillator is driven by an external force, it exhibits maximum response when the driving frequency aligns closely with the resonance frequency. This principle is evident in musical instruments.

Waves in Guitar Strings

Consider a guitar with various strings, each possessing different widths and tensions. Tensions can be adjusted, creating distinct frequencies for each string. Nodes, points where the string doesn't

move, are present at both ends, where the oscillation amplitude is zero. In between these nodes, the amplitude may vary depending on the way of excitation of the strings.

The shape of the oscillation depends on the way of exciting string. Two notable features of the guitar include its unique body shape. There is an air column in the middle of the guitar where resonance occurs when the strings set the body in vibration. By plucking or exciting a string, different tones are generated. The frequency of each tone is proportional to the square root of the tension divided by the mass per unit length, meaning thicker strings have more mass per unit length.

Adjusting tension in the strings of guitar influences frequency. Increasing tension increases frequency. So, the frequency of oscillation does depend upon the square root of tension.

This is because the speed of sound in all strings is equal to the square root of the tension divided by the mass per unit length.

Nodes are present at both ends. When exciting the middle string, a unique sound is produced. The guitarist controls frequency by altering the length, generating multiple frequencies from each string. Unlike pure tones, which consist of a single frequency, the guitar produces a variety of frequencies.

Mathematics of Complex Waveforms

When different frequencies coexist, we can comprehend this phenomenon through mathematics. As mentioned earlier, combining sine waves with varying amplitudes results in a complex waveform that, despite being periodic, appears unconventional compared to simple sine or cosine functions.

Altering the coefficients produces different shapes.

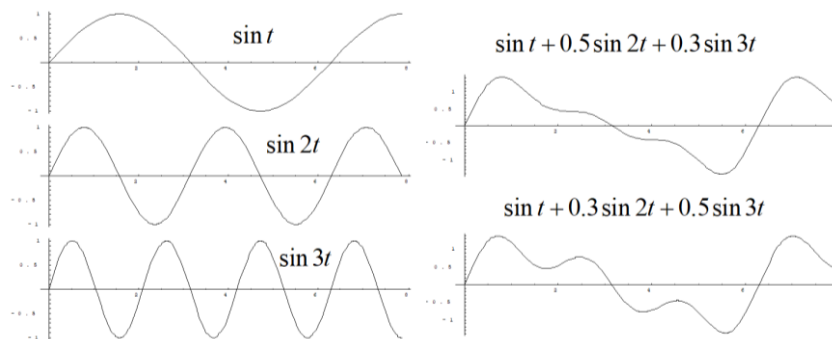


Figure 19. 2: Addition of sin waves i.e., $\sin t$, $\sin 2t$, and $\sin 3t$, each wave with different amplitudes (0.5 and 0.3) is obtained.

You can see cost with various frequencies and amplitudes as well.

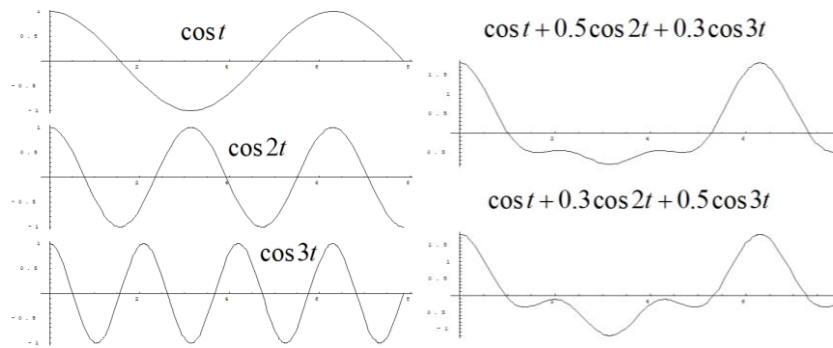


Figure 19. 3: Addition of cosine waves i.e., $\cos t$, $\cos 2t$, and $\cos 3t$.

A remarkable discovery attributed to Joseph Fourier two centuries ago says that any wave, regardless of its shape, can be represented as a sum of sine and cosine functions. This concept is widely recognized as Fourier analysis.

Instrument Tuning Software

There is software designed for tuning various musical instruments, ensuring accurate note generation by comparing with the instrument's natural nodes. Unlike a guitar, which produces multiple notes, the low-frequency note is known as the **fundamental frequency**, and the other frequency notes which appear after the low-frequency notes are termed as **overtones**.

It's important to note the distinction between music and pure tones. While pure tones lack appeal, music incorporates a fundamental frequency and overtones. Comparing the tones generated by the software with recorded guitar sounds reveals that the guitar produces a mix of fundamental frequencies and overtones, around 84 Hz.

To further illustrate this, harmonic analysis quantifies the amplitude of each harmonic, providing information about the coefficients of sine frequencies and others in decibels. The software takes an interesting step by performing Fourier analysis at each moment. As a person speaks, a graph on the screen displays the analysis in real-time.

Doppler Effect

The Doppler Effect is the phenomena in which relative motion between source and observer causes the observer to receive a frequency that is different from that emitted by the source. The phenomenon was discovered by Austrian physicist Christian Johann Doppler in 1842.

Case 1: Observer is moving towards a stationary source

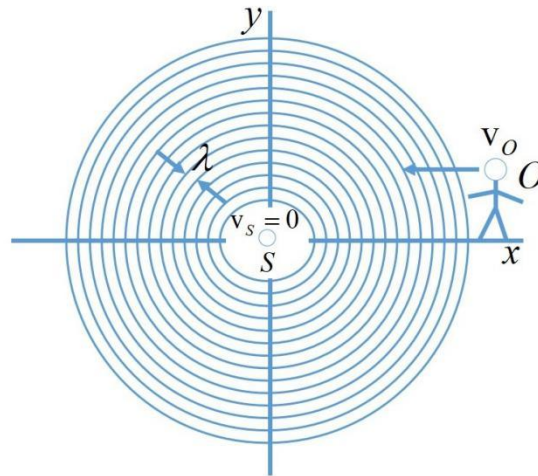


Figure 19. 4: Doppler effect with a stationary sound source (S) and a moving observer (O) travelling at a speed (v_o) towards the source. The circles show wavefronts moving across the medium separated by one wavelength.

If the observer was at rest, the number of waves received in time t would be t/T or vt/λ ($1/T = v = v/\lambda$). Where v is the speed of sound in the medium and λ and v are the wavelength frequency of sound wave. But if O is moving towards the source with speed v_o , the additional number of waves received is obviously $v_o t/\lambda$. By definition,

$$v' = \text{frequency actually heard} = \frac{\text{number of waves received}}{\text{unit time}}$$

$$\begin{aligned} v' &= \frac{\frac{vt}{\lambda} + \frac{v_o t}{\lambda}}{t} = \frac{v + v_o}{\lambda} \\ &= \frac{v + v_o}{v/v} = v \left(\frac{v + v_o}{v} \right) \end{aligned}$$

Frequency actually heard is

$$v' = v \left(1 + \frac{v_o}{v} \right)$$

The increase in frequency $v(v_o/v)$, brought on by the observer's motion, is added to the frequency (v) heard at rest to create the frequency (v') heard by the observer. We finally conclude that the observer hears a higher frequency (higher pitch) while moving towards the source.

Case 2: Observer is moving away from a stationary source

Waves are received at a reduced rate by an observer receding from the source, . Hence, number of waves received per unit time in this case is

$$v' = \frac{\frac{vt}{\lambda} - \frac{v_o t}{\lambda}}{t} = \frac{v - v_o}{\lambda} = \frac{v - v_o}{v/v} = v \left(\frac{v - v_o}{v} \right) = v \left(1 - \frac{v_o}{v} \right)$$

The above expression shows a *decrease* in frequency by an amount $v(v_o/v)$ corresponding to the waves that do not reach the observer in each unit of time because of the receding motion.

Since the observer is moving through the medium while the source is at rest with respect to it, the general relation existing in this situation is

$$v' = v \left(\frac{v \pm v_o}{v} \right)$$

where motion towards the source is represented by a plus sign, and motion away from the source is represented by a negative sign.

Case 3: Source is moving towards the observer at rest

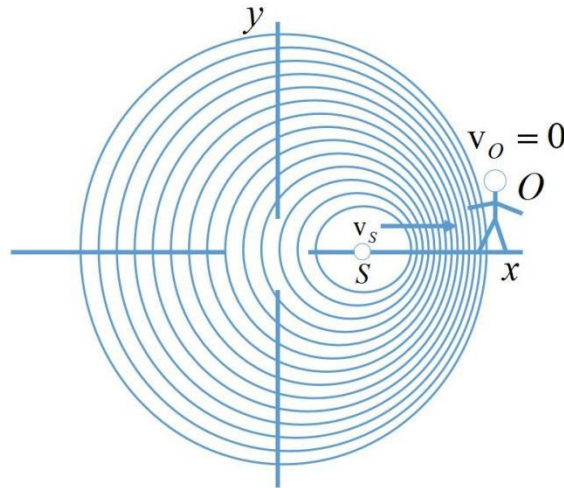


Figure 19. 5: Doppler effect with a source (S) moving at a speed (v_s) towards a stationary observer (O).

The waves are compressed by a quantity known as "doppler shift" ($\Delta\lambda = v_s/v$) as the source comes closer to the observer because same number of waves are contained in a smaller region depending upon the velocity of source v_s . So, the wavelength seen by the observer is

$$\lambda' = \lambda - \Delta\lambda = \frac{v}{v} - \frac{v_s}{v} = \frac{v - v_s}{v}$$

From this frequency received by the observer is

$$v' = \frac{v}{\lambda'} = \frac{v}{(v - v_s)/v}$$

$$v' = v \frac{v}{(v - v_s)}$$

Consequently, whenever the source is travelling in the direction of the observer, the observed frequency rises.

Case 4: Source is moving away from the observer at rest

The wavelength emitted by the source increases as it gets farther away from the observer, causing the observer to hear a lower frequency.

$$\lambda' = \lambda + \Delta\lambda = \frac{v}{v} + \frac{v_s}{v} = \frac{v + v_s}{v}$$

$$v' = \frac{v}{\lambda'} = \frac{v}{(v + v_s)/v} = v \frac{v}{(v + v_s)}$$

In light of this, the general relationship that holds when the source is moving through a medium and the observer is at rest is

$$v' = v \frac{v}{(v \pm v_s)}$$

where the minus sign holds for motion toward the observer and the plus sign holds for motion away from the observer.

Case 5: Both source and observer are moving

Let us consider that the source and observer are moving towards each other. Using eq. Doppler shift due to the motion of source is toward observer is

$$\lambda' = \frac{v - v_s}{v}$$

While observed frequency due to motion of observer towards source is

$$v' = \frac{v + v_o}{\lambda'}$$

Putting the value of λ' in above equation

$$v' = v \left(\frac{v + v_o}{v - v_s} \right)$$

Similarly, we can find the expression for the source and observer moving away from each other

$$v' = v \left(\frac{v - v_o}{v + v_s} \right)$$

Combining the above expressions, we can write in general,

$$v' = v \frac{v \pm v_o}{(v \mp v_s)}$$

In this expression, the upper signs ($+v_o$ and $-v_s$) refer to motion of one toward the other, and the lower signs ($-v_o$ and $+v_s$) refer to motion of one away from the other.

For all Doppler-effect problems we can conveniently say that:

The word *toward* is related to an *increase* in observed frequency while the words *away from* are related to a *decrease* in observed frequency.

Both Doppler formulas that we derived here can approximate electromagnetic waves when their relative speeds are significantly slower than the speed of light.

Applications of Doppler Effect

Remember, the Doppler shift is commonly used in sound emission, with applications extending to ultrasound in medical diagnoses. **Ultrasound** operates at several MHz, allowing for precise

visualization of small distances within the body due to variations in tissue density affecting sound frequency.

In ultrasound, a source emits sound, and reflections are captured by a microphone or probe for computer analysis. In terms of the Doppler effect, it's employed to estimate **blood flow speed** in veins.

Marine applications utilize sound waves to determine the speed and direction of submarines underwater, a technique known as sonar. During World War II, sonar proved valuable for submarine detection, showcasing the practicality of sound waves in underwater scenarios where radio waves fall short.

Physics-PHY101-Lecture 20

WAVE MOTION

Introduction to waves:

Wave motion is a fundamental concept in physics, describing the propagation of disturbances or oscillations through a medium or space. Waves manifest in various forms, encompassing mechanical waves like sound and ocean waves, electromagnetic waves such as light and radio waves, and even quantum waves. Wavelength, amplitude, and frequency are fundamental concepts in the study of wave motion. Whether observing the rhythmic undulations of water, the transmission of sound through air, or the propagation of light through space, understanding wave motion is crucial in analyzing the dynamics of the natural world across a diverse range of phenomena.

Waves, including ocean waves, light waves, X-rays, and radio waves, exhibit common characteristics. When two frequencies combine, they generate additional frequencies, known as **beats**. Additionally, when waves from different sources interact, they can either cancel each other out (destructive interference) or reinforce each other (constructive interference). These principles apply universally across various types of waves.

Difference between constructive and destructive interference:

Constructive Interference	Destructive Interference
1) When the crest of one wave meets with the crest of another wave (or trough to trough) then constructive interference takes place.	1) When the crest of the one wave is meet with the trough of another wave then destructive interference is takes place
2) In the constructive interference when two waves meet each other the two waves are added up each other.	2) In destructive interference when two waves meet each other they cancel out each other.
3) The condition for the constructive interference is that the phase difference between two waves must be zero.	3) The condition for destructive interference is that the phase difference between superimposing waves is 180° .
4) The intensity of waves is increased in the constructive interference.	4) The intensity of light wave decreases in destructive interference.
5) The waves in the constructive interference are in phase.	5) The waves in the destructive interference are in out of phase.

Longitudinal Waves:

In longitudinal waves, medium vibrations align with the wave's direction of motion. If one end of a spring is pushed, the resulting wave travels through the entire spring. An earthquake

generates longitudinal waves in the ground, where one-part pushes another forward, causing the waves to propagate.

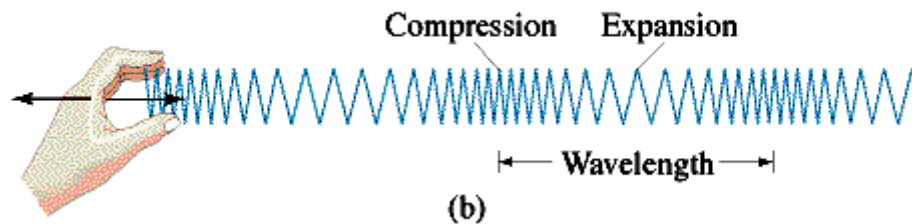


Figure 20. 1: Propagation of a longitudinal wave, showing compressions (high-pressure regions) and rarefactions (low-pressure regions) as the wave moves through a medium.

Transverse Waves:

In transverse waves, medium vibrations are perpendicular to the direction of motion. Transverse waves are evident in phenomena like water ripples when an object is placed on the water's surface, demonstrating upward and downward motion. In earthquake, both longitudinal and transverse waves are present. Light waves are another example of transverse waves.

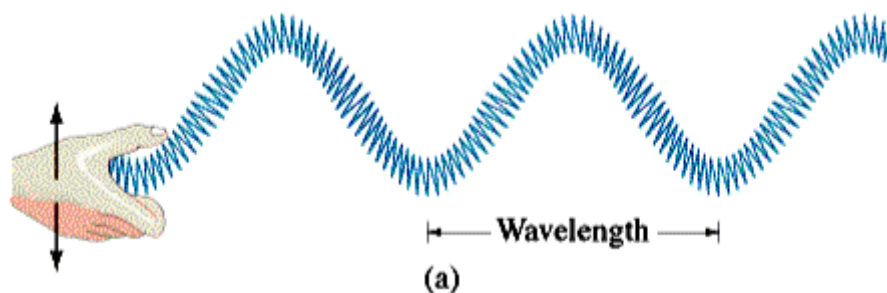


Figure 20. 2: Motion of a transverse wave, highlighting the perpendicular oscillation of particles relative to the direction of wave propagation within a medium.

Difference between Longitudinal and Transverse Waves:

<u>Longitudinal</u>	<u>Transverse</u>
The particles of the medium move in the same direction of the wave.	The particles of the medium are moving perpendicular to the direction of wave.
It acts in one dimension	It acts in two dimensions
The wave cannot be polarized or aligned.	The wave can be polarized or aligned
This wave can be produced in any medium such as gas, liquid or solid.	This wave can be produced in solid and liquid's surface.
Example: The earthquake P wave, spring waves and sound waves.	Example: Earthquake S wave, and light waves.

It is made of rarefactions and compressions.	It is made of troughs and crests.
--	-----------------------------------

Amplitude and Power in Waves:

The relationship between amplitude and power varies in different cases, making it challenging to provide a specific explanation. However, a general point can be made: "**The height of a wave is its amplitude,**" and "**the energy in a wave is proportional to the square of the amplitude.**" This implies that the power or energy of a wave is determined by squaring its amplitude. To illustrate, considering a harmonic oscillator with an extension x expressed as:

$$x = x_m \sin \omega t$$

The symbol " x_m " represents the amplitude, which is the maximum value a wave can attain. The potential energy $\left(\frac{1}{2}kx^2\right)$ and kinetic energy $\left(\frac{1}{2}mv^2 = \frac{1}{2}k\left(\frac{dx}{dt}\right)^2\right)$, when summed, are proportional to x_m^2 or the square of the amplitude. This relationship holds true for any wave, where the power or energy is directly proportional to the square of the amplitude.

Plane Waves

Let's consider generating sound waves using a piston, where the area of the piston is assumed to be infinite. By moving the piston back and forth, it produces planar waves that propagate uniformly in one direction. These waves exhibit infinite extent, with wavefronts spaced at a wavelength (λ). In this pressure wave, if the pressure is at a maximum at one wavefront, it will reach maximum again after one wavelength, creating an infinite train of plane waves moving in one direction.

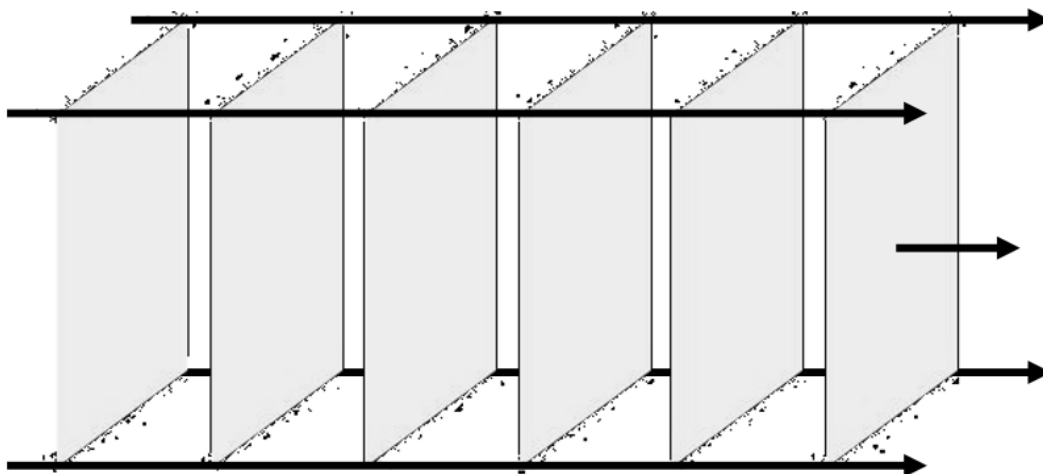


Figure 20. 3: Representation of plane waves. The planes represent wavefronts spaced one wavelength apart, and the arrows represent rays.

Spherical Waves

If sound is emitted spherically, as seen with a chirping sound, spherical waves are formed. Spherical waves decrease in intensity as they propagate, eventually dissipating to zero after traveling a certain distance. The rate of intensity decrease depends on the medium's energy absorption capacity. If the medium does not absorb energy, the produced energy remains within the sphere, creating different conditions for spheres with varying radii.

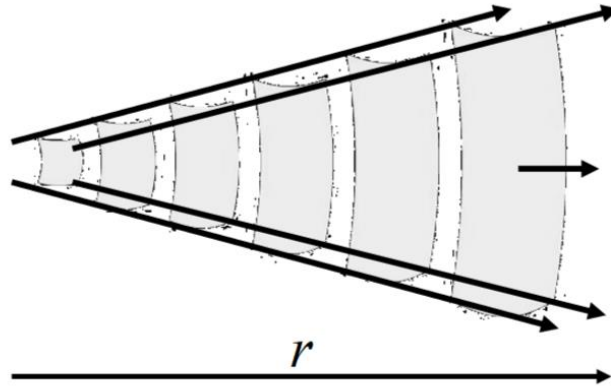


Figure 20. 4: Spherical wavefronts emanating from a point source, illustrating the outward propagation of energy in all directions uniformly.

$$\text{Power} \propto \frac{1}{r^2}$$

Let's consider two spheres with radii r_1 and r_2 , where the total power passing through both spheres is same. This implies that if power P pass through the first sphere, an equal amount passes through the second sphere. If there is no object in the middle absorbing sound, and only air is present, there is no decrease in intensity. The radiated power (P) can be expressed as,

$$P = 4\pi r^2 I \quad (\text{Area of sphere} = 4\pi r^2)$$

where “ I ” is the intensity. For the smaller sphere, the surface area is $4\pi r_1^2$, and the power (P_1) is:

$$P_1 = 4\pi r_1^2 I_1$$

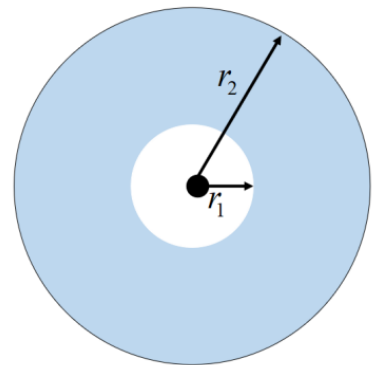
Similarly, for the larger sphere, with surface area $4\pi r_2^2$, the power (P_2) is:

$$P_2 = 4\pi r_2^2 I_2$$

$$P_1 = P_2 = P$$

$$\cancel{4\pi} r_1^2 I_1 = \cancel{4\pi} r_2^2 I_2$$

$$\frac{I_1}{I_2} = \frac{r_2^2}{r_1^2}$$



This is evidence that when r^2 increases, intensity—which is measured in W/m^2 —will also decrease.

Problem: Spherically sound waves are emitted uniformly in all directions from a point source, the radiated power P being 25 W. What is the intensity, and the sound level of the sound waves at distance $r = 2.5$ m from the source?

$$P = 25 \text{ W}, \quad r = 2.5 \text{ m}$$

$$P = 4\pi r^2 I$$

$$I = \frac{P}{4\pi r^2} = \frac{25}{4(3.14)(2.5)^2} = .32 \text{ W/m}^2$$

$$SL = 10 \log \frac{I}{I_0} = 10 \log \frac{.32}{10^{-12}} = 115 \text{ dB} \quad \therefore I_0 = 10^{-12} \text{ W / m}^2$$

Phase in Wave Cycles

Phase refers to the specific position in the cycle of a wave at a given point in time. Certainly, the sine wave follows a characteristic pattern, starting at zero, reaching its maximum at $\pi/2$, returning to zero at π , descending to negative values, hitting -1 at $3\pi/2$, and rising again to zero at 2π . The cosine wave is closely related to the sine wave, differing by a phase shift of $\pi/2$.

The concept of phase introduces a new dimension, representing the argument of a sine or cosine function. In a general wave like $\sin(kx - \omega t)$, adding a fixed angle to it introduces a constant phase difference, crucial in understanding wave behavior. The interaction of phases, whether added or subtracted, is significant, especially in the context of pure tones where they can either reinforce or cancel each other. Geometrically, phase can be visualized by examining the displacement (y) when x is zero.

$$y(x, t) = y_m \sin(kx - \omega t - \phi)$$

In the expression $(kx - \omega t - \phi)$ is termed as phase, with ϕ called the phase constant. The phase constant either moves forward or backward in space or time, and the displacement (y) can be expressed in two distinct manners.

$$y(x, t) = y_m \sin \left[k \left(x - \frac{\phi}{k} \right) - \omega t \right]$$

$$y(x, t) = y_m \sin \left[kx - \omega \left(t + \frac{\phi}{\omega} \right) \right]$$

Principle of superposition:

According to principle of superposition, if there are two separate sources producing waves with amplitudes y_1 and y_2 , the total amplitude is given by $y_1 + y_2$. This holds true in a linear system. The power obtained from this superimposed amplitude is proportional to $(y_1 + y_2)^2$.

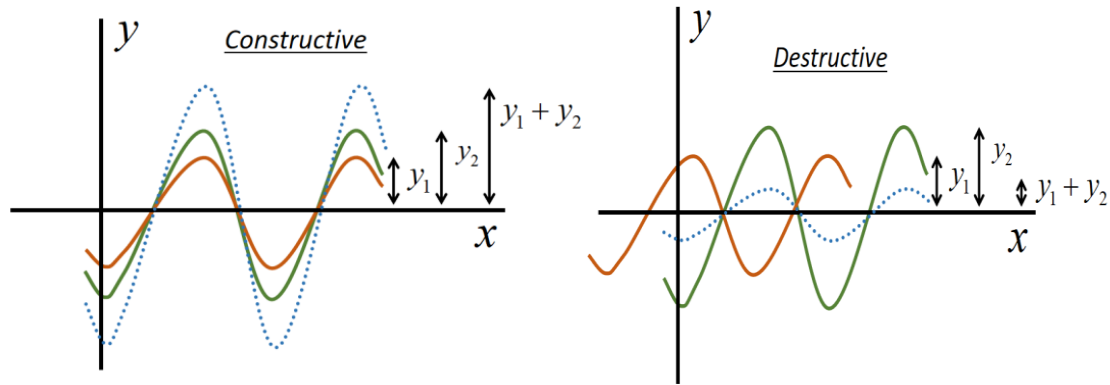


Figure 20. 5: Demonstration of constructive and destructive interference patterns: on the left, waves aligning in phase to produce amplification (constructive), and on the right, waves aligning out of phase leading to cancellation (destructive).

In the case of waves colliding, considering one-dimensional motion, the total amplitude is $y_1 + y_2$. However, it's important to note that at certain locations, this amplitude increases, while at others, it decreases when the waves overlap. This behavior remains consistent as long as the waves are separate and pass through each other while maintaining their original forms. The principle of superposition is applicable to all types of waves.

Example:

When two loudspeakers emit sine waves of a specific frequency or pure tone, and a person stands in front of them, the waves can interfere with each other as the person moves back and forth. Two waves ($y_1(x, t)$ and $y_2(x, t)$) having same magnitude y_m , same wave number k , same angular frequency ω but different phases (ϕ_1 and ϕ_2) are given by,

$$y_1(x, t) = y_m \sin(kx - \omega t - \phi_1)$$

And

$$y_2(x, t) = y_m \sin(kx - \omega t - \phi_2)$$

Resultant wave $y(x, t)$ by superposition principal is sum of two waves and can be expressed as

$$y(x, t) = y_1(x, t) + y_2(x, t)$$

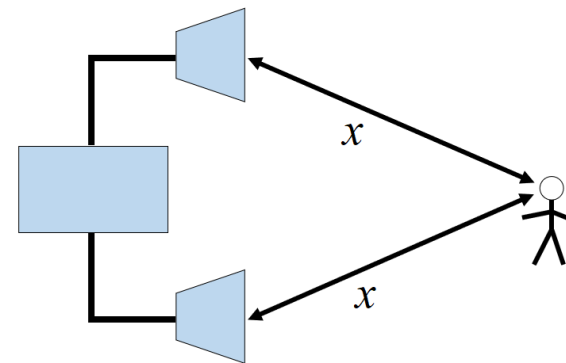
$$y(x, t) = y_m \sin(kx - \omega t - \phi_1) + y_m \sin(kx - \omega t - \phi_2)$$

$$y(x, t) = y_m [\sin(kx - \omega t - \phi_1) + \sin(kx - \omega t - \phi_2)]$$

By using trigonometric product identity given below,

$$\sin B + \sin C = 2 \cos\left(\frac{B - C}{2}\right) \sin\left(\frac{B + C}{2}\right)$$

$y(x, t)$ becomes,



$$\begin{aligned}
 y(x, t) &= y_m \left[2 \cos \left(\frac{(kx - \omega t - \phi_1) - (kx - \omega t - \phi_2)}{2} \right) \cdot \sin \left(\frac{(kx - \omega t - \phi_1) + (kx - \omega t - \phi_2)}{2} \right) \right] \\
 y(x, t) &= y_m \left[2 \cos \left(\frac{\phi_2 - \phi_1}{2} \right) \cdot \sin \left(\frac{2kx - 2\omega t - \phi_1 - \phi_1}{2} \right) \right] \\
 y(x, t) &= 2y_m \left[\cos \left(\frac{\phi_2 - \phi_1}{2} \right) \cdot \sin \left(kx - \omega t - \left(\frac{\phi_1 + \phi_1}{2} \right) \right) \right] \\
 y(x, t) &= \left[2y_m \cos \left(\frac{\Delta\phi}{2} \right) \right] \cdot \sin(kx - \omega t - \phi')
 \end{aligned}$$

Where $\Delta\phi = \phi_2 - \phi_1$

$$\phi' = \frac{(\phi_1 + \phi_2)}{2}$$

The first term can be considered as amplitude (as it constant for particular/fixed values of ϕ_1 and ϕ_2 and is independent of time t).

$$y(x, t) = \underbrace{\left[2y_m \cos \left(\frac{\Delta\phi}{2} \right) \right]}_{\text{amplitude}} \cdot \sin(kx - \omega t - \phi')$$

There are two distinct aspects of sine and cosine functions. For constructive interference, when the $\cos 0^\circ = 1$, the amplitude of sine is at its maximum. This occurs when two different frequencies combine, resulting in constructive interference. For constructive interference,

$$\frac{\Delta\phi}{2} = 0, \pi, 2\pi, \dots$$

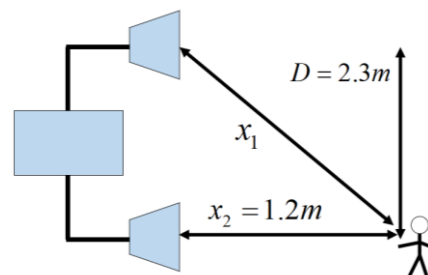
On the contrary, for destructive interference, when the cosine is zero (as seen with the $\cos 90^\circ = 0$), adding π again to phase, results in zero ($\cos 270^\circ = 0$), signifying destructive interference. If two waves cancel each other out, both amplitudes are equal, and this occurs when at the same distance from the loudspeaker. This principle applies more broadly without any specific limitations. For destructive interference,

$$\frac{\Delta\phi}{2} = \frac{\pi}{2}, \frac{3\pi}{2}, \dots$$

Problem: Two speakers, which are separated by a distance D of 2.3 m, emit a pure tone. The waves are in phase when they leave the speakers. For what wavelengths will the listener hear a minimum in the sound intensity, if the person is standing at the distance of 1.2 m from speaker 2?

To achieve destructive or constructive interference, we need to determine the required phase difference. Using Pythagoras theorem,

$$\text{Hypotenuse}^2 = \text{base}^2 + \text{perpendicular}^2$$



$$x_1^2 = x_2^2 + D^2$$

$$x_1 = \sqrt{x_2^2 + D^2} = 2.6m$$

$$x_1 - x_2 = (2.6 - 1.2)m = 1.4m$$

For destructive interference

$$\text{Phase difference} = \frac{\lambda}{2}, 3\frac{\lambda}{2}, 5\frac{\lambda}{2}, \dots$$

$$1.4m = \frac{\lambda}{2}, 3\frac{\lambda}{2}, 5\frac{\lambda}{2}, \dots$$

$$\lambda = 2.8m, 0.93m, 0.56m, \dots$$

Stationary Waves

Stationary waves are formed by the superposition of two waves with the same frequency and amplitude traveling in opposite directions.

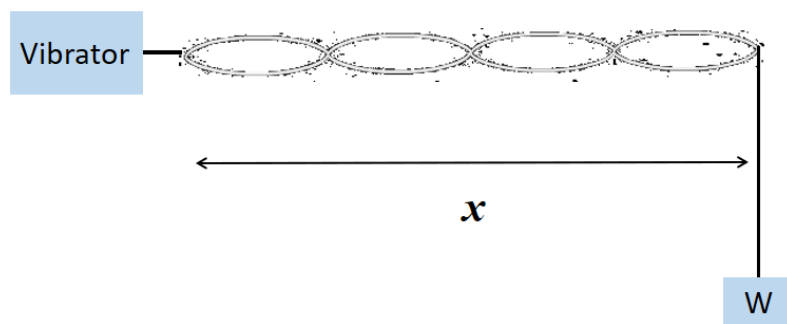


Figure 20. 6: A rope with a weight oscillating over a pulley, producing a standing wave pattern due to interference between waves traveling in opposite directions.

In a standing wave, some sections remain stationary (nodes), while adjacent sections experience maximum displacement (antinodes), forming a distinctive pattern. The resultant amplitude of stationary waves depends on the interference of the individual waves. The wave equations can be expressed as:

$$y_1(x, t) = y_m \sin(kx - \omega t)$$

$$y_2(x, t) = y_m \sin(kx + \omega t)$$

The difference in sign here shows that y_1 is the wave which is going in the positive x -direction while the other is in the negative x -direction.

The stationary wave is formed by the superposition of these two waves. So, the resultant displacement, $y(x, t)$, is given by:

$$\begin{aligned} y(x, t) &= y_1(x, t) + y_2(x, t) \\ &= y_m \sin(kx - \omega t) + y_m \sin(kx + \omega t) = y_m [\sin(kx - \omega t) + \sin(kx + \omega t)] \end{aligned}$$

Use the trigonometric identity $\sin(A + B) + \sin(A - B) = 2 \sin(A) \cos(B)$ to simplify the expression:

$$= \underbrace{[2y_m \sin kx]}_{\text{amplitude}} \cos \omega t$$

This is the equation for a stationary wave. It shows that the displacement (y) of the particles in the medium depends on both the spatial part ($\sin(kx)$) and the temporal part ($\cos(\omega t)$). The amplitude of the stationary wave is $2y_m$, which means it is twice the amplitude of the individual waves, leading to regions of constructive and destructive interference in the resulting wave pattern.

The locations of nodes occur at $kx = 0, \pi, 2\pi, 3\pi, \dots$, where the amplitude is zero. Conversely, at $kx = \pi/2, 3\pi/2, \dots$, there are antinodes, resulting in maximum amplitude. This proves that the standing waves formed exhibit both nodes and antinodes, and the distance between them can be calculated based on the points where the amplitude is at its maximum and minimum.

So far, we have only considered interference in terms of frequency. Now, let's explore what happens when two separate sources that are considered to be interfering generate pressure waves (sound waves) having same amplitude p_0 but different angular frequencies (ω_1 and ω_2).

$$\begin{aligned} \Delta p_1(t) &= p_0 \sin \omega_1 t \\ \Delta p_2(t) &= p_0 \sin \omega_2 t \\ \Delta p(t) &= \Delta p_1(t) + \Delta p_2(t) \\ &= p_0 (\sin \omega_1 t + \sin \omega_2 t) \quad \therefore \sin \alpha + \sin \beta = 2 \cos \left(\frac{\alpha - \beta}{2} \right) \sin \left(\frac{\alpha + \beta}{2} \right) \\ &= \left[2p_0 \cos \left(\frac{\omega_1 - \omega_2}{2} t \right) \right] \sin \left(\frac{\omega_1 + \omega_2}{2} t \right) \end{aligned}$$

The argument of sin is $(\omega_1 + \omega_2)/2$ representing the average of frequency, while the argument of cosine is $(\omega_1 - \omega_2)/2$. This difference in frequencies is called **beat frequency**.

$$\begin{aligned} \bar{\omega} &= \frac{\omega_1 + \omega_2}{2} \\ \omega_{\text{diff}} &= \frac{\omega_1 - \omega_2}{2} \quad \therefore p_o = p_m \\ \Delta p(t) &= [2\Delta p_m \cos \omega_{\text{diff}} t] \sin \bar{\omega} t \\ \omega_{\text{beat}} &= 2\omega_{\text{diff}} = \omega_1 - \omega_2 \end{aligned}$$

Consider two audio generators fixed to separate amplifiers. Turning on one generator produces a sound, and when switched to the second generator, a different sound is heard. Our ears cannot easily discern the frequency difference between them. However, when both generators are turned on simultaneously, two frequencies are generated: the beat frequency and the sum of

frequencies. As we observe, when the frequencies of the generators are nearly equal, a beat frequency is noticeable, characterized by a slow modulation.

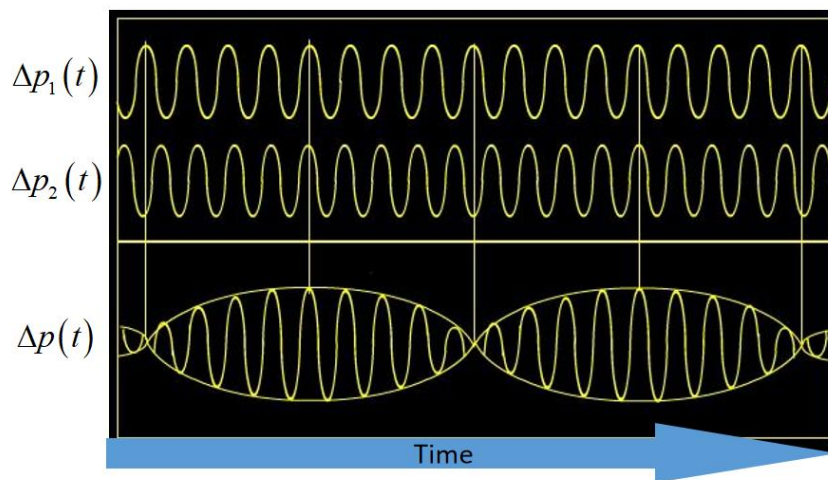


Figure 20. 7: Phenomenon of beats formation, showing the periodic variation in amplitude resulting from the interference of two waves of slightly different frequencies.

Applications of Beats Phenomenon:

The beat phenomenon helps in tuning musical instruments where the frequencies of two tones become nearly equal, producing an audible beat. Another application is in radio modulation. Since audio frequencies range from 20 cycles/s to 20,000 cycles/s, and radio waves operate in kHz and MHz, the beat phenomenon is used for modulation. By adding an audio frequency to a carrier wave, a new frequency is formed, such as combining a 20 kHz frequency with a 20 mega cycle radio frequency. This creates a high-frequency wave suitable for efficient radiation, and upon reaching the radio, it undergoes demodulation.

Speed of sound

A mass attached to a spring exhibits oscillations when pulled and released. The frequency (f) of oscillation is determined by $f = \frac{1}{2\pi} \sqrt{\frac{k}{m}}$, where k is the spring constant (representing the force with which the spring tries to return to its original position based on its stiffness) and m is the mass. Sound waves obey a similar principle when a particle is displaced from its position; it strives to return to its original position. The force exerted in this process is calculated using bulk modulus (B). The velocity of sound (v) is related to bulk modulus and density (ρ) through the formula $v = \sqrt{\frac{B}{\rho}}$. Sound speed varies in different mediums, being faster in liquids and even more so in solids.

Medium	Sound speed (ms^{-1})
Gases	
Air (0°C)	330

Helium	965
Hydrogen	1274
Liquids	
Water (0°C)	1402
Solids	
Aluminium	6420
Steel	5941

Pulse Propagation:

Sound typically propagates in the form of pulses (a short burst of a wave or a localized disturbance) rather than pure tones. Pulse shapes can vary, gradually diminishing or ending at different points. At time $t = 0$, pressure distribution is observed as a function of x . The shape of the pulse is represented by $f(x)$. Moving with the pulse at the speed of sound (v) without changing $f(x)$ is crucial.

At time, the original shape of the function $f(x)$ remains the same.

$$y(x, 0) = f(x)$$

At time t it is displaced by an amount vt

$$y(x, t) = f(x') = f(x - vt)$$

$$\text{where, } x' = x - vt$$

$$x - vt = \text{constant}$$

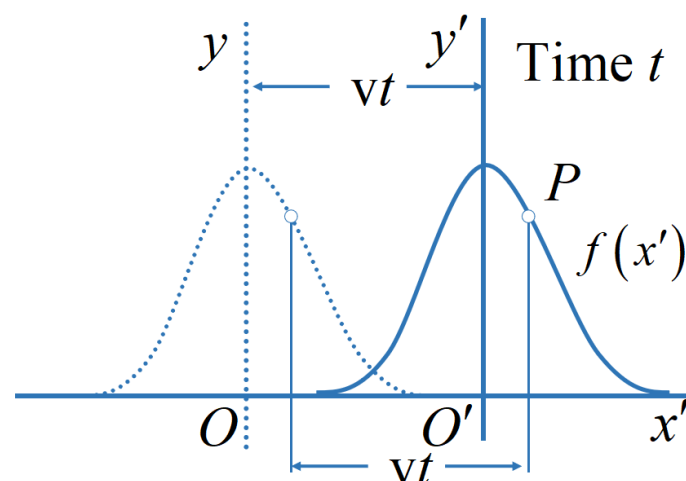


Figure 20. 8: Propagation of a single pulse without change in shape along a medium.

It implies that if you move with speed v , the pulse will continue to move with you, preserving its shape.

$$x - vt = \text{constant}$$

Differentiate w.r.t time

$$\frac{dx}{dt} - v = 0$$

$$\frac{dx}{dt} = v$$

The speed at which the pulse maintains its shape is called **phase velocity**. Phase velocity is independent of frequency, meaning it remains constant regardless of the frequency of the pure tone. Uniform speed for each frequency is crucial to avoid variations in arrival times.

Dispersion in Medium:

Dispersion occurs when there is a correlation between frequency and speed, leading to different frequencies arriving at varying times. **Dispersive medium** is one where different waves move at different speeds. In such a medium, a pulse loses its shape over time, undergoing alterations in its shape. Short peak pulses will spread and flatten over time in a dispersive medium. In cold air, the speed of sound is approximately 330 ms^{-1} . If a moving car emits a sound wave that reaches you, you can deduce that the car is approaching if its speed is slower than the speed of sound (330 m/s). But what happens if it's moving faster? In such a scenario, it is termed supersonic motion.

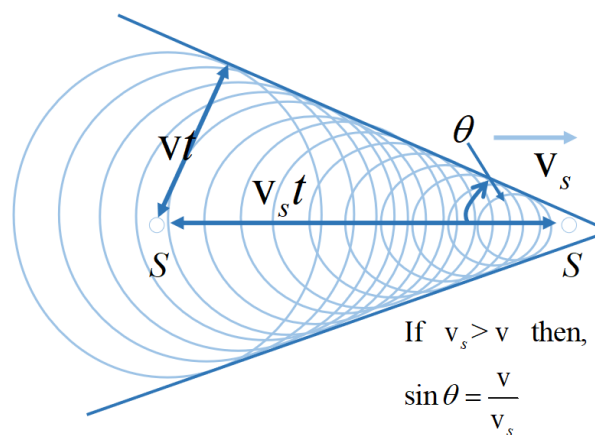


Figure 20. 9: Formation and propagation of shock waves.

Shock Waves:

Supersonic motion occurs when the source moves at a speed surpassing that of sound. This leads to the generation of shock waves or shock fronts, resulting in explosion-like noises. The associated angles (denoted as θ) involve sine θ and cosine θ . The quantification of these angles and their trigonometric functions is expressed through what are known as Mach numbers. **Mach numbers** serve as a numerical representation of the ratio between the speed of a moving object and the speed of sound, providing a standardized measure for describing the intensity and characteristics of supersonic motion. For example, if an object can fly two or three times faster than the speed of sound, it is a Mach Two or Mach Three aircraft.

Applications of Shock Waves:

Shock waves carry substantial energy and find applications in various fields. One practical application is the use of ultrasound shock waves to break down kidney stones, offering a non-surgical solution. The example of using ultrasound shock waves illustrates the potential benefits derived from a profound understanding of wave dynamics.

LECTURE 21

Gravity

Introduction:

Two events were considered very important during early times:

1. Why every thing fall towards the earth.
2. Why stars revolve.

It was assumed that these two occurrences had no connection.

Aristotle proposed that because Earth is the mother of everything, it attracts all other objects to itself. Approximately 2000 years ago, Ptolemy coined the idea of Earth being the centre of the universe and that all celestial objects rotate around it. In contrast, in the 16th century, Nicholas Copernicus proposed that the Sun is the universe's centre and that the Earth and other celestial bodies orbit around it. In 17th century, Galileo Galilei took his observations and verified the theory of Nicholas Copernicus using his own telescope.

In this lecture we will study the concept of gravity and it's effect on the objects.

Newton's Law of Universal Gravitation:

It states that the force of attraction between two masses m_1 and m_2 is:

$$F \propto \frac{m_1 m_2}{r^2} \quad (21.1)$$

and is directed along the line joining the two bodies. Putting in a constant of proportionality,

$$F = G \frac{m_1 m_2}{r^2} \quad (21.2)$$

Now let's be a bit careful of the direction of the force. Looking at the **fig. (21.1)** below,

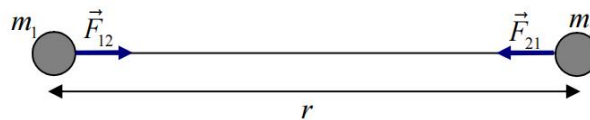


Fig. (21.1): Two masses m_1 and m_2 Exerting force on each other

$\vec{F}_{21}$ = Force on m_1 by m_2 , $\vec{F}_{12}$ = Force on m_2 by m_1 , Both forces are equal in magnitude $|\vec{F}_{12}| = |\vec{F}_{21}| = F$, but opposite in direction which can be represented by Newton's Third Law, $\vec{F}_{12} = -\vec{F}_{21}$.

The Gravitational Constant:

The gravitational constant G is a very small quantity and needs a very sensitive experiment. An early experiment to find G involved suspending two masses and measuring the attractive force. Suppose the distance between these two masses is L . From the **fig. (21.2)** we can work out an expression for the gravitational torque:

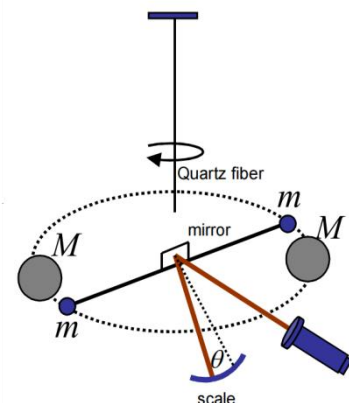


Fig. (21.2): Finding

$$\begin{aligned}\tau &= r \times F \\ &= rF \sin \theta\end{aligned}\quad (21.3)$$

Where F is the gravitational force between two masses, $r = \frac{L}{2}$ and $\theta = 90^\circ$. We will multiply this expression by 2 because there are two masses m_1 and m_2 .

$$\tau = 2 \left(\frac{GmM}{r^2} \right) \frac{L}{2} \quad (21.4)$$

A thread provides the restoring torque $\kappa\theta$. κ is the torsion in a thread. The deflection θ can be measured by observing the beam of the light reflected from the small mirror. In equilibrium the torques balance,

$$\frac{GmML}{r^2} = \kappa\theta \quad (21.5)$$

Hence,

$$G = \frac{\kappa\theta r^2}{mML} \quad (21.6)$$

How to find κ ?

In **Lecture No. 16**, we learned that a mass suspended on a string will oscillate and its oscillating frequency depends on torque. Suppose the mass is suspended, it will oscillate and κ can be found from observing the period of free oscillations,

$$T = 2\pi\sqrt{\frac{I}{k}} \implies \kappa = \frac{2\pi^2 I}{T^2} \quad (21.7)$$

with $I = \frac{mL^2}{2}$. The modern value is $G = 6.67259 \times 10^{-11} \text{Nm}^2/\text{kg}^2$.

Mass, Volume and Density of the Earth:

The magnitude of the force with which the Earth attracts a body of mass m towards its centre is,

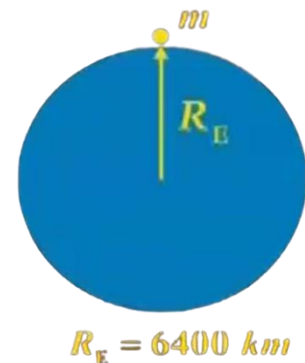
$$F = G \frac{mM_E}{R_E^2} \quad (21.8)$$

where $R_E = 6400 \text{ km}$ is the radius of the Earth and M_E is the mass. It does not depend on the quality of matter, i.e. iron, wood, leather, etc. Objects feel the force in proportion to their masses. According to Newton's 2nd law of motion, if a body falls freely, then it will accelerate with gravitational acceleration. So,

$$F = mg = \frac{GmM_E}{R_E^2} \quad (21.9)$$

We measure g , the acceleration due to gravity, as 9.8m/s^2 . From this we can immediately deduce the Earth's mass by substituting $R_E = 6400 \text{ km}$ and $G = 6.67259 \times 10^{-11} \text{Nm}^2/\text{kg}^2$:

$$\begin{aligned}M_E &= \frac{gR_E^2}{G} \\ &= \boxed{5.97 \times 10^{24} \text{kg}}\end{aligned} \quad (21.10)$$



What a remarkable achievement!

We can still do more. We can find the volume of the Earth as:

$$\begin{aligned}V_E &= \frac{4}{3}\pi R_E^3 \\ &= \boxed{1.08 \times 10^{21} \text{m}^3}\end{aligned} \quad (21.11)$$

Hence the density of the Earth is:

$$\begin{aligned}\rho_E &= \frac{V_E}{M_E} \\ &= \boxed{5462 \text{ kgm}^{-3}}\end{aligned}$$

So this is 5.462 times greater than the density of water and tells us that the earth must be quite dense inside.

The Gravitational Potential:

The gravitational potential is an important quantity. It is the work done in moving a unit mass from infinity to a given point R, and equals:

$$\begin{aligned}V(r) &= \frac{U(r)}{m} \\ &= -\frac{GM}{R}\end{aligned}\quad (21.12)$$

Proof:

We need to find the work done in bringing a unit mass from the surface of the Earth to a point at infinity (A point where net force is zero). According to conservation of energy,

$$dV = -Fdr \quad (21.13)$$

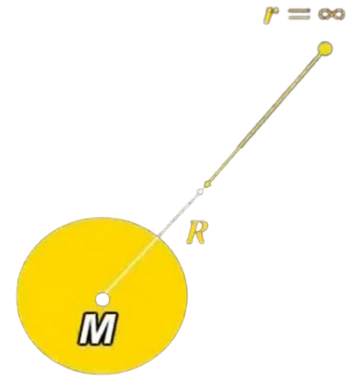
Integrate both sides:

$$\int_{V(R)}^0 dV = - \int_R^\infty F(r)dr \quad (21.14)$$

$$0 - V(R) = GM \int_R^\infty \frac{dr}{r^2} \quad (21.15)$$

$$V(R) = -GM \left[\frac{1}{r} \right]_R^\infty \quad (21.16)$$

$$V(R) = -\frac{GM}{R} \quad (21.17)$$



Negative indicates that the work is done on the object. This work is now stored in the object in the form of gravitational potential energy.

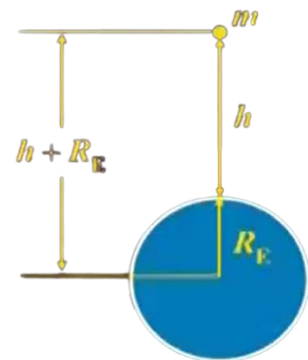
Change in Potential Energy:

Using Eq. (21.17), let us calculate the change in potential energy ΔU when we raise a body of mass to a height above the Earth's surface. Let R_E be the radius of the earth and $h+R_E$ is the distance from the center of the Earth as shown in the figure. Hence potential energy ΔU at height h from the surface of the earth will be:

$$\Delta U = U(R_E + h) - U(R_E) \quad (21.18)$$

$$\Delta U = GMm \left(\frac{1}{R_E} - \frac{1}{R_E + h} \right) \quad (21.19)$$

$$\begin{aligned}&= GMm \left(1 - \frac{1}{1 + \frac{h}{R_E}} \right) \\ &= GMm \left(1 - \left(1 + \frac{h}{R_E} \right)^{-1} \right)\end{aligned}\quad (21.20)$$



Now suppose that the distance is much smaller than the Earth's radius. So, for $h \ll R_E$, we can use **binomial approximation**¹.

¹ Binomial approximation:

$$(1 + x)^a \approx 1 + ax$$

when, x and a are real numbers and $|x| < 1$

$$\left(1 + \frac{h}{R_E}\right)^{-1} = \left(1 - \frac{h}{R_E}\right) \quad (21.21)$$

So we find,

$$GMm \left(1 - \left(1 + \frac{h}{R_E}\right)^{-1}\right) = GMm \left(1 - \left(1 - \frac{h}{R_E}\right)\right) \quad (21.22)$$

$$= m \left(\frac{GM}{R_E}\right) h \quad (21.23)$$

$$= mgh \quad (21.24)$$

Escape Velocity:

We can use the expression for potential energy and the law of conservation of energy to find the minimum velocity needed for a body to escape the Earth's gravity. Far away from the Earth, the potential energy is zero, and the smallest value for the kinetic energy is zero. Requiring that:

$$(K.E + P.E)_{r=R} = (K.E + P.E)_{r=\infty} \quad (21.25)$$

Gives,

$$\frac{1}{2}mv_e^2 - \frac{GMm}{R_E} = 0 + 0 \quad (21.26)$$

From this,

$$v_e = \sqrt{\frac{2GM}{R_E}} \quad (21.27)$$

$$= \sqrt{2gR_E} \quad (21.28)$$

Putting in some numbers we find that for the Earth,

$v_e = 11.2 \text{ km/s}$ and for the Sun, $v_e = 618 \text{ km/s}$.

For a Black Hole, the escape velocity is so high that nothing can escape, even if it could move with the speed of light! (Nevertheless, Black Holes can be observed because when matter falls into them, a certain kind of radiation is emitted.)

$$v_e = \sqrt{\frac{2GM}{R}}$$

Put $v_e = c$

$$c = \sqrt{\frac{2GM}{R}}$$

$$c^2 = \frac{2GM}{R}$$

$$R = \frac{2GM}{c^2}$$

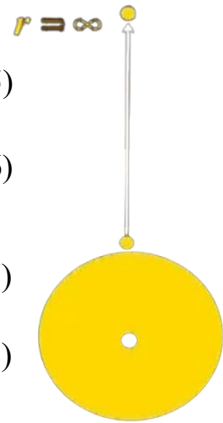
This value of R is known as Schwarzschild radius of a black hole. If a photon (particle of light) is put into the Schwarzschild radius, it will not be able to escape and ultimately it will fall into the black hole. It is interesting to note that the Schwarzschild radius is outside the black hole.

Revolution of a Satellite:

A satellite is in circular orbit over the Earth's surface. The condition for equilibrium,

$$\frac{mv_0^2}{r} = \frac{GMm}{r^2} \quad (21.29)$$

$$= \sqrt{\frac{GM}{r}} \quad (21.30)$$



If R_E is the Earth's radius, and h is the height of the satellite above the ground, then $r = R_E + h$. Hence,

$$v_0 = \sqrt{\frac{GM}{R_E + h}} \quad (21.31)$$

For $h \ll R_E$, we can approximate,

$$v_0 = \sqrt{\frac{GM}{R_E}} \quad (21.32)$$

$$= \sqrt{gR_E} \quad \because \left(g = \frac{GM}{R_E^2}\right) \quad (21.33)$$

We can easily calculate the time for one complete revolution,

$$T = \frac{2\pi}{\omega} \quad (21.34)$$

$$= \frac{2\pi r}{v_o} \quad (21.35)$$

$$= 2\pi r \sqrt{\frac{r}{GM}} \quad (21.36)$$

$$= \frac{2\pi}{\sqrt{GM}} r^{\frac{3}{2}} \quad (21.37)$$

This gives the important result, observed by Kepler nearly 3 centuries ago that,

$$T^2 = \frac{4\pi^2}{GM} r^3 \quad (21.38)$$

Or $T^2 \propto r^3$

Total Energy of a Satellite:

What is the total energy of a satellite moving in a circular orbit around the earth? Clearly, it has two parts, kinetic and potential. Remember that the potential energy is negative. So,

$$E = K.E + P.E \quad (21.39)$$

$$= \frac{1}{2}mv_0^2 - \frac{GM_E m}{r} \quad (21.40)$$

But, $v_0^2 = \frac{GM}{r}$ as we saw earlier and therefore,

$$E = \frac{1}{2} \frac{GMm}{r} - \frac{GMm}{r} \quad (21.41)$$

$$= -\frac{1}{2} \frac{GMm}{r} \quad (21.42)$$

Note that the magnitude of the potential energy is larger than the kinetic energy. If it wasn't, the satellite would not be bound to the Earth!

The negative sign indicates that the work has to be done on the satellite when taken from a point on the Earth to a point at infinity.

Geostationary Satellite:

Geo means "Earth" and stationary means "at rest". A satellite which revolve around the earth with rotation frequency equal to that of the Earth, (1 rotation = 24 hr). It appears to be stationary with respect to the Earth.

Radius (the distance from center of the Earth) of this satellite can be found:

$$T^2 = \frac{4\pi^2}{GM} r^3 \quad (21.43)$$

Rearranging to find r. Multiply and divide by R

$$r^3 = \left(\frac{GM}{R} \right) \frac{RT^2}{4\pi^2} \quad (21.44)$$

$$= g \frac{RT^2}{4\pi^2} \quad (21.45)$$

Taking cube root on both sides:

$$r = \sqrt[3]{g \frac{RT^2}{4\pi^2}} \quad (21.46)$$

$$= \boxed{42.33 \times 10^6 \text{ m}}$$

Height of the satellite from the surface of earth:

$$r - R = 35.93 \times 10^6 \text{ m}$$

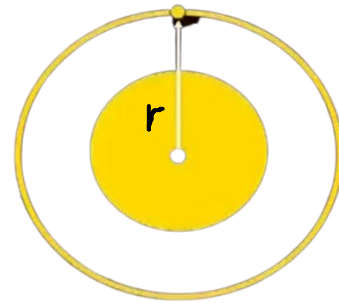


Fig. (21.3): Satellite at a distance r from the centre of the Earth

Kepler's Laws:

German astronomer Johannes Kepler deduce a mathematical model for the motion of the planets. Kepler's complete analysis of planetary motion is summarized in three laws known as Kepler's laws.

- Kepler's 1st Law:**

It states that: All planets move in elliptical orbits with the Sun at one focus.

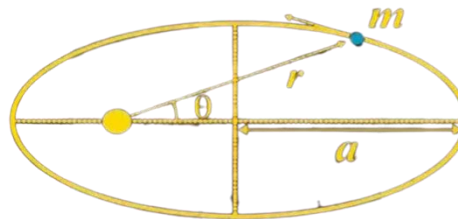


Fig. (21.4): Motion of planets in elliptical orbit

It is also known as Law of Orbits.

- Kepler's 2nd Law:**

A famous discovery of the astronomer Johann Kepler some 300 years ago says that, the line joining a planet to the Sun sweeps out equal areas in equal intervals of time.

We can easily see this from the conservation of angular momentum. Call ΔA the area swept out in time Δt . Then from the figure we can see that,

$$\Delta A = \frac{1}{2} r(r\Delta\theta) \quad (21.47)$$

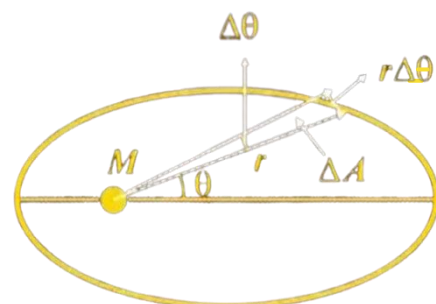
Divide this by Δt and then take the limit where it becomes very small.

$$\frac{dA}{dt} = \lim_{\Delta t \rightarrow 0} \frac{\Delta A}{\Delta t} \quad (21.48)$$

$$= \frac{1}{2} r^2 \left(\lim_{\Delta t \rightarrow 0} \frac{\Delta\theta}{\Delta t} \right) \quad (21.49)$$

$$= \frac{1}{2} r^2 \omega \quad (21.50)$$

Recall from [Lecture No. 13](#), [Angular](#)



momentum² is the momentum multiplied by the perpendicular distance.

$$\begin{aligned}
 \frac{dA}{dt} &= \frac{1}{2} r^2 \omega \\
 &= \frac{1}{2} r^2 \left(\frac{v}{r} \right) && \because (v = r\omega) \\
 &= \frac{1}{2} r m v && \because (\text{Multiply and Divide by } m) \\
 &= \frac{1}{2} r p && \because (p = mv) \\
 &= \frac{L}{2m} && \because (L = rp)
 \end{aligned}$$

Since L is a constant, let's see why. Angular momentum changes only when torque acts on it. Remember from **previous lectures** that:

$$\frac{dL}{dt} = \tau$$

If $\tau = 0$ then $\frac{dL}{dt} = 0$ and L is a constant. $\tau = 0$ because the force acting on the Sun due to the planet, is central means it is acting along the radial direction towards the center of the Sun, hence:

$$\begin{aligned}
 \vec{r} \times \vec{F} &= \left(\frac{GMm}{r^2} \right) \vec{r} \times \hat{r} \\
 &= 0
 \end{aligned}$$

We have proved one of Kepler's laws (with so little effort)!

• Kepler's 3rd Law:

Also known as Law of Periods. It states that: The square of the period of revolution of any planet is proportional to the cube of the planet (21.52) distance from the Sun. Lets prove this result for circular orbits.

Proof:

Suppose a planet orbits around the sun. The gravitational force acting towards the center of the sun will be equal to the **centripetal force**³ acting on the planet. Hence:

$$\begin{aligned}
 F_g &= F_c \\
 \frac{GMm}{r^2} &= m\omega r^2
 \end{aligned}$$

Where $\omega = \frac{2\pi}{T}$.

$$\frac{GM}{r^2} = \left(\frac{2\pi}{T} \right)^2 r, \text{ or} \quad (21.53)$$

$$T^2 = \left(\frac{4\pi^2}{GM} \right) r^3 \quad (21.54)$$

Example:

Suppose a planet revolves around a sun in elliptical orbit. If it has a speed v_1 at a distance d_1 from the Sun. What will be its speed at distance d_2 ?

Solution:

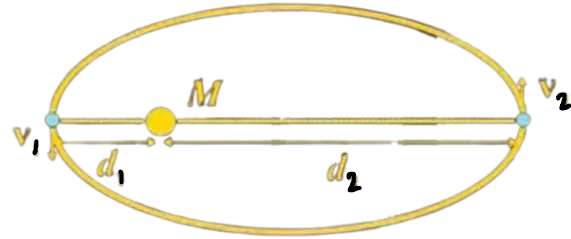
Using the law of conservation of angular momentum:

² Angular momentum: $L = r \times p$

³ Centripetal force: $F = \frac{mv^2}{r} = m\omega r^2$. More detail in Lecture No. 11

$$mv_1d_1 = mv_2d_2 \quad (21.55)$$

$$v_1 = \frac{v_2d_2}{d_1} \quad (21.56)$$



According to conservation of energy:

$$-\frac{GMm}{d_1} + \frac{1}{2}mv_1^2 = -\frac{GMm}{d_2} + \frac{1}{2}mv_2^2 \quad (21.57)$$

Substituting v_1 in above equation to find v_2 ,

$$\begin{aligned} -\frac{GMm}{d_1} + \frac{1}{2}m\left(\frac{v_2d_2}{d_1}\right)^2 &= -\frac{GMm}{d_2} + \frac{1}{2}mv_2^2 \quad (21.58) \\ \frac{1}{2}m\frac{v_2^2d_2^2}{d_1^2} - \frac{1}{2}mv_2^2 &= \frac{GMm}{d_1} - \frac{GMm}{d_2} \\ \frac{1}{2}mv_2^2\left(\frac{d_2^2}{d_1^2} - 1\right) &= GMm\left(\frac{1}{d_1} - \frac{1}{d_2}\right) \\ \frac{1}{2}mv_2^2\left(\frac{d_2^2 - d_1^2}{d_1^2}\right) &= GMm\left(\frac{d_2 - d_1}{d_1d_2}\right) \\ v_2^2 &= \frac{2GMm}{m}\left(\frac{d_1^2(d_2 - d_1)}{d_1d_2(d_2^2 - d_1^2)}\right) \end{aligned}$$

Using $a^2 - b^2 = (a + b)(a - b)$ in denominator:

$$\begin{aligned} v_2^2 &= 2GM\left(\frac{d_1(d_2 - d_1)}{d_2(d_2 - d_1)(d_2 + d_1)}\right) \\ v_2^2 &= 2GM\left(\frac{d_1}{d_2(d_2 + d_1)}\right) \\ v_2 &= \sqrt{\frac{2GMd_1}{d_2(d_1 + d_2)}} \end{aligned}$$

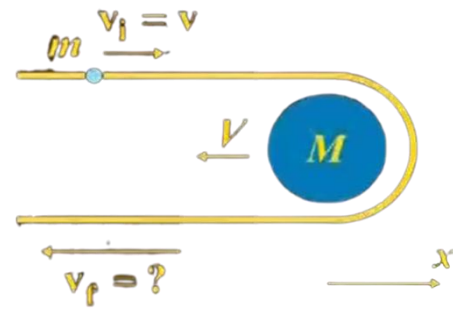
Substituting value in Eq. (21.55)

$$\begin{aligned} mv_1d_1 &= mv_2d_2 \\ &= m\sqrt{\frac{2GMd_1}{d_2(d_1 + d_2)}}d_2 \\ &= m\sqrt{\frac{2GMd_1d_2^2}{d_2(d_1 + d_2)}} \\ &= m\sqrt{\frac{2GMd_1d_2}{(d_1 + d_2)}} \end{aligned}$$

The Sling Shot Effect:

Sling shot effect is a non-impact collision used to give an extra boost to spacecraft. This collision is one dimensional elastic collision.

Figure shows a planet with mass M and velocity V . A satellite of mass m moving with velocity v_i is directed to pass near the Planet to increase its velocity to v_f . Such that $M \gg m$ and V remains constant.



Energy and momentum remain conserved during elastic collision in one dimension. In Lecture No. 10, we already proved that the relative velocity before collision is equal to the relative velocity after collision.

$$v_i + V = -(v_f + V) \quad (21.59)$$

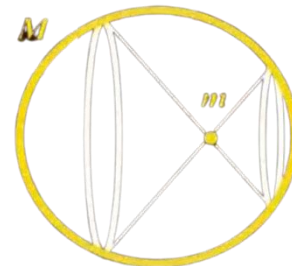
$$v_f = -(v + 2V) \quad (21.60)$$

It is important to note that “A uniformly dense spherical shell attracts an external point mass as if the mass of the shell were concentrated at its center”, as shown in the figure below.



- Gravitational effect of spherical distribution of matter:**

Another important point to note is, “A uniform spherical shell of matter exerts no gravitational force on a particle located inside it.” In other words, the sum of all the forces acting on a mass m at any point inside a uniform spherical shell is, zero.



Variation in Gravitational Acceleration (g):

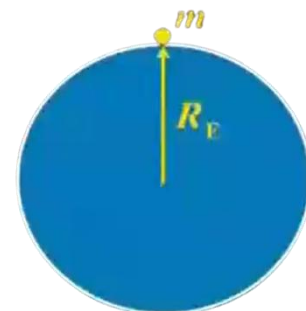
- At the Surface of the Earth:**

The value of g at the surface of the earth can be found using Newton's law of universal gravitation. The magnitude of force with which Earth attracts the body towards its centre is:

$$F = mg = \frac{GM_E m}{R_E^2} \quad (21.61)$$

$$g = \frac{GM_E}{R_E^2} \quad (21.62)$$

Note that the value of g will be constant all over the surface for a perfectly spherical Earth because it is inversely proportional to R_E^2 .



- At some height h above the surface:**

The magnitude of force with which Earth attracts the body towards its centre is:

$$F = \frac{GM_E m}{(h + R_E)^2} \quad (21.63)$$

According to Newton's second law:

$$F = mg_h = \frac{GM_E m}{(h + R_E)^2} \quad (21.64)$$

$$g_h = \frac{GM_E}{(h + R_E)^2} \quad (21.65)$$

From here, we can find the effective value of g:

$$g_h = \frac{GM_E}{(h + R_E)^2} \quad (21.66)$$

$$= \frac{GM_E}{R_E^2 \left(1 + \frac{h}{R_E}\right)^2} \quad (21.67)$$

$$= \frac{\left(\frac{GM_E}{R_E^2}\right)}{\left(1 + \frac{h}{R_E}\right)^2} \quad (21.68)$$

$$= \frac{g}{\left(1 + \frac{h}{R_E}\right)^2} \quad (21.69)$$

For $h \ll R_E$, or $\frac{h}{R_E} \ll 1$

$$g_h = g \left(1 + \frac{h}{R_E}\right)^{-2} \quad (21.70)$$

$$= g \left(1 - \frac{2h}{R_E}\right) + \dots \quad (21.71)$$

For $h = 0$, $g_h = g$

As discussed in the section of the sling shot effect, when a body of mass m is on the surface or above the surface of the Earth, the Earth works like a point particle, as if its total mass is concentrated at its centre.

- **Below the surface of the Earth:**

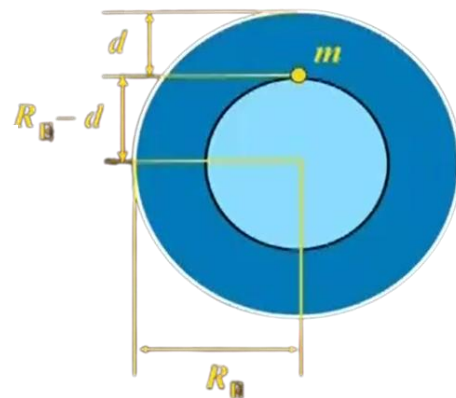
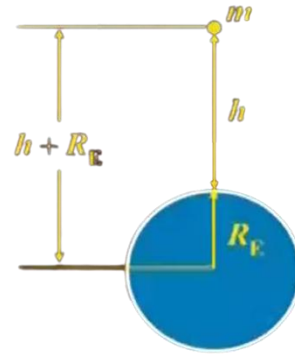
As the particle lies inside the shell of radius d , there is no force on the particle due to the shell. The only force exerted comes from the sphere of radius $R_E - d$. The mass of this smaller sphere is:

$$M'_E = \rho_E \left(\frac{4}{3}\pi(R_E - d)^3\right) \quad (21.72)$$

But the density of this sphere is:

$$\rho_E = \frac{M_E}{\left(\frac{4}{3}\pi R_E^3\right)} \quad (21.73)$$

Now we can find out the effective acceleration due to gravity. The force acting on the mass m will be:



$$F = mg_d = \frac{GM'_E m}{(R_E - d)^2} \quad (21.74)$$

$$g_d = \frac{G [\rho_E (\frac{4}{3}\pi(R_E - d)^3)]}{(R_E - d)^2} \quad \therefore (\text{Substitute Eq. (21.72)}) \quad (21.75)$$

$$= G\rho_E \left(\frac{4}{3}\pi(R_E - d) \right) \quad (21.76)$$

$$= G\rho_E \frac{4}{3}\pi R_E \left(1 - \frac{d}{R_E} \right) \quad \therefore (\text{Take } R_E \text{ common}) \quad (21.76)$$

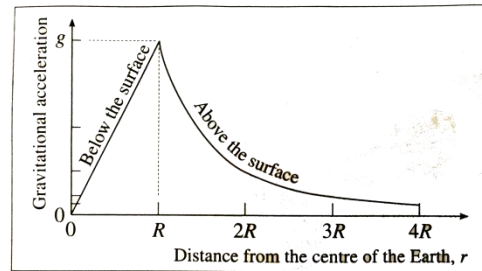
Now multiply and divide by R_E^2

$$g_d = \frac{G\rho_E \frac{4}{3}\pi R_E^3 \left(1 - \frac{d}{R_E} \right)}{R_E^2} \quad \therefore (\text{Multiply and Divide by } R_E^2)$$

$$= \frac{GM_E}{R_E^2} \left(1 - \frac{d}{R_E} \right) \quad \therefore (\text{Substitute } M_E \text{ from Eq. (21.73)})$$

$$g_d = g \left(1 - \frac{d}{R_E} \right) \quad \therefore (\text{Substitute } g = \frac{GM_E}{R_E^2})$$

From the above equation, we realize that $g_d = g$ when $d = 0$. If we increase d then g_d decreases which implies that effective gravitational acceleration decreases as we go below the surface of the Earth. At the center of the Earth where $d = R_E$ the value of $g_d = 0$. Hence there is no gravitational force at the center of the Earth as shown in the figure.



Variation of the gravitational acceleration due to the Earth with distance from the centre of the Earth

Lecture 23

Electrostatics II

Introduction to Charges:

In the last lecture, we discussed the fundamental concept of charges. Charges are entities that exert forces on each other. These forces are governed by certain principles. One key fact is that charge is conserved, meaning the total amount of charge remains constant in any closed system.

Coulomb's Law:

Coulomb's Law describes the force between two-point charges. If there are two charges, q_1 and q_2 , separated by a distance r , the force F between them is given by:

$$F = k \frac{q_1 q_2}{r^2}$$

where k is Coulomb's constant. This force is a vector quantity, meaning it has both magnitude and direction. The direction of the force is along the line connecting the two charges. Importantly, if charge q_1 exerts a force on charge q_2 , charge q_2 exerts an equal and opposite force on charge q_1 .

Electric Fields:

Previously we have discussed the concept of electric fields. When a charge is present in an electric field, it experiences a force. The strength of this force depends on the strength of the field. If a charge Q is placed in an electric field E , the force F acting on the charge is:

$$F = Q E$$

Continuous Charge Distributions:

In previous studies, the focus has been on point charges, which are idealized models where the entire charge is assumed to be concentrated at a single point. However, in many real-world scenarios, charges are distributed over a larger region of space rather than being concentrated

at a single point. Understanding how to deal with these continuous charge distributions is crucial for accurately calculating the electric fields they produce.

Summing Up Contributions from Small Elements:

When dealing with continuous charge distributions, the electric field at any point in space is the result of contributions from all infinitesimal charge elements within the distribution. This is expressed as:

$$E = \Delta E_1 + \Delta E_2 + \Delta E_3 + \cdots + \Delta E_n$$

Here, ΔE_n represents the infinitesimal electric field contribution from the n-th small charge element. Summing these contributions gives the total electric field E .

Transition to Integration:

Since the charge distribution is continuous, the sum of the contributions from all infinitesimal charge elements is represented by an integral:

$$E = \sum \Delta E_n$$

This summation can be approximated by an integral for continuous distributions:

$$E = \int dE$$

Vector Components of the Electric Field:

The electric field $\vec{E}$ is a vector quantity, meaning it has both magnitude and direction. It can be decomposed into its Cartesian components along the x, y, and z axes:

$$\vec{E} = E_x \hat{i} + E_y \hat{j} + E_z \hat{k}$$

Where:

- E_x , E_y and E_z are the components of the electric field in the x, y, and z directions, respectively.

- $\hat{i}$, $\hat{j}$, and $\hat{k}$ are the unit vectors in the x, y, and z directions.

Calculating Each Component:

To find the total electric field, you need to integrate the contributions to each component over the entire charge distribution:

$$E_x = \int dE_x$$

$$E_y = \int dE_y$$

$$E_z = \int dE_z$$

These integrals take into account the continuous distribution of charge and calculate the contributions to the electric field in each direction.

Linear Charge Distribution:

Consider a line of charge where charges are continuously distributed along its length. To find the electric field at a point P due to this line of charge, we break the line into small segments and sum the contributions of each segment.

Suppose we have a line of charge with charge density λ , which is charge per unit length. If we take a small segment of length ds , the charge dq on this segment is:

$$dq = \lambda ds$$

By summing the contributions of all these small segments, we can determine the total electric field at point P .

Surface and Volume Charge Distributions:

In two dimensions, we consider surface charge distributions. Here, charges are spread over a surface, such as a sheet of paper. The charge density σ is charge per unit area. For a small area element dA , the charge dq is:

$$dq = \sigma dA$$

In three dimensions, we consider volume charge distributions. Here, charges are distributed throughout a volume. The charge density ρ is charge per unit volume. For a small volume element dV , the charge dq is:

$$dq = \rho dV$$

Calculating Electric Fields for Different Distributions:

To find the electric field due to a continuous charge distribution, we integrate over the entire distribution. For a line charge, the electric field E at a point can be calculated using:

$$E = \int \frac{k dq}{r^2} \hat{r}$$

where dq is the charge element, r is the distance from the charge element to the point, and $\hat{r}$ is the unit vector pointing from the charge element to the point.

Electric Field Due to a Ring of Charge:

Now, consider a ring of charge with radius R . We want to find the electric field along the axis of the ring. By symmetry, we only need to consider the z-component of the electric field. If we break the ring into small segments, each segment contributes a small electric field dE :

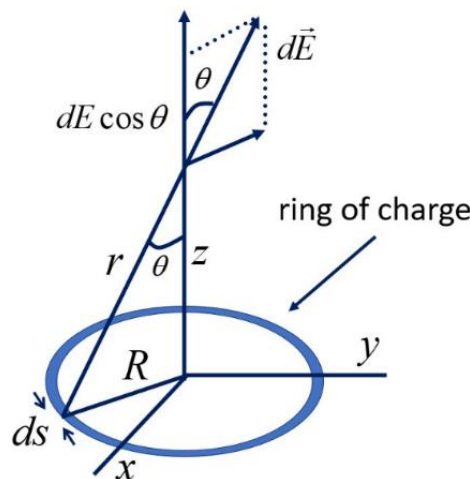


Fig 23. 1: The electric field of a charged ring points along the ring axis, since field components perpendicular to the axis cancel in pairs.

Starting with an infinitesimal electric field element dE :

$$dE = \frac{1}{4\pi\epsilon_0} \frac{\lambda ds}{r^2} = \frac{\lambda ds}{4\pi\epsilon_0 (z^2 + R^2)}$$

This expression gives the infinitesimal electric field dE due to a small charge segment ds of the ring, where:

- λ is the linear charge density (charge per unit length).
- ds is the infinitesimal segment length of the ring.
- r is the distance from the charge segment to the point where the field is being calculated.
- ϵ_0 is the permittivity of free space.

The distance r is given by $r = \sqrt{z^2 + R^2}$, where z is the distance along the axis from the centre of the ring to the point of interest, and R is the radius of the ring.

Component of dE along the z-axis dE_z :

$$dE_z = dE \cos \theta$$

Here, $\cos \theta$ is the cosine of the angle between the electric field vector and the z-axis. This angle θ can be calculated using the geometry of the ring.

Expression for $\cos \theta$:

$$\cos \theta = \frac{z}{r} = \frac{z}{(z^2 + R^2)^{1/2}}$$

This step uses the definition of $\cos \theta$ in terms of the z-component and the hypotenuse r .

Substituting $\cos \theta$ into dE_z :

$$dE_z = \frac{z\lambda ds}{4\pi\epsilon_0 (z^2 + R^2)^{3/2}}$$

This equation represents the infinitesimal component of the electric field along the z-axis due to a small segment ds of the charged ring.

Integrating dE_z around the ring:

$$E_z = \frac{z\lambda}{4\pi\epsilon_0(z^2 + R^2)^{3/2}} \int ds$$

Since the ring is symmetrical, we integrate dE_z over the entire circumference of the ring, where $\int ds = 2\pi R$, the circumference of the ring.

Solving the integral:

$$E_z = \frac{z\lambda(2\pi R)}{4\pi\epsilon_0(z^2 + R^2)^{3/2}}$$

Simplifying, we obtain:

$$E_z = \frac{qz}{4\pi\epsilon_0(z^2 + R^2)^{3/2}}$$

Here, $q = \lambda(2\pi R)$ is the total charge of the ring.

Electric field for $z \gg R$:

When the point of interest is far along the z-axis $z \gg R$, the expression simplifies. Under this approximation:

$$E_z \approx \frac{qz}{4\pi\epsilon_0 z^3} = \frac{q}{4\pi\epsilon_0 z^2}$$

This is the electric field along the axis of the ring when the observation point is far from the ring. It resembles the electric field of a point charge q at a distance z . The steps provide the derivation of the electric field along the axis of a charged ring, starting from the contribution of a small segment and integrating over the entire ring, leading to a simplified form for distances much larger than the ring radius.

Electric Field Due to a Disk of Charge:

Consider a uniformly charged disk with charge density σ . To find the electric field at a point along the axis of the disk, we divide the disk into concentric rings and integrate over these rings:

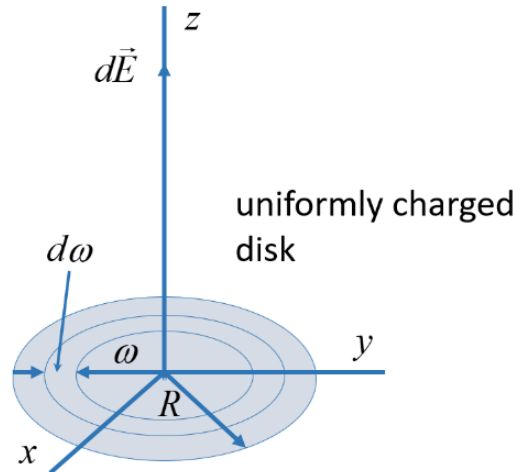


Fig 23. 2: A uniformly charged disk of radius R . The field on the z -axis due to an infinitesimally small ring element at a radius ω with thickness $d\omega$.

To find the electric field at a point along the axis of a uniformly charged disk, we divide the disk into concentric rings and integrate over these rings. Let's break down the given steps in detail:

Surface Charge Density and Elemental Charge:

- The disk has a uniform surface charge density σ , which is the charge per unit area.
- Consider an infinitesimally small ring element at a radius ω with thickness $d\omega$.
- The area of this ring element dA is given by the circumference of the ring ($2\pi\omega$) times the thickness ($d\omega$).

$$dA = 2\pi\omega d\omega$$

- The charge dq on this ring element is:

$$dq = \sigma dA = \sigma(2\pi\omega)d\omega$$

Electric Field Contribution by the Ring Element:

To find the electric field dE_z at a point along the axis (distance z from the disk centre) due to this ring element, we use Coulomb's law.

The vertical component dE_z of the electric field due to the ring element is:

$$dE_z = \frac{z\sigma 2\pi\omega d\omega}{4\pi\epsilon_0 (z^2 + \omega^2)^{3/2}}$$

Simplifying the expression:

$$dE_z = \frac{\sigma z}{4\epsilon_0} (z^2 + \omega^2)^{-3/2} (2\omega) d\omega$$

Integrating Over All Rings:

To find the total electric field E_z along the axis, integrate dE_z from $\omega=0$ to $\omega=R$ (the radius of the disk):

$$E_z = \int dE_z = \frac{\sigma z}{4\epsilon_0} \int_0^R (z^2 + \omega^2)^{-3/2} (2\omega) d\omega$$

Solving the Integral:

Evaluate the integral:

$$\int_0^R \frac{2\omega d\omega}{(z^2 + \omega^2)^{3/2}}$$

Using a suitable substitution ($\omega = z \tan \theta$), this integral evaluates to:

$$\int_0^R \frac{2\omega d\omega}{(z^2 + \omega^2)^{3/2}} = \frac{2}{z^2} \left[\sqrt{z^2 + \omega^2} \right]_0^R = \frac{2}{z^2} (\sqrt{z^2 + R^2} - z)$$

Final Electric Field Expression:

Plugging the evaluated integral back:

$$E_z = \frac{\sigma z}{4\epsilon_0} \cdot \frac{2}{z^2} \left(\sqrt{z^2 + R^2} - z \right) = \frac{\sigma}{2\epsilon_0} \left(1 - \frac{z}{\sqrt{z^2 + R^2}} \right)$$

Special Cases:**1. When $z \ll R$:**

If $z \ll R$, the distance z is much smaller than the radius R , implying the point is very close to the disk:

$$E_z = \frac{\sigma}{2\epsilon_0} \left(1 - \frac{z}{\sqrt{z^2 + R^2}} \right)$$

$$E_z = \frac{\sigma}{2\epsilon_0}$$

This is the same as the electric field near an infinite sheet of charge.

2. When $z \gg R$:

If $z \gg R$, the distance z is much larger than the radius R , implying the point is far from the disk:

$$E_z = \frac{\sigma}{4\epsilon_0} \frac{R^2}{z^2}$$

Since $q = \sigma\pi R^2$ (total charge on the disk):

$$E_z = \frac{q}{4\pi\epsilon_0 z^2}$$

This is the same as the electric field due to a point charge q at a distance z . This process involves breaking the disk into infinitesimal rings, calculating the electric field contribution from each ring, and integrating these contributions to find the total electric field along the axis

of the disk. The results match known cases for very close and very far distances, providing confidence in the calculations.

Electric Field Due to infinite Line of Charge:

To calculate the electric field due to an infinite line of charge, consider a line of charge distributed along the z -axis. We want to find the electric field at a point P on the y -axis. Here's a step-by-step explanation of the process:

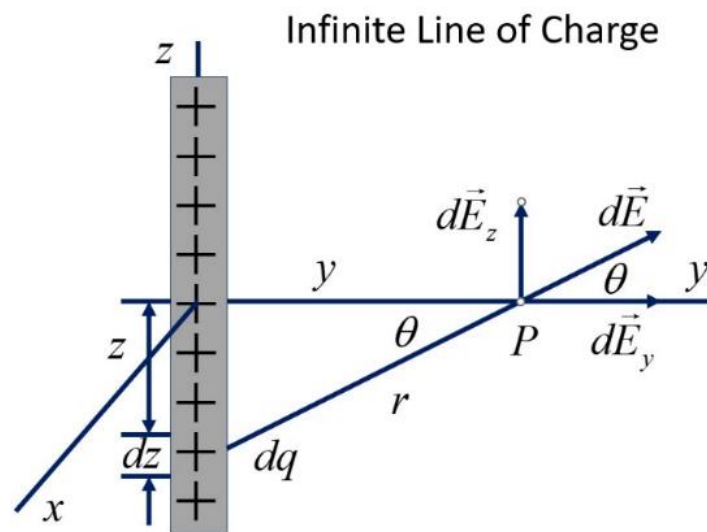


Fig 23. 3: The field of a charged line is the vector sum of the fields from all the individual charge elements dq along the line.

Linear Charge Density:

Let the linear charge density be λ , which is the charge per unit length.

Electric Field Contribution by an Elemental Segment:

Consider a small segment of the line of charge dz at a distance z from the origin. The charge dq on this segment is:

$$dq = \lambda dz$$

The distance r from this segment to the point on the y -axis at distance y is given by:

$$r = \sqrt{y^2 + z^2}$$

The electric field dE due to this charge segment at the point is:

$$dE = \frac{1}{4\pi\epsilon_0} \frac{dq}{r^2} = \frac{1}{4\pi\epsilon_0} \frac{\lambda dz}{y^2 + z^2}$$

Components of the Electric Field:

The electric field dE can be decomposed into two components: dE_y along the y-axis and dE_z along the z-axis. Due to symmetry, the z-components from opposite segments cancel out, leaving only the y-components. The y-component dE_y is:

$$dE_y = dE \cos \theta = \frac{1}{4\pi\epsilon_0} \frac{\lambda dz}{y^2 + z^2} \frac{y}{\sqrt{y^2 + z^2}} = \frac{1}{4\pi\epsilon_0} \frac{\lambda y dz}{(y^2 + z^2)^{3/2}}$$

Integration Over the Entire Line:

To find the total electric field E_y , integrate dE_y from $z = -\infty$ to $z = \infty$:

$$E_y = \int_{z=-\infty}^{z=\infty} dE_y = \frac{\lambda y}{4\pi\epsilon_0} \int_{z=-\infty}^{z=\infty} \frac{dz}{(y^2 + z^2)^{3/2}}$$

Simplifying the Integral:

Use the substitution $z = y \tan \theta$, hence $dz = y \sec^2 \theta d\theta$:

$$E_y = \frac{\lambda y}{4\pi\epsilon_0} \int_{-\pi/2}^{\pi/2} \frac{y \sec^2 \theta d\theta}{y^3 \sec^3 \theta} = \frac{\lambda y}{4\pi\epsilon_0 y^2} \int_{-\pi/2}^{\pi/2} \cos \theta d\theta$$

Simplifying further:

$$E_y = \frac{\lambda}{4\pi\epsilon_0 y} [\sin \theta]_{-\pi/2}^{\pi/2} = \frac{\lambda}{4\pi\epsilon_0 y} (1 - (-1)) = \frac{\lambda}{2\pi\epsilon_0 y}$$

Final Result and Symmetry:

The electric field due to an infinite line of charge at a point on the y-axis is given by:

$$E = \frac{\lambda}{2\pi\epsilon_0 y}$$

Given the cylindrical symmetry of the problem, this result can be generalized to any point at a perpendicular distance r from the line of charge:

$$E = \frac{\lambda}{2\pi\epsilon_0 r}$$

By breaking the infinite line of charge into infinitesimal segments, calculating their contributions to the electric field, and integrating over the entire length of the line, we find that the electric field at a perpendicular distance r from an infinite line of charge is inversely proportional to the distance r and depends on the linear charge density λ .

Corona Discharge:

Corona discharge is an electrical discharge brought about by the ionization of a fluid such as air surrounding a conductor that is electrically charged. It occurs when the electric field around the conductor becomes sufficiently strong to ionize the surrounding air or other gaseous medium, but not strong enough to cause a complete electrical breakdown or spark.

Conditions for Corona Discharge:

- **High Voltage:** The conductor must be at a high voltage relative to its surroundings.
- **Sharp Points or Edges:** Corona discharge is more likely to occur at sharp points or edges on the conductor because the electric field is concentrated in these areas.
- **Gas Type and Pressure:** The surrounding gas and its pressure influence the onset of corona discharge. Air is the most common medium, but other gases like SF₆ (sulphur hexafluoride) are used in high-voltage equipment to suppress corona.

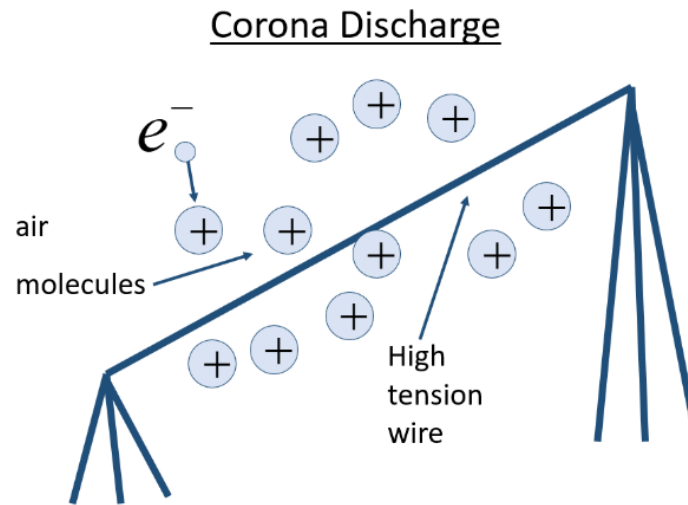


Fig 23. 4: The hissing noise at high voltage transmission lines is due to corona discharge. Corona discharge occurs due to high voltage which causes surrounding medium (air) to ionized.

Electric Flux:

Electric flux is a measure of the number of electric field lines passing through a surface. If we have an electric field E passing through a surface area A at an angle θ , the flux Φ_E is given by:

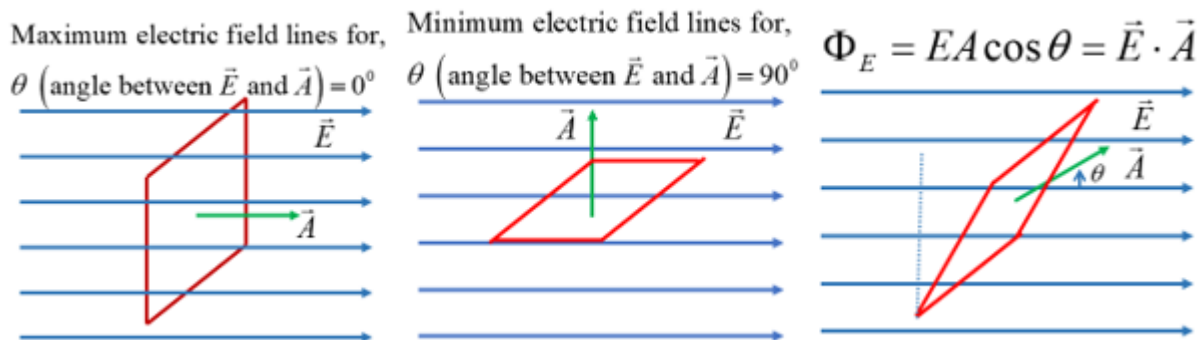


Fig 23. 5: Field lines for a uniform electric field through an area A for different angles to the field. Because the number of lines that go through the shaded area A is the same as the number that go through A , we conclude that the flux through A is equal to the flux through A and is given by $\Phi_E = EA \cos \theta$.

$$\Phi_E = EA \cos \theta$$

For a curved surface, we sum the flux through small area elements dA :

$$\Phi_E = \int E \cdot dA$$

$$\Phi_E = \sum \vec{E}_i \cdot \Delta \vec{A}_i$$

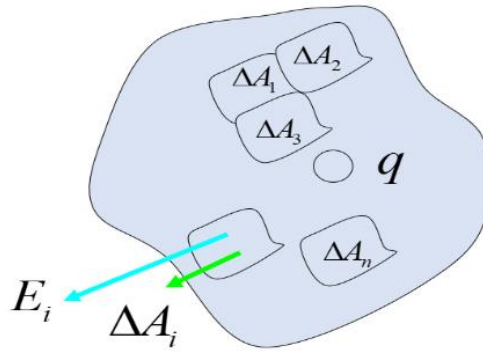


Fig 23. 6: Flux for a curved surface will be the sum of the flux through small area elements.

Gauss's Law:

Gauss's Law is a fundamental principle in electromagnetism that relates the electric flux through a closed surface to the charge enclosed by that surface. It is one of Maxwell's four equations, which form the foundation of classical electromagnetism.

Gauss's Law states that the net electric flux through any closed surface is equal to the charge enclosed divided by the permittivity of the medium. Mathematically, it is expressed as:

$$\Phi_E = \int E \cdot dA = \frac{Q_{\text{enclosed}}}{\epsilon_0}$$

where:

- $\int E \cdot dA$ is the surface integral of the electric field E over a closed surface S .
- dA is a vector representing an infinitesimal area on the surface S , pointing outward.
- Q_{enclosed} is the total electric charge enclosed within the surface S .
- ϵ_0 is the permittivity of free space (vacuum permittivity).

Explanation:

Gauss's Law essentially states that the total electric flux through a closed surface is proportional to the total charge enclosed within that surface. The electric flux is a measure of the number of electric field lines passing through a given area. If more charge is enclosed, more field lines emanate from or terminate at the charges, resulting in greater flux.

1. Electric Field of a Point Charge:

Consider a point charge q located at the origin. The electric field $\vec{E}$ at a distance r from the charge is given by Coulomb's Law:

$$E = \frac{q}{4\pi\epsilon_0 r^2} \hat{r}$$

where $\hat{r}$ is the unit vector pointing radially outward from the charge.

2. Flux Through a Spherical Surface:

Imagine a spherical surface of radius r centered around the charge. The surface area of the sphere is $4\pi r^2$. The electric field is radially symmetric and has the same magnitude at all points on the surface.

The flux Φ_E through the spherical surface is:

$$\Phi_E = \int E \cdot d\vec{A}$$

Since E and $d\vec{A}$ are parallel and $|E|$ is constant over the sphere, this simplifies to:

$$\Phi_E = E \int d\vec{A} = E \times 4\pi r^2$$

Substituting $E = \frac{q}{4\pi\epsilon_0 r^2}$:

$$\Phi_E = \left(\frac{q}{4\pi\epsilon_0 r^2} \right) \times 4\pi r^2 = \frac{q}{\epsilon_0}$$

3. Generalization:

For a general distribution of charges, consider a closed surface S enclosing a total charge Q_{enclosed} . By the superposition principle, the total flux through S is the sum of the fluxes due to each individual charge. Hence:

$$\Phi_E = \int E \cdot dA = \frac{Q_{\text{enclosed}}}{\epsilon_0}$$

This concludes the derivation, showing that the electric flux through any closed surface is proportional to the enclosed charge, as stated by Gauss's Law.

Applications:

1. **Symmetry:** Gauss's Law is particularly powerful for calculating electric fields of symmetric charge distributions (e.g., spherical, cylindrical, or planar symmetry).
2. **Conductors:** Helps determine the distribution of charge on conductors and the resulting electric field.
3. **Capacitors:** Used to analyse the electric field and potential in capacitors.
4. **Electrostatics:** Fundamental in solving electrostatic problems and understanding the behaviour of electric fields in various configurations.

Electric Field Due to infinite Parallel Plates:

application of Gauss's Law in finding the electric field between two infinite parallel plates with uniform surface charge densities:

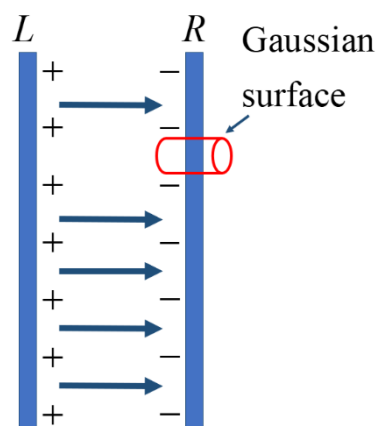


Fig 23. 7: Electric field $\vec{E}$ between two infinite parallel plates with uniform surface charge densities. A cylindrical Gaussian surface surrounds a parallel plate.

1. Electric Flux Definition:

$$\Phi = \int \vec{E} \cdot d\vec{A}$$

This integral defines the electric flux Φ through a closed surface, where $\vec{E}$ is the electric field and $d\vec{A}$ is a differential area vector on the surface.

2. Decomposing the Flux:

$$\Phi = \int_{\text{outer cap}} \vec{E} \cdot d\vec{A} + \int_{\text{inner cap}} \vec{E} \cdot d\vec{A} + \int_{\text{side walls}} \vec{E} \cdot d\vec{A}$$

This equation breaks down the total electric flux through the Gaussian surface into three parts:

- The flux through the outer cap.
- The flux through the inner cap.
- The flux through the side walls.

3. Evaluating Each Component:

$$\Phi = 0 + EA + 0$$

For the chosen Gaussian surface (a cylindrical "surface" that straddles one of the plates):

- The flux through the outer cap is zero because the electric field outside the plates cancels out.
- The flux through the inner cap is EA because the electric field E is perpendicular to the surface and uniform over the area A of the cap.
- The flux through the side walls is zero because the electric field is parallel to these surfaces and doesn't contribute to the flux.

4. Relating Flux to Charge:

$$\varepsilon_0 \Phi = q$$

According to Gauss's Law, the electric flux Φ through a closed surface is equal to the enclosed charge q divided by the permittivity of free space ε_0 .

5. Enclosed Charge in Terms of Surface Charge Density:

$$\varepsilon_0 EA = \sigma A$$

The charge q enclosed by the Gaussian surface is the surface charge density σ times the area A of the inner cap.

6. Solving for the Electric Field:

$$E = \frac{\sigma}{\varepsilon_0}$$

Simplifying this equation gives the electric field E between the plates.

Electric Field Due to a long, straight wire:

Consider a straight wire with a constant charge distributed along its length with charge density λ , we use a cylindrical Gaussian surface with radius r and length L .

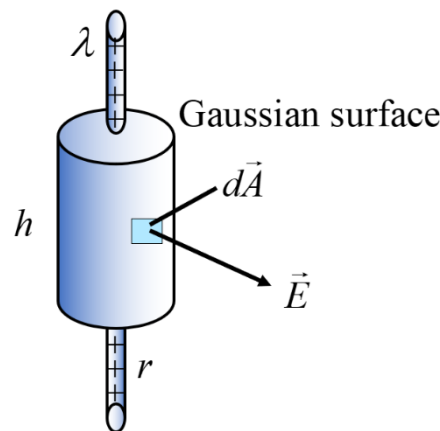


Fig 23. 8: A cylindrical Gaussian surface surrounds a line charge.

1. **Gaussian Surface:** We create an imaginary cylindrical Gaussian surface around the wire to simplify our calculations. The surface is closed, and we can choose its shape to our advantage.
2. **Calculating the Electric Flux:**
 - The flux through the cylindrical surface is the product of the electric field and the surface area of the cylinder.

1. **Gauss's Law:**

$$\epsilon_0 \int \vec{E} \cdot d\vec{A} = q$$

This is the integral form of Gauss's Law, where ϵ_0 is the permittivity of free space, $\vec{E}$ is the electric field, $d\vec{A}$ is a differential area element, and q is the charge enclosed by the Gaussian surface.

2. **Choosing the Gaussian Surface:** For a long, straight wire with a uniform linear charge density λ (charge per unit length), we choose a cylindrical Gaussian surface with:
 - Radius r
 - Height h This surface is coaxial with the wire.
3. **Calculating the Electric Flux:** The flux through the Gaussian surface is:

$$\epsilon_0 \int \vec{E} \cdot d\vec{A}$$

Since the electric field $\vec{E}$ is radial and perpendicular to the surface of the cylinder, and is uniform at a distance r from the wire, this simplifies to:

$$\epsilon_0 E (2\pi r h)$$

Here, $2\pi r$ is the circumference of the cylinder's base, and h is the height of the cylinder.

4. **Enclosed Charge:** The charge enclosed by the Gaussian surface is:

$$q = \lambda h$$

Here, λ is the linear charge density, and h is the height of the cylinder.

5. **Applying Gauss's Law:** According to Gauss's Law:

$$\epsilon_0 E (2\pi r h) = \lambda h$$

Simplifying this equation to solve for $\vec{E}$:

$$E = \frac{\lambda}{2\pi\epsilon_0 r}$$

For a point charge q at the centre of a sphere of radius r , the electric field is radial and has the same magnitude at every point on the surface. The flux through the sphere is:

$$\oint \vec{E} \cdot d\vec{A} = E \cdot 4\pi r^2 = \frac{q}{\epsilon_0}$$

Solving for E , we get:

$$E = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2}$$

This result is consistent with Coulomb's Law.

Example.3: Electric Field Due to Infinite Sheet of Charge:

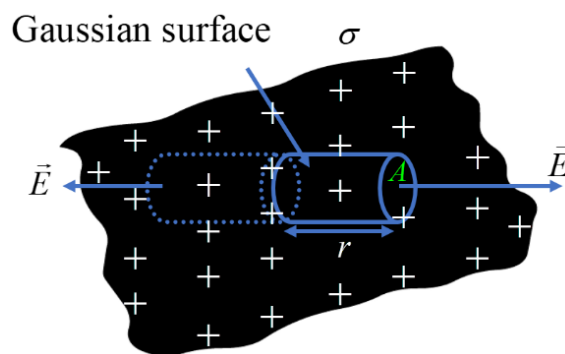


Fig 23. 9: A cylindrical Gaussian surface penetrating an infinite sheet of charge. (b) A cross section of the same Gaussian cylinder. The flux through each end of the Gaussian surface is EA . There is no flux through the cylindrical surface.

We use Gauss's Law to determine the electric field near an infinite plane of charge.

1. **Gauss's Law:**

$$\epsilon_0 \int \vec{E} \cdot d\vec{A} = q$$

This is the integral form of Gauss's Law, where ϵ_0 is the permittivity of free space, $\vec{E}$ is the electric field, $d\vec{A}$ is a differential area element on the Gaussian surface, and q is the charge enclosed by the surface.

2. **Choosing the Gaussian Surface:** For an infinite plane of charge with a surface charge σ (charge per unit area), we choose a Gaussian surface in the shape of a rectangular box (a "surface") that extends symmetrically on both sides of the plane.
3. **Calculating the Electric Flux:** The total electric flux through the Gaussian surface is the sum of the flux through the two caps (top and bottom) of the surface.

$$\epsilon_0 (EA + EA)$$

Since the electric field $\vec{E}$ is perpendicular to the plane and uniform over each cap with area $\vec{A}$, the flux through each cap is EA . There are two such caps, so the total flux is $EA + EA$.

4. **Enclosed Charge:** The charge enclosed by the Gaussian surface is:

$$q = \sigma A$$

Here, σ is the surface charge density, and $\vec{A}$ is the area of the plane enclosed by the Gaussian surface.

5. **Applying Gauss's Law:** According to Gauss's Law:

$$\epsilon_0 (EA + EA) = \sigma A$$

Simplifying this equation to solve for $\vec{E}$:

$$2\varepsilon_0 EA = \sigma A$$

$$E = \frac{\sigma}{2\varepsilon_0}$$

This derivation shows how Gauss's Law can be used to find the electric field near an infinite sheet of charge. The result is that the electric field is uniform and directed perpendicular to the plane, with a magnitude given by $\frac{\sigma}{2\varepsilon_0}$.

Electric Field Due to a Spherical Shell:

We consider a spherical shell with charge distributed only on its surface and calculate the electric field both inside and outside the shell using Gauss's Law.

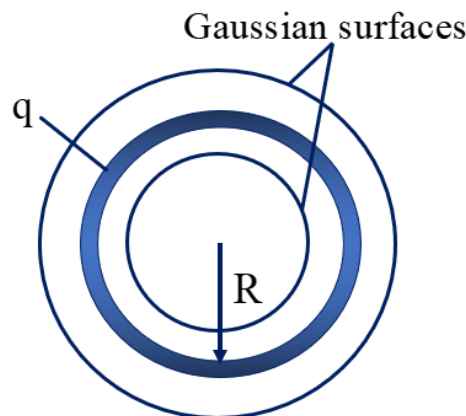


Fig 23. 10: A spherical Gaussian surface surrounds a spherical shell of radius R and charge q.

1. Inside the Shell:

- **Gauss's Law:** Gauss's Law states that the total electric flux through a closed surface is proportional to the charge enclosed by that surface.

$$\int \vec{E} \cdot d\vec{A} = \frac{q_{\text{enc}}}{\varepsilon_0}$$

- **Applying Gauss's Law Inside the Shell:**

- For a point inside the spherical shell (at radius r where $r < R$, with R being the radius of the shell), consider a Gaussian surface that is a sphere of radius r .
- Since the charge is distributed only on the surface, the enclosed charge q_{enc} within this Gaussian surface is zero.

$$q_{\text{enc}} = 0$$

- Therefore, the electric flux is zero:

$$\int \vec{E} \cdot d\vec{A} = E \cdot 4\pi r^2 = 0$$

- This implies that the electric field inside the shell is zero:

$$E = 0 \quad (\text{spherical shell, } r < R)$$

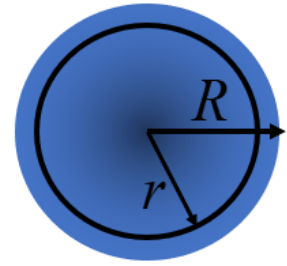


Fig 23. 11: A spherical Gaussian surface that is a radius r inside a spherical shell of radius R and charge q .

2. Outside the Shell

- **Applying Gauss's Law Outside the Shell:**

- For a point outside the spherical shell (at radius r where $r > R$), consider a Gaussian surface that is a sphere of radius r .
- The enclosed charge q_{enc} is the total charge q on the spherical shell.

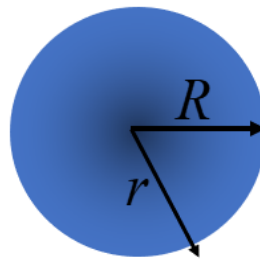


Fig 23. 12: A spherical Gaussian surface that is a radius r surrounds a spherical shell of radius R and charge q .

$$q_{\text{enc}} = q$$

- According to Gauss's Law, the electric flux through this surface is:

$$\int \vec{E} \cdot d\vec{A} = 4\pi r^2 E = \frac{q}{\epsilon_0}$$

- Solving for the electric field E :

$$E = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2}$$

- Thus, the electric field outside the shell behaves as if all the charge were concentrated at the centre of the shell:

$$E = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \quad (\text{spherical shell, } r > R)$$

This result shows the electric field due to a spherical shell of charge follows the same pattern as the electric field due to a point charge for points outside the shell and is zero inside the shell.

Conductors and Electric Fields:

In conductors, the electric field inside is zero because the charges redistribute to cancel any internal fields. When a charge is placed inside a conducting cavity, the charges on the conductor rearrange to neutralize the effect, resulting in no electric field within the cavity.

1. Conducting Sphere:

- If a charge is placed inside a hollow conducting sphere, the charges on the inner surface of the conductor will rearrange to cancel the field within the cavity.

2. Conductor Surface:

- If a charge is placed on a conductor, it will move to the surface to minimize the repulsive forces between like charges. The electric field just outside the surface of a conductor is perpendicular to the surface and proportional to the surface charge density.

Discontinuity of the Electric Field at a Charged Surface:

When there is a surface charge density σ on a boundary, the electric field experiences a discontinuity as you move across the surface. Gauss's Law can be used to quantify this discontinuity.

Applying Gauss's Law to a Surface Charge:

Consider a small Gaussian surface that straddles the surface with area A . The surface has one face just above the surface and one face just below the surface. The electric field on each side of the surface is represented as E_{n1} (above the surface) and E_{n2} (below the surface).

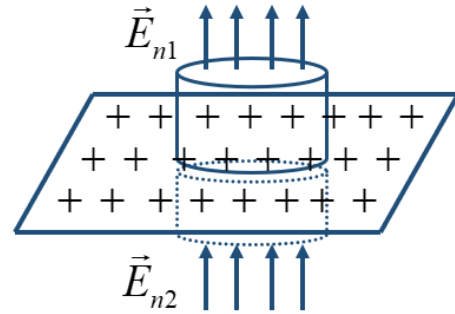


Fig 23. 13: An infinite sheet of charge, with a Gaussian surface straddling the sheet.

Gauss's Law states:

$$\int \vec{E} \cdot d\vec{A} = \frac{q_{\text{enc}}}{\epsilon_0}$$

For the surface, the only contribution to the flux is through the flat faces of the surface (since the sides are perpendicular to the electric field and contribute nothing to the flux). The charge enclosed q_{enc} is the surface charge density σ times the area A .

$$q_{\text{enc}} = \sigma A$$

The electric flux through the surface is the sum of the flux through the top and bottom faces:

$$\int \vec{E} \cdot d\vec{A} = E_{n2}A - E_{n1}A$$

By Gauss's Law:

$$E_{n2}A - E_{n1}A = \frac{\sigma A}{\epsilon_0}$$

Dividing by the area A :

$$E_{n2} - E_{n1} = \frac{\sigma}{\epsilon_0}$$

This equation tells us that the difference between the electric field just outside (E_{n1}) and just inside (E_{n2}) a charged surface is directly proportional to the surface charge density σ , with the proportionality constant being $\frac{1}{\epsilon_0}$.

Potential Energy and Electric Fields:

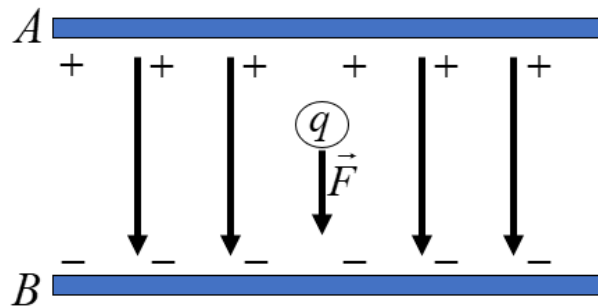


Fig 23. 14: Electric field (E) between two infinite parallel plates with uniform surface charge densities ($+\sigma$ and $-\sigma$). A charge q moves from the positively charged plate to the negatively charged plate, experiencing a force F due to the electric field.

Electric potential energy is the energy stored in an electric field due to the positions of charges. Moving a charge against an electric field requires work, which increases the potential energy of the system. This concept is analogous to gravitational potential energy.

1. Relationship Between Potential Energy and Work:

The change in electric potential energy (ΔU) between two points A and B is related to the work done (W_{ab}) to move a charge from point A to point B:

$$U_B - U_A = W_{ab}$$

$$\Delta U = -W_{ab}$$

2. Work Done by Electrostatic Force:

The work done by an electrostatic force $\vec{F}$ when moving a charge from point A to point B is given by:

$$U_b - U_a = -\int_a^b \vec{F} \cdot d\vec{s}$$

3. Electrostatic Force and Potential Energy:

The electrostatic force is conservative and can be represented by a potential energy function. Comparing with gravitational force:

$$F = G \frac{m_1 m_2}{r^2} \quad \text{and} \quad F = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^2}$$

Where G is the gravitational constant and ϵ_0 is the permittivity of free space.

4. Electric Field and Force:

The relationship between the electric field $\vec{E}$ and the force $\vec{F}$ on a charge q is:

$$\vec{E} = \frac{\vec{F}}{q}$$

For a gravitational field $\vec{g}$ and a mass m :

$$\vec{g} = \frac{\vec{F}}{m}$$

5. Calculation of Potential Energy:

Consider a charge q_2 in the electric field created by another charge q_1 . The work done to move q_2 from distance r_a to r_b is:

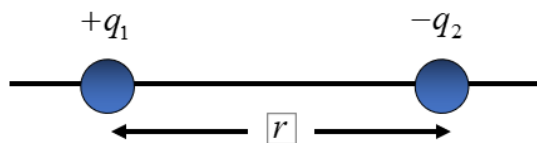


Fig 23. 15: Electric field E experiencing by charge q_2 due to charge q_1 at a distance r .

$$U_b - U_a = -q_2 \int_{r_a}^{r_b} E_x dr$$

For a point charge, the electric field E due to q_1 at a distance r is:

$$E = \frac{1}{4\pi\epsilon_0} \frac{q_1}{r^2}$$

Thus,

$$U_b - U_a = -\frac{1}{4\pi\epsilon_0} q_1 q_2 \int_{r_a}^{r_b} \frac{dr}{r^2}$$

$$U_b - U_a = \frac{1}{4\pi\epsilon_0} q_1 q_2 \left(\frac{1}{r_b} - \frac{1}{r_a} \right)$$

6. Reference Point and Potential Energy:

If we take the reference point $r_a \rightarrow \infty$ where $U_a \rightarrow 0$, then at a distance $r_b = r$, the potential energy $U_b = U(r)$ is:

$$U(r) = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r}$$

7. Comparison with Gravitational Potential Energy:

The electric potential energy $U(r)$ is analogous to the gravitational potential energy:

$$U_{\text{electric}}(r) = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r}$$

$$U_{\text{gravitational}}(r) = -G \frac{m_1 m_2}{r}$$

Electric potential energy is the energy stored due to the relative positions of charges. Moving a charge in an electric field requires work, changing the potential energy. This work is path-independent because the electrostatic force is conservative. The potential energy between two-point charges is inversely proportional to the distance between them, analogous to gravitational potential energy.

Summary

In this lecture, we have explored the foundational concepts of electrostatics, including charges, electric fields, continuous charge distributions, and Gauss's Law. We have seen how these principles are applied in real-world problems and practical applications. Mastery of these topics is essential for a deeper understanding of electromagnetism and its applications in technology and nature.

Lecture 24

Electric Potential

Let's start understanding the forces between charges and how they relate to electric potential energy.

The Basics of Electric Charges and Forces

In electrostatics, we deal with charges that can be either positive or negative. When there are two positive or two negative charges, a force arises between them. This force is repulsive, meaning the charges push away from each other. Conversely, a positive and a negative charge will attract each other. The force between these charges is crucial in understanding electric potential energy.

Work and Potential Energy

When we bring one charge close to another, we must exert a force to overcome the repulsive, as shown in fig. 24.1, or attractive force between them. This exertion of force is called work. As we do work to move a charge, potential energy is added to the system. This potential energy is what gives rise to the concept of electric potential.

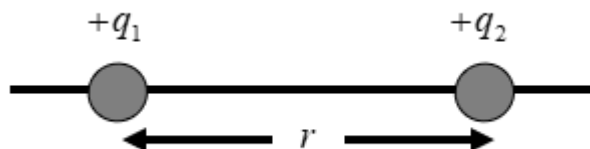


Figure 24.1: Two opposite charges repel each other.

To illustrate this, imagine pressing a spring. Pressing the spring requires work against the force it exerts, storing potential energy in the compressed spring. Similarly, when we bring two charges close together, we do work against the electric force, thereby increasing the electrical potential energy.

Visualizing Electric Potential Energy

Consider two charges being brought close to each other. A simple animation, given in ppt file, can help visualize this process. Imagine a monkey holding one charge and bringing it close to another fixed charge. The monkey must do work to bring the charges closer, but if the monkey

releases the charge, all the work done is recovered as kinetic energy. This demonstrates that no energy is lost, and all the energy stored in the system can be recovered.

Calculating Electric Potential Energy

Let's imagine one charge is fixed in place, and we want to bring another charge, a unit charge ($q = 1$), from a far distance to a specific distance R . We need to calculate the total work required for this process. This calculation is divided into two parts.

1. **Starting from Infinity:** We begin by considering the unit charge at infinity, where the electric potential energy is considered zero.
2. **Moving to Distance R :** We move the charge to a distance R , calculating the work done step by step.

The work done in moving a unit charge from infinity to a distance R is given by the integral of the force over the distance:

$$W = \int_{\infty}^R E dr$$

Here, E represents the electric field, and dr is the infinitesimal distance. The electric field E is derived from Coulomb's law:

$$E = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2}$$

Substituting this into our integral:

$$W = \frac{1}{4\pi\epsilon_0} q \int_{\infty}^R \frac{dr}{r^2}$$

This integral simplifies as:

$$W = \frac{1}{4\pi\epsilon_0} q \int_{\infty}^R r^{-2} dr$$

$$W = \frac{1}{4\pi\epsilon_0} q \left[\frac{r^{-2+1}}{-2+1} \right]_{\infty}^R$$

$$W = \frac{1}{4\pi\epsilon_0} q \left[\frac{r^{-1}}{-1} \right]_{\infty}^R$$

$$W = \frac{1}{4\pi\epsilon_0} q \left[-\frac{1}{r} \right]_{\infty}^R$$

Evaluating this from infinity to R:

$$W = \frac{q}{4\pi\epsilon_0} \left(\frac{1}{R} - \frac{1}{\infty} \right)$$

$$W = \frac{q}{4\pi\epsilon_0} \frac{1}{R} \quad \because \frac{1}{\infty} = 0$$

Potential Energy Between Two Charges

If we have two charges, q_1 and q_2 , the potential energy U between them when separated by a distance r is:

$$U(r) = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r}$$

We assume the potential energy is zero when the charges are infinitely far apart. This simplifies our calculations and provides a reference point.

Units of Electric Potential

The unit of electric potential is the volt (V), which is equivalent to one joule per coulomb (J/C). When you multiply the electric potential (in volts) by the charge (in coulombs), you get the energy (in joules).

Electron Volt

A useful unit in atomic and nuclear physics is the electron volt (eV). One electron volt is the energy gained or lost by an electron when it moves through an electric potential difference of one volt. This is equal to 1.6×10^{-19} joules.

Applications of Electric Potential

In atomic physics, the electron volt is particularly significant. For instance, to ionize a hydrogen atom (i.e., to remove an electron from a proton), you need to provide 13.6 electron volts of energy. In nuclear physics, the binding energy of protons and neutrons in a nucleus is often measured in million electron volts (MeV).

Movement of Charges in Electric Potential

A positive charge naturally moves from a region of higher potential to a region of lower potential, whereas a negative charge moves in the opposite direction. This behaviour can be visualized with a simple experiment. If you place a charge in an electric field, the charge will move according to the potential difference between two points.

Work Done by Electric Field

Consider a charge moving from point A to point B in an electric field. The work done by the electric field on the charge can be calculated. For instance, if the electric field does 8 joules of work on a charge of 2 coulombs moving from point A to point B, as shown in fig. 24.2, the potential difference is:

$$\Delta V = -\frac{W}{q}$$

$$\Delta V = -\frac{8}{2} = -4V$$

The negative sign indicates a decrease in potential.

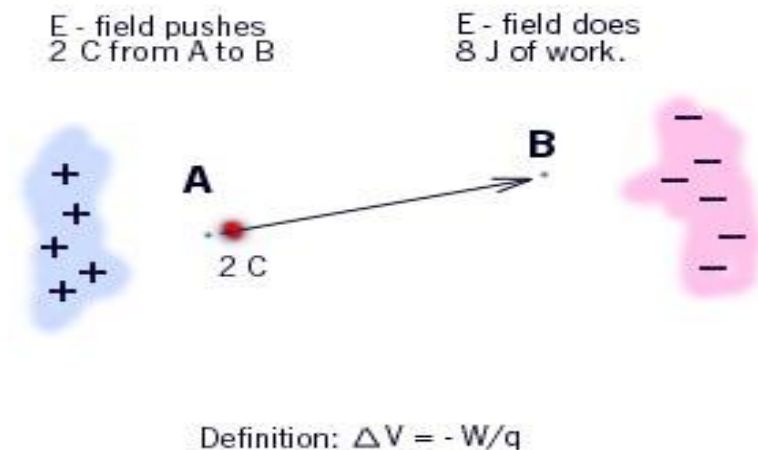


Figure 24.2: Change in potential as charge moves from A to B.

Conservative and Non-Conservative Forces

The force between two charges is conservative, meaning the work done in moving a charge between two points is path independent. This property allows us to define potential energy and potential differences accurately. In contrast, non-conservative forces, such as friction, depend on the path taken.

Electrostatic Potential Energy in Multi-Charge Systems

Understanding the electrostatic potential energy in systems of multiple point charges is fundamental in electrostatics. Consider a system of three-point charges q_1 , q_2 , and q_3 , as shown in fig. 24.3. The total electrostatic potential energy $U(r)$ of this system is given by

$$U(r) = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r_{12}} + \frac{1}{4\pi\epsilon_0} \frac{q_1 q_3}{r_{13}} + \frac{1}{4\pi\epsilon_0} \frac{q_2 q_3}{r_{23}}$$

$$U(r) = \frac{1}{4\pi\epsilon_0} \left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}} \right)$$

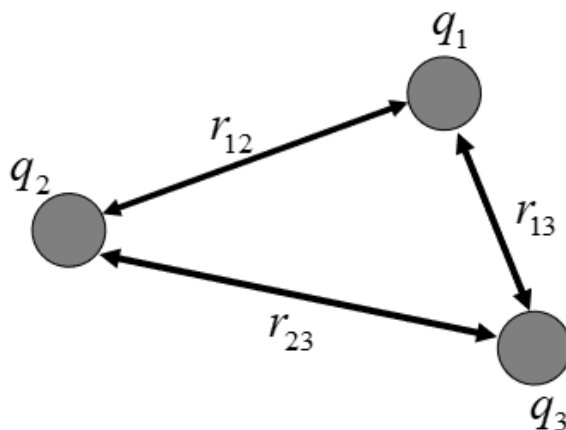


Figure 24.3: Electrostatic potential energy for three charges.

Dipole

A dipole consists of two equal, but opposite charges separated by a distance d , as shown in fig. 24.4. The dipole moment p is defined as:

$$P = q \cdot d$$

Dipoles are significant in many molecules where there is a natural separation of charge, such as in water (H_2O).

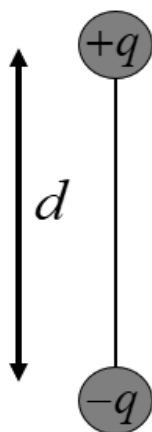


Figure 24.4: A dipole separated by distance d .

To calculate the potential V at a point P due to a dipole, consider the distances from the charges to the point P , as shown in fig. 24.5:

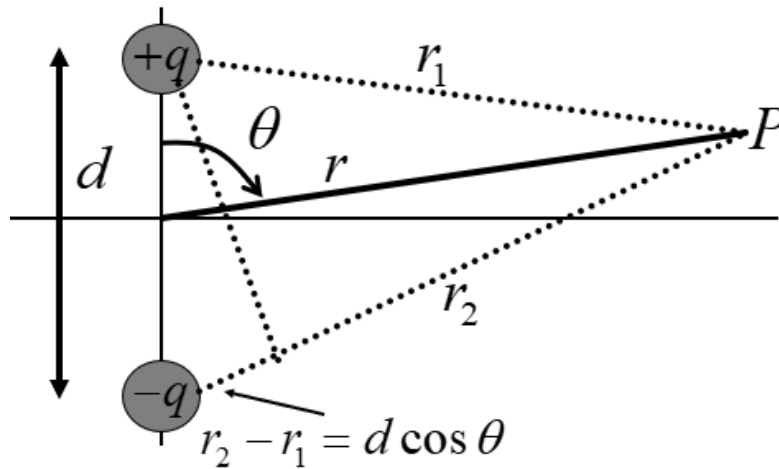


Figure 24.5: Potential at point P due to a dipole.

$$\begin{aligned}
 V_P &= V_1 + V_2 \\
 &= \frac{1}{4\pi\epsilon_0} \left(\frac{q}{r_1} + \frac{-q}{r_2} \right) \\
 &= \frac{q}{4\pi\epsilon_0} \frac{r_2 - r_1}{r_1 r_2}
 \end{aligned}$$

For points far from the dipole, where r is the distance from the centre of the dipole to the point P and θ is the angle between the dipole axis and the line joining the centre of the dipole to the point P :

$$\begin{aligned}
 &\text{For } r \gg d \\
 &r_2 - r_1 \approx d \cos \theta \\
 &r_1 r_2 \approx r^2 \\
 &V \approx \frac{q}{4\pi\epsilon_0} \frac{d \cos \theta}{r^2} = \frac{1}{4\pi\epsilon_0} \frac{p \cos \theta}{r^2} \quad \because p = q \cdot d \\
 &\text{Note that } V = 0 \text{ at } \theta = \frac{\pi}{2}
 \end{aligned}$$

Ring of Charges

Consider a uniformly charged ring lies in xy-plan with its center at origin, as shown in fig. 24.6, have a total charge q and radius R . To find the potential at a point P along the axis of the ring, use the symmetry of the problem.

The potential due to a point charge dq at a distance r is given by:

$$dV = \frac{1}{4\pi\epsilon_0} \frac{dq}{r} \quad \text{eq(1)}$$

For a small element of charge dq on the ring, at coordinates $(R\cos\theta, R\sin\theta, 0)$, the distance to point P is:

$$r = \sqrt{R^2 + z^2} \quad \text{eq(2)}$$

Since the ring is uniformly charged, the charge density λ (charge per unit length) is:

$$\lambda = \frac{q}{2\pi R}$$

An infinitesimal charge element is:

$$dq = \lambda R d\theta = \frac{q}{2\pi R} R d\theta = \frac{q}{2\pi} d\theta \quad \text{eq(3)}$$

Integrate dV over the entire ring:

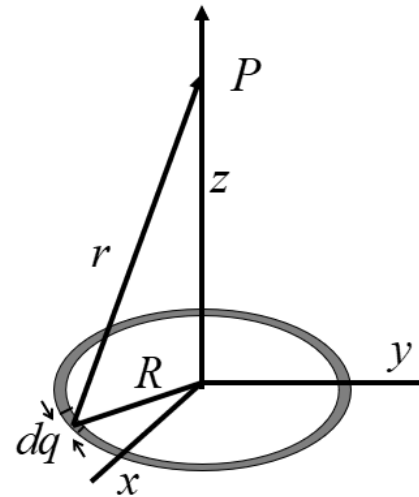


Figure 24.6: Potential due to a charged ring at perpendicular axis.

$$V = \int dV$$

using eq(1) and eq(2), we get

$$V = \int_0^{2\pi} \frac{1}{4\pi\epsilon_0} \frac{dq}{\sqrt{R^2 + z^2}}$$

using eq(3), we get

$$V = \int_0^{2\pi} \frac{1}{4\pi\epsilon_0} \frac{\frac{q}{2\pi} d\theta}{\sqrt{R^2 + z^2}}$$

$$V = \frac{1}{4\pi\epsilon_0} \frac{q}{2\pi\sqrt{R^2 + z^2}} \int_0^{2\pi} d\theta$$

$$V = \frac{1}{4\pi\epsilon_0} \frac{q}{2\pi\sqrt{R^2 + z^2}} 2\pi \quad \because \int_0^{2\pi} d\theta = 2\pi$$

$$V = \frac{q}{4\pi\epsilon_0\sqrt{R^2 + z^2}}$$

So, the potential at a point P on the axis of the ring is:

$$V = \frac{1}{4\pi\epsilon_0} \frac{q}{\sqrt{R^2 + z^2}}$$

Disk of Charges

For a uniformly charged disk lies in the xy-plane with its centre at the origin have surface charge density σ and radius R, as shown in fig. 24.7, we want to find the electric potential V at a point P on the axis perpendicular to the disk at a distance z from the centre of the disk.

Consider an infinitesimal ring of radius ω and thickness $d\omega$ within the disk. The charge dq on this ring is:

$$dq = \sigma \cdot 2\pi\omega \cdot d\omega$$

The distance from any point on this ring to point P is:

$$r = \sqrt{\omega^2 + z^2}$$

The potential dV due to this ring at point P is:

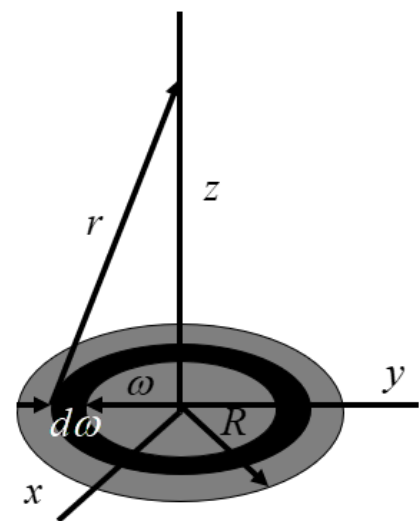


Figure 24.7: Potential due to a disk of uniform surface charge density at perpendicular axis.

$$dV = \frac{1}{4\pi\epsilon_0} \frac{dq}{\sqrt{\omega^2 + z^2}}$$

$$dV = \frac{1}{4\pi\epsilon_0} \frac{\sigma \cdot 2\pi\omega \cdot d\omega}{\sqrt{\omega^2 + z^2}}$$

$$dV = \frac{\sigma}{2\epsilon_0} \frac{\omega d\omega}{\sqrt{\omega^2 + z^2}}$$

$$V = \int_0^R \frac{\sigma}{2\epsilon_0} \frac{\omega d\omega}{\sqrt{\omega^2 + z^2}}$$

$$\text{let } u = \omega^2 + z^2 \text{ then } (du = 2\omega d\omega) \text{ and } \omega d\omega = \frac{du}{2}$$

$$V = \frac{\sigma}{2\epsilon_0} \int_{z^2}^{R^2+z^2} \frac{du}{2\sqrt{u}}$$

$$V = \frac{\sigma}{4\epsilon_0} \left[2\sqrt{u} \right]_{z^2}^{R^2+z^2} = \frac{\sigma}{2\epsilon_0} \left(\sqrt{R^2 + z^2} - z \right)$$

So, the potential at point P on the axis perpendicular to the disk is:

$$V = \frac{\sigma}{2\epsilon_0} \left(\sqrt{R^2 + z^2} - z \right)$$

Understanding Electric Potential:

Imagine a uniform electric field. This means the strength of the electric field doesn't change as you move through it. Now, consider a scenario where we have two points, a and b, in this field. If we move a charge (let's call it q) from point b to point a, as shown in fig. 24.8, we have to do work against the electric field. This work done by the electric field is the force pushing the charge multiplied by the distance it moves.

The electric potential difference ΔV between points a and b is given by: $V_b - V_a = EL$ and work done $W_{ab} = q_0 EL$ where:

- E is the strength of the electric field,
- L is the distance between points a and b,
- W_{ab} is the work done.

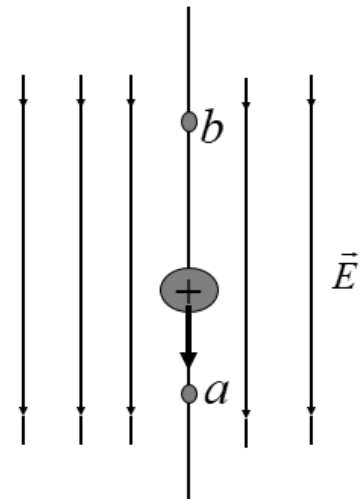


Figure 24.8: Motion of a charge in an equipotential surface from point a to b.

Equipotential Surfaces:

As we move around in space, the potential (electric potential energy per unit charge) changes. However, there are surfaces where the electric potential is the same everywhere. These are called equipotential surfaces, as shown in fig. 24.9. For instance, around a single point charge or between two oppositely charged plates, every point on an equipotential surface has the same electric potential.

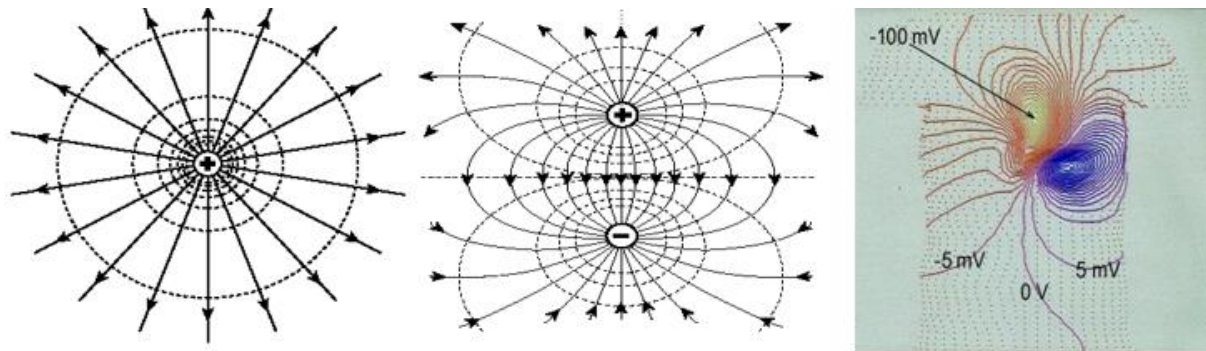


Figure 24.9: Equipotential surfaces for three different situations.

Creating Potential Differences:

To create a potential difference between two points, we need a device like a battery. Batteries use chemical reactions (like those between carbon and zinc electrodes in sulfuric acid, as shown in fig. 24.10) to generate a potential difference. This difference in potential allows us to move charges and generate electric current.

The potential difference ΔV produced by a battery can be calculated based on the chemistry involved and the physical setup of the battery.

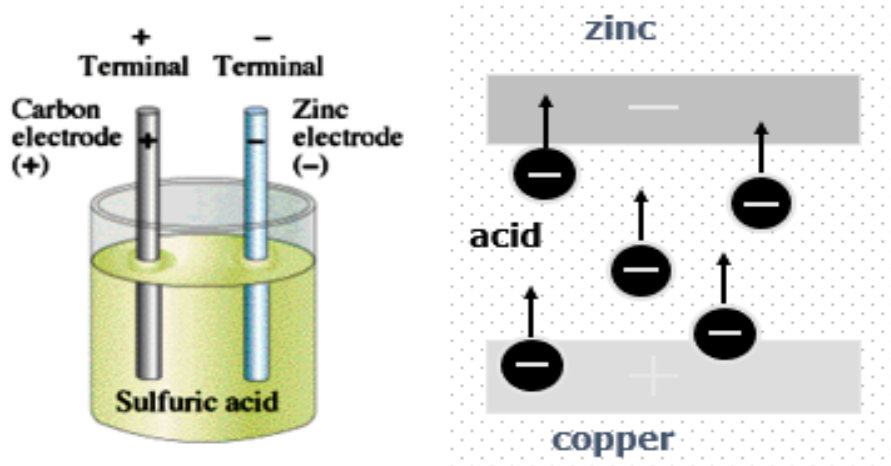


Figure 24.10: Carbon and zinc electrodes in sulfuric acid generates potential differences.

Various Methods to Create Potential Difference:

Potential differences can also be created by mechanical means, such as in the Van de Graaff generator, where charges are moved using belts and rollers against electric forces. Similarly, natural processes like ionization due to rising air currents can create potential differences, leading to phenomena like lightning.

Problem:

Consider two conducting balls with different charges. When connected by a wire, as shown in fig. 24.11, charge redistributes between them until they reach the same potential. The charge Q on each ball is proportional to its radius r and the potential V it reaches.

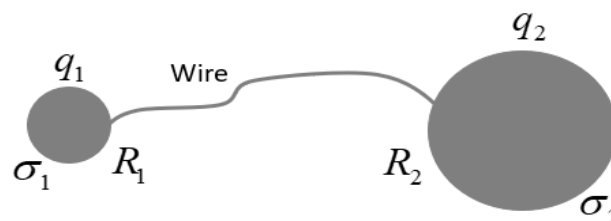


Figure 24.11: Sphere of different radiuses having different surface charge densities are connected with conducting wire.

Solution:

as we know, surface charge density for a sphere is

$$\sigma = \frac{q}{4\pi R^2}$$

for each sphere

$$\sigma_1 = \frac{q_1}{4\pi R_1^2} \text{ and } \sigma_2 = \frac{q_2}{4\pi R_2^2}$$

$$\Rightarrow q_1 = \sigma_1 4\pi R_1^2 \text{ and } q_2 = \sigma_2 4\pi R_2^2$$

using it in eq(4)

$$\frac{\sigma_1 4\pi R_1^2}{\sigma_2 4\pi R_2^2} = \frac{R_1}{R_2}$$

$$\frac{\sigma_1}{\sigma_2} = \frac{R_1 R_2^2}{R_2 R_1^2}$$

$$\Rightarrow \frac{\sigma_1}{\sigma_2} = \frac{R_2}{R_1}$$

as given $R_2 > R_1$ so we can say, $\sigma_1 > \sigma_2$

Thus, the electric field at the surface of the smaller sphere is greater than that at the surface of the larger sphere:

$$E_{\text{small sphere}} > E_{\text{large sphere}}$$

PHY101-Lecture# 25

Capacitors and Current

Capacitors and Energy Storage

In this lecture, we discuss **capacitors**, which store electrical energy. Capacitors are used in various devices like camera flashes, where they store energy and release it quickly. They consist of two conducting plates, and when a potential difference is created between them, charge gets stored.

Basics of Capacitors

1. Structure and Working:

- Capacitors consist of two plates with a potential difference between them. When charge is transferred from one plate to the other, an **electric field** is created between the plates.
- This results in energy being stored in the capacitor as charges are moved against the electric field.

2. Charge and Potential Relationship:

- The amount of **charge** q stored in a capacitor is proportional to the **potential difference** V between the plates:

$$Q \propto V$$

- Introducing a constant of proportionality C , we get the formula for **capacitance**:

$$Q = CV$$

- Capacitance C is defined as “the amount of charge stored per unit potential difference”.

$$C = \frac{Q}{V}$$

Parallel Plate Capacitor

1. Capacitance Calculation:

- Consider a parallel plate capacitor with plates of area A and separated by a distance d as shown in fig.25.1.
- Using **Gauss's law**, the **electric flux** Φ through the surface is:

$$\Phi = E \cdot A$$

- According to Gauss's law:

$$\Phi = \int \vec{E} \cdot d\vec{A}$$

$$\Phi = \frac{Q_{\text{enclosed}}}{\epsilon_0} \quad \therefore \Phi = E \cdot A$$

$$E = \frac{Q}{\epsilon_0 A}$$

where ϵ_0 is the permittivity of free space.

- The **electric field** E is related to the potential difference V given by:

$$V = Ed$$

- Substituting the value of E into this equation, the capacitance C of a parallel plate capacitor is:

$$V = \frac{Q}{\epsilon_0 A} d$$

$$C = \frac{Q}{V} = \epsilon_0 \frac{A}{d}$$

- This shows that capacitance increases with the plate area A and decreases with the distance d between the plates.

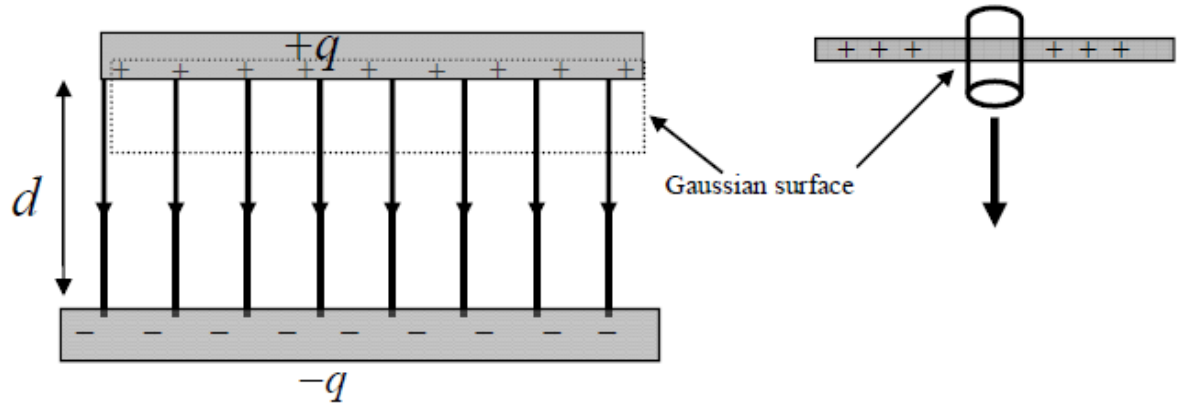


Fig.25.1: Illustration of the electric field (E) between the plates of a parallel plate capacitor, with the plates separated by distance d , a uniform electric field established across the plates, and charge distributed over the plate area A .

Units and Practical Examples

- **Units of Capacitance:**
- The unit of capacitance is the **farad** (F), defined as:

$$1 \text{ farad} = 1 \frac{\text{coulomb}}{\text{volt}}$$

- The farad is a large unit, so capacitors are commonly measured in microfarads (μF), nano-farads (nF), or picofarads (pF).

$$1 \text{ microfarad} = 10^{-6} \text{ farad}$$

$$1 \text{ nanofarad} = 10^{-9} \text{ farad}$$

$$1 \text{ picofarad} = 10^{-12} \text{ farad}$$

- **Example Calculation:**

How big is 1 Farad?

- For a **1 farad** capacitor with plates 1 mm apart, the required area A is approximately:

Choose $d = 1\text{mm}$

$$C = \epsilon_0 \frac{A}{d}$$

Calculate area,

$$A = \frac{Cd}{\epsilon_0}$$

$$A = \frac{1\text{F} \times 0.001\text{m}}{8.85 \times 10^{-12}} = 1.1 \times 10^8 \text{m}^2$$

- This large value is impractical for everyday capacitors.

Types of Capacitors

Capacitors are in various shapes and sizes, such as **spherical** or **cylindrical** capacitors, but the basic principle of charge storage and potential difference remains the same.

- **Air Capacitors:** These consist of two plates separated by air, where changing the area of the plates (by moving them closer or farther apart) changes the capacitance. This type was traditionally used for tuning radios.
- **Electrolytic Capacitors:** These are much larger in size, measured in microfarads (μF), and are used for storing larger amounts of charge.

Charging and Discharging Capacitors:

- A battery is used to charge capacitors by connecting the positive terminal to one plate and the negative to the other. Once charged, a capacitor can store energy that can be discharged later.
- **Discharging:** The stored energy can be discharged by connecting the plates to a circuit. A spark occurs as the charge flows back through the wires.

Analogy of Storing Charge:

- The process of charging and discharging a capacitor is compared to filling and emptying a water tank. A pump (battery) fills the tank (capacitor), and water (charge) can be released (discharged) when needed.

Capacitor Charging Behavior:

- During charging, the current decreases as the capacitor becomes fully charged. The current also depends on the resistance in the circuit, and if a large resistance is present, the capacitor will discharge slowly.

Parallel and Series Connections:

- **Parallel Connection:**

If capacitors are connected in parallel, the total capacitance increases as shown in fig.25.2.

The formula for total capacitance is the sum of individual capacitances as given:

For each capacitor,

$$q_1 = C_1 V, \quad q_2 = C_2 V$$

Where V is the potential
between terminals a and b

$$q = q_1 + q_2$$

$$C_{eq} V = C_1 V + C_2 V$$

$$\Rightarrow C_{eq} = C_1 + C_2$$

$$C_{eq} = \sum C_n \quad (\text{For } n \text{ capacitors})$$

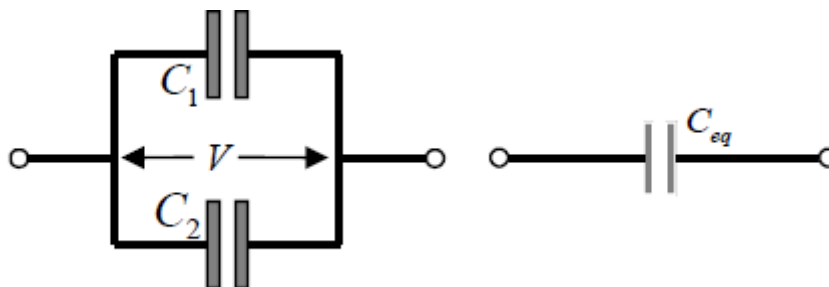


Fig.25.2: Representation of a parallel plate capacitor and its equivalent capacitance.

- **Series Connection:** In a series connection, the total capacitance is calculated differently, as the voltage is divided among the capacitors as shown in fig.25.3.

$$\begin{aligned}
 V_1 &= \frac{q}{C_1}, & V_2 &= \frac{q}{C_2} \\
 V &= V_1 + V_2 \\
 \frac{q}{C_{eq}} &= \frac{q}{C_1} + \frac{q}{C_2} \\
 \Rightarrow \frac{1}{C_{eq}} &= \frac{1}{C_1} + \frac{1}{C_2} \\
 \frac{1}{C_{eq}} &= \sum \frac{1}{C_n} \quad (\text{For } n \text{ capacitors})
 \end{aligned}$$

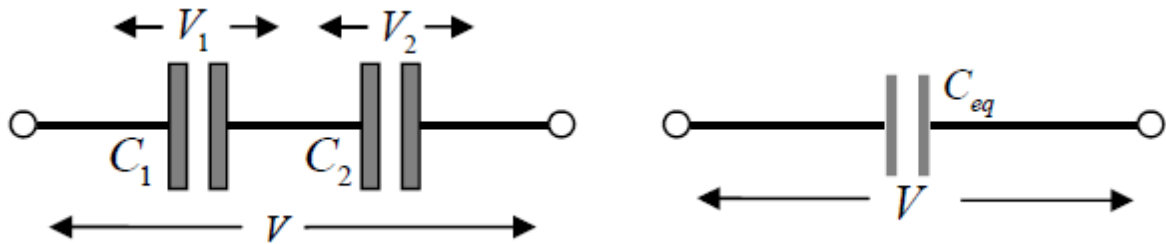


Fig.25.3: Representation of a series plate capacitor and its equivalent capacitance.

Energy Stored in Capacitors:

When a battery is connected to a capacitor, positive and negative charges appear on the opposite plates. Some energy has been transferred from the battery to the capacitor and now been stored in it. When the capacitor is discharged, the energy is recovered. Now let us calculate the energy required to charge a capacitor from zero to volts.

The amount of energy required to transfer a small charge dq to the plates is

$$dU = Vdq$$

where V is the voltage at a time when the charge is $q = CV$. As time goes on, the total charge increases until it reaches the final charge Q (at which point the voltage becomes V). So,

$$\begin{aligned}
 dU &= Vdq = \frac{q}{C} dq \\
 U &= \int dU = \int_0^Q \frac{q}{C} dq \\
 U &= \frac{Q^2}{2C}
 \end{aligned}$$

where C is the capacitance and V is the voltage across the capacitor. The energy increases quadratically with voltage. This energy is analogous to work done in transferring charge between the plates of a capacitor.

Energy Density:

Q: Where is the energy stored in a capacitor?

Answer:

It is present in the electric field in the volume between the two plates.

The energy stored per unit volume (energy density) in a capacitor is given by:

$$U = \frac{1}{2} CV^2$$

$$u = \frac{U}{Ad} = \frac{\frac{1}{2} CV^2}{Ad}$$

$$C = \frac{\epsilon_0 A}{d}$$

$$u = \frac{\epsilon_0}{2} \left(\frac{V}{d} \right)^2 = \frac{1}{2} \epsilon_0 E^2$$

where E is the electric field between the plates, and ϵ_0 is the permittivity of free space. The energy density is proportional to the square of the electric field.

Electric Field and Energy:

- The energy stored in a capacitor is proportional to the square of the electric field.
- When a dielectric is added, it reduces the electric field, decreasing the potential difference and increasing capacitance.

Electric Current:

- Current is the flow of charge, typically electrons in a wire.

$$i = \frac{dq}{dt}$$

$$\text{constant current: } q = i \times t$$

$$\text{else: } q = \int i dt$$

- **Drift velocity** of electrons is a key factor in current, and the flow of electrons is affected by collisions within the conductor as shown in fig.25.4.

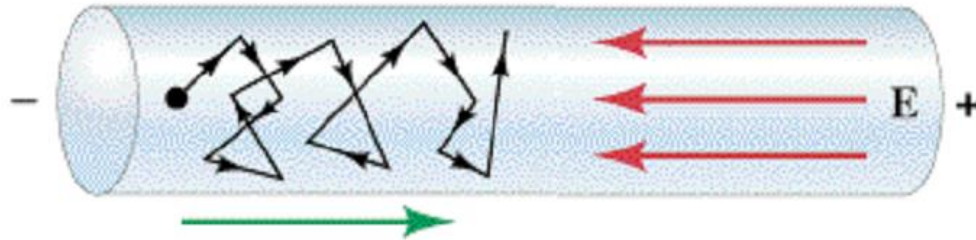


Fig.25.4: Visualization of drift velocity of electrons in a conductor, illustrating how collisions within the conductor influence the flow of charge. Current is represented by the movement of electrons.

- The **current** is measured in amperes (A).
- **Current: units and sizes:**

$$1 \text{ ampere} = \frac{1 \text{ coulomb}}{\text{second}}$$

$$1 \text{ milliampere} = 1 \text{ ma} = 10^{-3} \text{ A}$$

$$1 \text{ microampere} = 1 \mu\text{A} = 10^{-6} \text{ A}$$

$$1 \text{ nanoampere} = 1 \text{ nA} = 10^{-9} \text{ A}$$

$$1 \text{ picoampere} = 1 \text{ pA} = 10^{-12} \text{ A}$$

- Currents can vary widely, from large currents in car batteries to tiny currents (nano or pico-amperes) in integrated circuits.

Conservation of Current:

- Current is conserved in a circuit: if it splits, the total current before and after the split remains the same.
- The current flowing through a resistor depends on the applied voltage and the resistance (Ohm's Law): $I = V/R$.
- This law holds true within certain limits, and excessive current can cause a resistor to overheat or break.

Capacitor's Function:

- The result concludes with an analogy to a spring, where energy is stored in the capacitor due to the electric field, just like energy is stored in a spring due to its extension.

Coulomb's Law with Dielectrics:

Coulomb's Law with dielectrics states that the electrostatic force between two charges is reduced by a factor of the dielectric constant (k) of the medium. The force is given by,

$$F = \frac{1}{4\pi\epsilon_0\epsilon_r} \frac{q_1q_2}{r^2}$$

ϵ_r = relative permittivity

$$E_{\text{medium}} = \frac{1}{4\pi\epsilon_0\epsilon_r} \frac{q}{r^2}$$

$$E_{\text{medium}} = \frac{E_{\text{vacuum}}}{\epsilon_r}$$

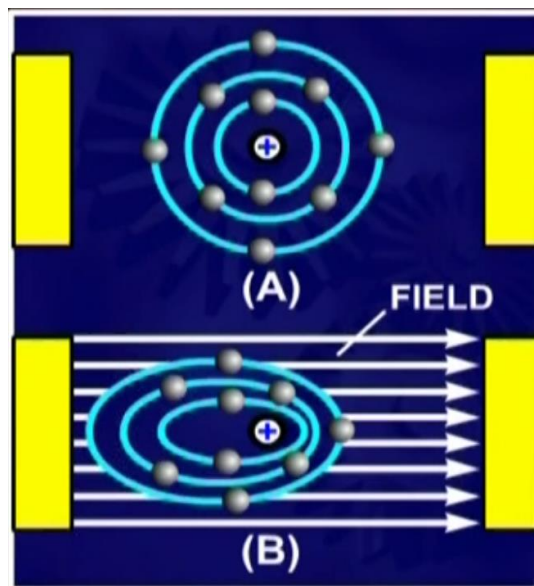


Fig.25.5: Distortion of electron orbital paths due to electrostatic force.

Effect of Dielectric Materials on Capacitance:

- The capacitance of a capacitor increases with the insertion of dielectric materials, which reduce the electric field and potential difference between plates.
- Dielectric materials, like water or plastic, polarize when an electric field is applied, affecting the electric field inside the material and influencing the capacitance.

- The **relative permittivity** of materials (also called the dielectric constant) determines how much the dielectric material can reduce the electric field, increasing capacitance.

Dielectric Constant and Polarization:

- The dielectric constant ϵ_r is greater than 1 and depends on the material. It determines how much the material can reduce the electric field.
- When a dielectric material (like water or plastic) is placed between plates, the molecules within it align with the electric field, reducing the overall electric field between the plates.

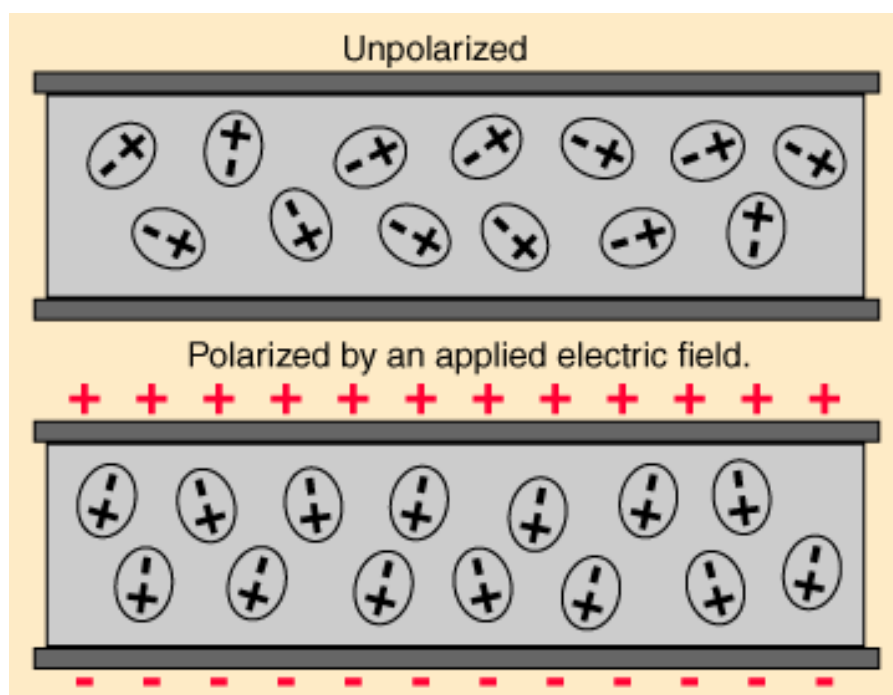


Fig.25.6: Electric field decreases in dielectrics due to polarization.

Practical Use of Dielectrics:

- Capacitors with high dielectric constants (such as electrolytic capacitors) can store much more charge in a smaller volume compared to air-based capacitors, but these capacitors may not always be perfect due to the small conductivity of the dielectric, which can lead to leakage of charge.

Lecture#26

ELECTRIC POTENTIAL ENERGY - II

Electric Current and Electric Circuits

Electric current and electric circuits are fundamental concepts in the study of electricity. Electric current refers to the flow of electric charge, and the magnitude of this current depends on the amount of charge passing through a given point in a conductor over time. The greater the movement of charge, the higher the current in the system. An electric circuit is a complete and closed path through which electric charge flows. In a broader sense, the term "circuit" can refer to any closed pathway through which objects, such as vehicles, can move. However, in the context of electricity, an electric circuit specifically refers to a pathway that allows charged particles to move and eventually return to their starting point. The significance of electric circuits and current in modern life is profound, as most technological advancements rely on a thorough understanding and application of these principles.

To better understand electric current, it can be compared to the flow of water in a closed-loop system. Consider a pump connected to a network of pipes. The pump exerts a force that drives the water through the pipes, eventually returning it to the starting point. If the pipes are wide, water can flow with ease, but if they are narrow, greater force is required to push the same amount of water through in a given time. A similar concept applies to electricity. In an electrical circuit, if the conducting wire has a smaller cross-sectional area, the flow of charge encounters greater resistance, making it more difficult for current to pass through. Thus, narrowing a wire increases its electrical resistance. Electric current I is formally defined as the amount of electric charge dq that passes through a conductor in a given period of time dt .

$$I = \frac{dq}{dt} = \frac{q}{t}$$

Thus, in terms of units

$$1 \text{ Ampere} = \frac{1 \text{ Coulomb}}{1 \text{ second}}$$

Resistance, on the other hand, is defined as the ratio of the potential difference V across a resistor to the current I flowing through it as shown in figure 26.1. In an electrical circuit, if one end of a resistor is at a higher electric potential compared to the other end, a potential difference is established, driving the flow of current. For current to flow from one end of the

resistor to the other, the potential at the starting point 'a' must be greater than the potential at the endpoint 'b', similar to how water naturally flows from a higher elevation to a lower elevation due to gravitational potential.

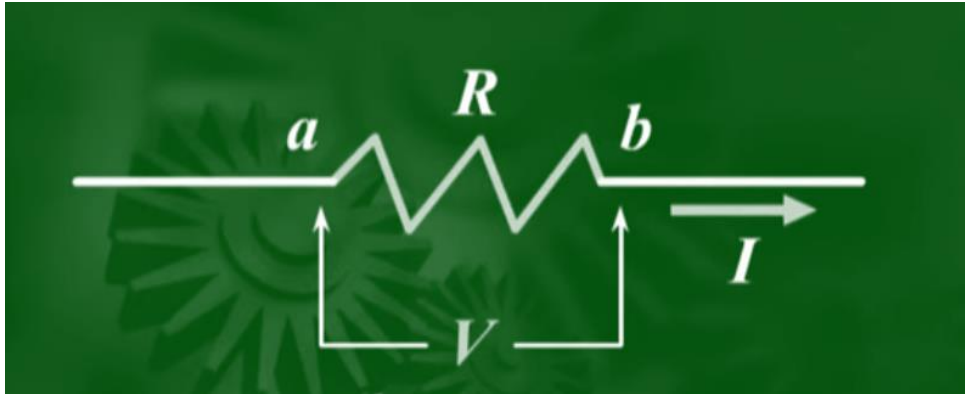


Figure 26.1: Current I flow through Resistor R because of potential difference V across resistor. a is at higher potential than b so current I is flowing from a toward b .

The direction of electric current is conventionally considered to be from a region of higher potential to a region of lower potential. It is important to note that this conventional current flow assumes the movement of positive charges, even though in many practical scenarios, such as in metallic conductors, the actual charge carriers are negatively charged electrons moving in the opposite direction.

Resistance is commonly expressed as the ratio of the potential difference across a resistor to the current passing through it.

$$R = \frac{V}{I}$$

However, this relationship alone does not define Ohm's law. Ohm's law specifically applies only when the resistance of the material remains constant under varying potential differences and resulting currents. In other words, if the current through a material increases linearly with an applied potential difference, the material is said to obey Ohm's law.

In practical applications, the resistance of a material may change with factors such as temperature. For example, when a current is passed through a wire, it generates heat due to resistance. As the potential difference across the wire increases, more current flows, and the wire's temperature rises. If excessive potential difference is applied, the wire may overheat and eventually burn out, preventing any further current flow. In such cases, the current is no longer

proportional to the applied potential difference, meaning that Ohm's law no longer holds under these extreme conditions.

Equivalent Resistance in Series and Parallel Circuits

The concept of equivalent resistance is crucial in analysing electrical circuits containing multiple resistors. When resistors are connected in different configurations, such as series or parallel, their combined effect can be represented by a single equivalent resistor that simplifies circuit analysis.

Resistors in Series

When two resistors are connected in series as shown in figure 26.2, the total or equivalent resistance is the sum of the individual resistances. In a series configuration, the same current I flow through all resistors since there is only a single path for the charge to travel. The total voltage V across the series combination is equal to the sum of the voltage drops across each individual resistor.

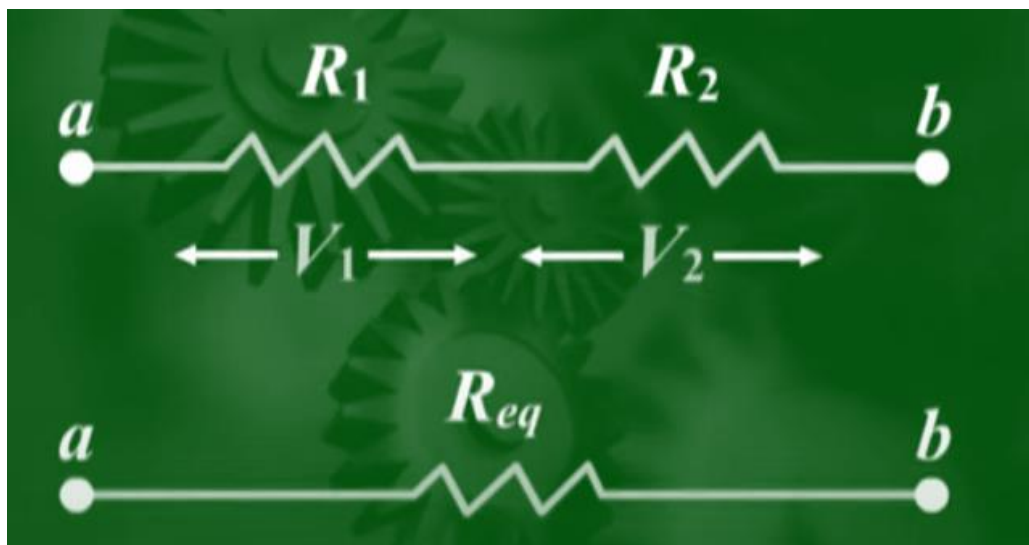


Figure 26.2: Resistor in Series

$$V_1 = IR_1, \quad V_2 = IR_2$$

$$V = V_1 + V_2$$

$$IR_{eq} = IR_1 + IR_2$$

I can be cancelled from both sides of equation so,

$$R_{eq} = R_1 + R_2$$

The equivalent resistance in a series circuit can be determined by adding the resistance values of the individual resistors. This means that if two resistors with resistances R_1 and R_2 are connected in series, their equivalent resistance will be the sum of the two, resulting in a total resistance that is greater than either of the individual resistances.

Resistors in Parallel

When resistors are connected in parallel, the situation differs significantly from the series configuration. In a parallel circuit, the resistors provide multiple paths for current to flow, meaning that the total current entering the combination is divided among the parallel branches. However, the voltage across each resistor remains the same. The conservation of charge principle applies here, ensuring that the total current flowing into the parallel network is equal to the sum of the currents flowing through each individual resistor. The equivalent resistance in a parallel circuit is determined using the reciprocal relationship of the individual resistances.

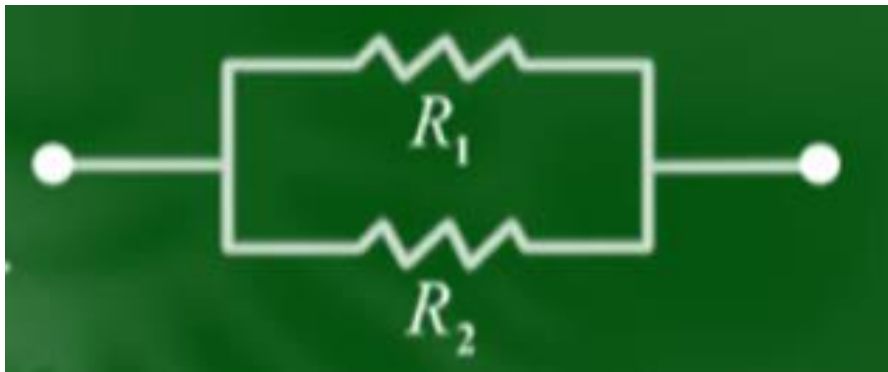


Figure 26.3: Resistors in Parallel

$$I_1 = \frac{V}{R_1}, \quad I_2 = \frac{V}{R_2}$$

$$I = I_1 + I_2$$

$$\frac{V}{R_{eq}} = \frac{V}{R_1} + \frac{V}{R_2}$$

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2}$$

$$R_{eq} = \frac{R_1 R_2}{R_1 + R_2}$$

For two resistors R_1 and R_2 connected in parallel, the reciprocal of the equivalent resistance is equal to the sum of the reciprocals of the individual resistances. This results in an equivalent resistance that is always less than the smallest resistor in the parallel network.

In practical terms, when resistors are connected in parallel, the total resistance decreases because the current has multiple paths to flow through. This reduced resistance allows more current to pass through the circuit compared to a single resistor of the same value.

Comparison with Capacitors

The behavior of resistors in series and parallel can be contrasted with that of capacitors. When capacitors are connected in parallel, their capacitance increases, as the total charge-storing capability of the system is enhanced. In contrast, when capacitors are connected in series, the total capacitance decreases. This behavior is opposite to that of resistors, where resistance increases in series and decreases in parallel. The reduction in resistance in parallel circuits occurs because the presence of multiple paths allows charge to flow more easily, thereby reducing the overall opposition to current flow.

Electromotive Force

The movement of electric charge within a circuit is a fundamental concept in physics, leading to the question of what drives this movement. Understanding the driving force behind the motion of electrons helps in comprehending the working principles of electrical circuits. Historically, certain terms used in physics can be misleading or inaccurately named, and one such example is the term Electromotive Force (EMF). Despite its name, EMF is not actually a force in the conventional sense but rather a measure of energy supplied to charge carriers in a circuit.

To illustrate this concept, consider an analogy with water flowing through a pipe. Water moves through the pipe because of an applied force, typically provided by a pump. Similarly, in an electrical circuit, electrons move through a conductor due to the presence of electromotive force. However, the term force in EMF can be misleading, as it does not represent a physical force but rather the energy required to move charge through the circuit.

A useful analogy to understand EMF is to imagine a person lifting balls from the ground and placing them onto a table. The balls roll off the table, experience friction, lose energy, and eventually reach the ground again. The person then lifts them back onto the table, repeating the process. In this analogy, the person providing energy to lift the balls acts as the source of EMF,

supplying the necessary energy to keep the process going. Similarly, in an electrical circuit, the EMF provides the energy required to move charge through various components, compensating for energy losses due to resistance and other factors.

Formally, electromotive force is defined as the energy provided per unit charge to move it around a closed circuit. This means that if twice the amount of charge is moved, twice the amount of energy is required. The source of EMF in an electrical circuit can vary, with different physical processes generating the required potential difference.

The sources of EMF can vary, depending on different physical processes that generate the required potential difference. One of the most common sources is chemical reactions, as seen in batteries where interactions between electrodes and electrolytes create a potential difference that drives electrons from one electrode to another. Another source of EMF is magnetism, where a changing magnetic field induces an electromotive force in a conductor, forming the basis of electrical generators and transformers. Light can also generate EMF through the photovoltaic effect, where certain materials, when exposed to light, produce electrical energy. This principle is used in solar panels, which convert sunlight into electricity to power satellites and other remote applications. Furthermore, heat energy can generate EMF through the thermoelectric effect, where differences in temperature across certain materials induce a potential difference. This phenomenon is widely applied in thermocouples for temperature measurement and power generation in space applications. Regardless of the source, the conversion of one form of energy into electrical energy results in the generation of electromotive force, which then drives current through the circuit.

Kirchhoff's Voltage Law (KVL)

Once charge carriers begin to move through the circuit under the influence of electromotive force, they encounter various components such as resistors and capacitors. As the charge moves through these elements, energy is dissipated or stored. A fundamental principle governing this behavior is Kirchhoff's Voltage Law (KVL), which states that the sum of all potential differences around a closed loop in a circuit is equal to zero. This law can be understood by considering a circuit where charge starts its journey at a certain potential and moves through different circuit elements such as resistors and capacitors. As it progresses, it experiences drops in potential across each component. However, by the time it completes the loop and returns to the starting point, the total potential must sum to zero. This means that the energy supplied by the EMF source is entirely accounted for by the energy dissipated or stored in the circuit.

elements. Mathematically, Kirchhoff's Voltage Law is expressed as the sum of all voltage drops and EMF sources in a closed loop being equal to zero. In practical terms, this law ensures that no energy is lost or created within an electrical circuit, aligning with the principle of conservation of energy.

In the simplest case of an electrical circuit as shown by figure 26.4, consider a battery that provides an electromotive force (EMF) and a single resistor connected in series with it. The circuit consists of four points: a, b, c and d, forming a closed loop.

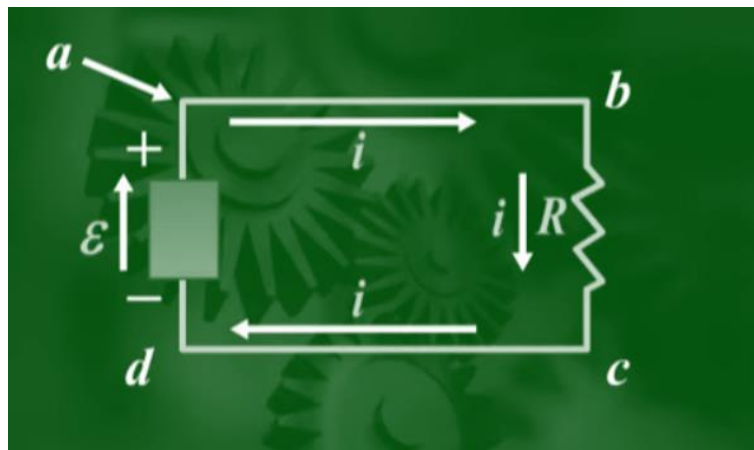


Figure 26.4: A simple circuit as an example for Kirchhoff's law

Starting from point a and moving towards point b, the wire is considered to have negligible resistance, resulting in no potential difference between these points. As we move from point b to point c, the resistor comes into play, and a voltage drop occurs due to the resistance. Since the resistor opposes the flow of current, the potential at point b is higher than at point c, and the voltage drop across the resistor is given by the product of the current and resistance. Continuing from point c to point d, since there is no additional resistance in the connecting wire, there is no change in potential. Finally, moving from d back to point a, the battery provides the electromotive force, which increases the potential by an amount equal to the EMF. Applying Kirchhoff's Voltage Law (KVL), which states that the sum of potential differences in a closed loop must equal zero, we find that the sum of the voltage drops across the resistor and the EMF of the battery results in a balanced equation.

$$V_a - iR + \varepsilon = V_a$$

$$iR + \varepsilon = 0$$

$$\varepsilon = iR$$

Simplifying this equation leads to the well-known relation that the EMF of the battery is equal to the voltage drop across the resistor, confirming Ohm's Law. This simple example demonstrates the fundamental principle of Kirchhoff's Voltage Law, which is applicable to more complex electrical circuits. The law asserts that if we start from any point in the circuit and traverse through all the components to return to the same point, the algebraic sum of all potential differences encountered must equal zero. This principle is rooted in the conservative nature of the electric field, meaning that the work done in moving a charge around a closed loop is zero. In physics, forces can be classified as conservative or non-conservative. Conservative forces, such as gravitational force, allow the definition of potential energy as a function of position. For a conservative force, if an object is moved around a closed path and returns to its starting position, no net work is done. In electrical circuits, the electric force behaves conservatively, which is why Kirchhoff's Voltage Law holds true.

The figure 26.5 illustrates another application of Kirchhoff's Voltage Law (KVL) in a simple electrical circuit. In this setup, there is a source of electromotive force (EMF) connected to two resistances.

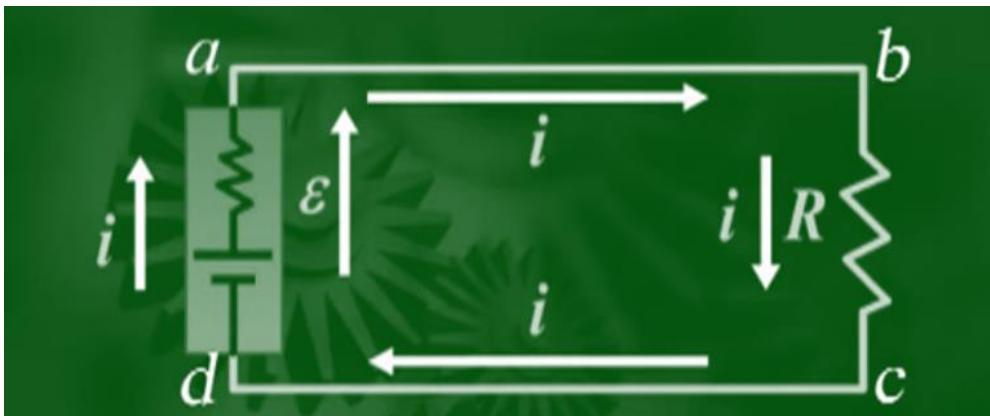


Figure 26.5: A circuit where EMF source has internal resistance r .

The first resistance is intrinsic to the EMF source, often referred to as the internal resistance, while the second is an external resistance connected in the circuit. A current flows through the entire loop, and the objective is to determine its magnitude. The internal resistance of the EMF source arises from the physical properties of its components. For example, in a typical battery (such as a 1.5V or 9V battery), the internal resistance is due to the materials and electrolytes present between its terminals. Even though there is no visible resistor inside the battery, this inherent resistance prevents an infinite current from flowing when the terminals are directly connected without an external load. Therefore, in any circuit containing a real-world EMF

source, the internal resistance must be accounted for when analysing the current flow. To solve for the current, Kirchhoff's Voltage Law is applied to the circuit. According to KVL, the sum of potential differences across all elements in a closed loop is equal to zero. Here, the total EMF of the source is opposed by the voltage drops across the internal resistance and the external resistance. This relationship can be expressed mathematically to calculate the current as follows.

$$V_a + \varepsilon - ir - iR = V_a$$

$$\varepsilon - ir - iR = 0$$

$$\varepsilon = i(r + R)$$

$$i = \frac{\varepsilon}{r + R}$$

While this example deals with a simple, single-loop circuit, Kirchhoff's laws can be extended to more complex configurations. In circuits with multiple loops, KVL can be applied individually to each loop to analyse current flow and potential differences. Similarly, resistors, capacitors, or other circuit elements can be incorporated into the loop, and Kirchhoff's laws will still hold true for the system.

RC Circuit as an example of KVL

The figure 26.6 illustrate the behavior of a capacitor in a circuit and its relation to current flow, potential drops, and time dependence. The circuit components include a battery (as a source of EMF), a capacitor, a resistor, and, in some cases, a light-emitting diode (LED).

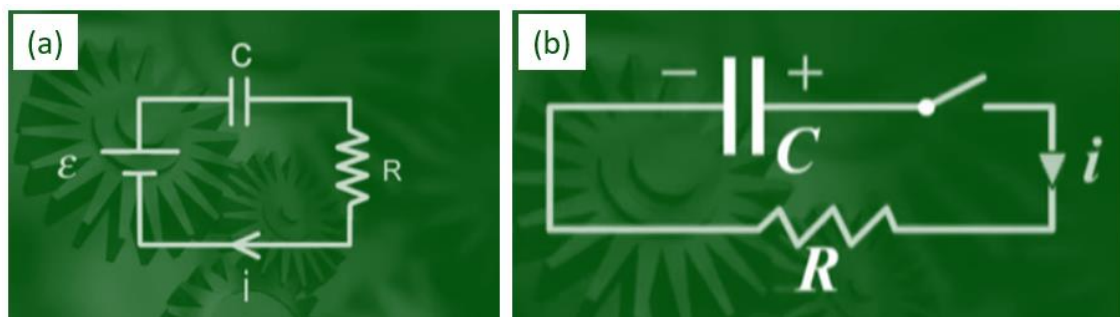


Figure 26.6: (a) charging and (b) discharging of capacitor

When the battery is connected to the circuit containing the capacitor, current initially flows, allowing the capacitor to charge. This charging process is analogous to filling a water tank: as water is pumped into the tank, the flow continues until the tank is full. Similarly, once the

capacitor is fully charged, the current stops flowing. This is because the potential difference across the capacitor becomes equal to the EMF of the battery, leaving no net driving force for further current. Kirchhoff's Voltage Law (KVL) governs this process. According to KVL, the sum of potential drops in a closed-circuit loop is zero. In this circuit, the potential drops occur across the capacitor and the resistor. The voltage across the capacitor is proportional to the charge stored on it, while the voltage drops across the resistor is proportional to the current. These relationships lead to a mathematical equation describing the circuit, which can be solved to determine how the charge and current vary over time. For circuit depicted by figure 26.6(a)

$$\varepsilon - \frac{q}{C} - iR = 0$$

If the charged capacitor is disconnected from the battery and connected to an LED, the stored charge creates a current that briefly powers the LED. The LED lights up momentarily, as the current diminishes with time until the capacitor is fully discharged. This process illustrates that the charge on the capacitor decreases exponentially, governed by the time constant of the circuit. For circuit depicted by figure 26.6(b)

$$\frac{q}{C} + iR = 0$$

Differentiating whole equation with respect to t i.e. d/dt

$$\frac{1}{C} \frac{dq}{dt} + \frac{di}{dt} R = 0$$

As

$$\frac{dq}{dt} = i$$

Equation becomes

$$\frac{i}{C} + \frac{di}{dt} R = 0$$

$$\frac{di}{dt} R = -\frac{i}{C}$$

$$\frac{di}{dt} = -i \frac{1}{RC}$$

$$i = i_0 e^{-\frac{t}{RC}}$$

The time constant (denoted by RC) is a critical parameter that defines how quickly the capacitor charges or discharges. Physically, it depends on the product of the resistance and capacitance. A larger capacitance (analogous to a larger water tank) or higher resistance (analogous to a narrower pipe or a partially opened tap) results in a longer time constant, meaning the capacitor will take more time to charge or discharge. This behavior of the capacitor can be described using differential equations. The solution to these equations shows that the current decreases exponentially over time, starting at an initial value determined by the circuit configuration. This mathematical model aligns with the physical intuition of how charge and current evolve in RC circuits, offering insights into both transient and steady-state behavior.

Kirchhoff's Current Law (KCL)

In electrical circuit analysis, there are two fundamental laws that govern the behavior of currents and voltages. The first law, commonly referred to as Kirchhoff's Voltage Law (KVL), states that the sum of all potential drops around a closed loop in a circuit is equal to zero. This principle is based on the conservation of energy, ensuring that the total energy gained and lost by charges in a complete loop balance out.

The second law, known as Kirchhoff's Current Law (KCL), states that the total current entering a junction in a circuit is equal to the total current leaving the junction. This is a direct consequence of the conservation of electric charge, which implies that charge cannot accumulate or disappear at any point in the circuit. Together, these laws provide a comprehensive framework for analysing and understanding electrical circuits.

KVL and KCL example

This circuit analysis shown below (for circuit in figure 26.7) demonstrates the use of Kirchhoff's laws to calculate current distribution in a circuit containing resistors and voltage sources. The circuit includes three resistors and two batteries providing electromotive force (emf). The current in the circuit splits into three paths: one through the first resistor, another through the second resistor, and a third through the resistor between the branches.

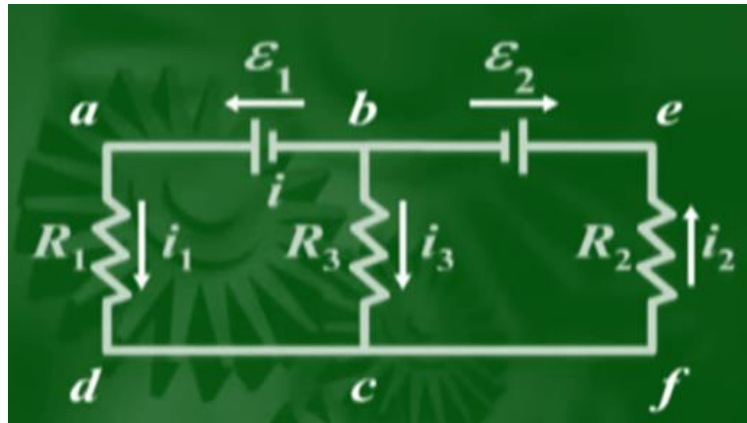


Figure 26.5: A circuit as an example of KVL and KCL

To analyse the circuit, Kirchhoff's Current Law (KCL) is first applied. This law states that the total current entering a junction equals the total current leaving it. At the key junction in this circuit, the current entering divides into two parts, ensuring that the sum of these two currents is equal to the current exiting the junction.

When analysing electrical circuits using Kirchhoff's laws, it is crucial to carefully apply the rules for assigning signs to voltage drops and rises. While traversing a circuit loop in the direction of the current, the voltage drops across a resistor should be considered negative. This is because the current flows from a higher potential to a lower potential across the resistor. On the other hand, when you encounter an electromotive force (EMF) source, the sign depends on the direction of traversal. If you move from the negative terminal to the positive terminal of the EMF source, it represents a gain in potential, and the EMF should be assigned a positive value.

Applying Kirchhoff's laws is fundamentally straightforward, relying more on systematic practice than on memorization of formulas. These laws, based on the principles of energy conservation and charge conservation, provide an intuitive method for analysing even complex circuits. With consistent practice, one can confidently solve various types of intricate circuit problems, ranging from basic setups to advanced configurations. The key to mastering these laws is a clear understanding of the physical principles they represent and a structured approach to solving circuit loops.

Next, Kirchhoff's Voltage Law (KVL) is used. This law states that the sum of all voltages around a closed loop in a circuit must be zero. By applying KVL to different loops in the circuit, equations are written to relate the emf of the batteries to the voltage drops across the resistors.

The voltage drops are proportional to the currents flowing through the resistors, according to Ohm's law.

$$i_1 + i_3 = i_2$$

$$\varepsilon_1 - i_1 R_1 + i_3 R_3 = 0$$

$$-i_3 R_3 - i_2 R_2 + \varepsilon_2 = 0$$

Value of three variables (i_1 , i_2 and i_3) can be determined as we have three equations. By combining the principles of KCL and KVL, the relationships between the currents and resistances in the circuit can be determined, enabling the calculation of the individual current values through each resistor. This approach is fundamental in circuit analysis and illustrates the power of Kirchhoff's laws in solving complex electrical networks.

Power in Circuits

When applying Kirchhoff's laws, which are closely related to the concept of potential, one can explore how much work is done by a source of electromotive force (EMF) when charges move through a circuit. To illustrate this, consider a scenario where a person lifts a heavy object repeatedly, using energy to move it against gravity. Ultimately, the work done in this process transforms into another form of energy, such as heat. In electrical circuits, a similar phenomenon occurs, but it involves charges. Charges do work as they move through the circuit, and the amount of work done depends on two factors: the magnitude of the EMF provided by the source and the quantity of charge involved.

When a charge moves through a potential difference created by an EMF source, work is performed. The work done on the charge is directly proportional to both the value of the EMF and the amount of charge. For instance, when a charge is raised through a potential difference, the work done can be calculated by multiplying the EMF by the charge. To understand this concept, consider a resistor within the circuit. Inside the resistor, charges experience a potential difference, requiring work to move from one end to the other. This work is done by the EMF source, which provides the energy required to push the charges through the resistor.

The relationship between the work done, EMF, and charge can be further refined. Work is defined in terms of the small charge transferred across the potential difference. The potential difference, expressed in volts, measures the energy transferred per unit charge. For instance, if one coulomb of charge passes through a potential difference of one volt, the work done is equal

to one joule. This relationship defines a volt as one joule per coulomb. When current flows through a resistor, the potential difference across the resistor causes energy dissipation in the form of heat. The energy dissipation depends on both the current flowing through the resistor and its resistance.

$$\varepsilon = \frac{dW}{dq} \quad \left(1 \text{ volt} = \frac{1 \text{ joule}}{1 \text{ coulomb}} \right)$$

$$dW = \varepsilon \cdot dq = \varepsilon \cdot i \cdot dt \quad \left(\text{as } i = \frac{dq}{dt} \right)$$

$$dW = \varepsilon \cdot i \cdot dt = (i \cdot R) \cdot i \cdot dt = i^2 \cdot R \cdot dt \quad \left(\text{as } i = \frac{\varepsilon}{R} \text{ or } \varepsilon = iR \right)$$

As

$$P = \frac{dW}{dt} = \frac{i^2 \cdot R \cdot dt}{dt} = i^2 R$$

$$P = \left(\frac{V}{R} \right)^2 \cdot R = \frac{V^2}{R}$$

$$P = i^2 R = i(iR) = iV$$

The power dissipated in a resistor, which represents the rate of energy conversion, can be calculated using multiple relationships. Power is the work done per unit time. It is proportional to the square of the current multiplied by the resistance or, alternatively, to the square of the potential difference divided by the resistance. Another expression for power involves the product of current and potential difference. These relationships highlight that the energy consumed in a resistor per unit time is directly connected to the electrical properties of the circuit. In practical applications, the units used to measure this energy dissipation are joules for work and watts for power. One watt is defined as one joule of work done per second. Power dissipation in devices such as home appliances is often measured in kilowatts (thousand watts). Understanding these fundamental principles is essential for analysing electrical circuits and quantifying energy usage effectively.

How Electric Current Flows

In understanding the flow of current through a material, several fascinating phenomena are revealed. If it were possible to observe the inner workings of a conductor, one would see that the charge carriers responsible for current are primarily negatively charged electrons. While

there are no visible moving positive charges in typical conductors, the electrons themselves do not move in a smooth, straight path. Instead, they constantly experience collisions and scatter in various directions as they make their way through the material.

A closer examination, perhaps using animation shown in video lecture, illustrates how charge carriers move within a conductor, such as a copper wire. When a current flows through the wire, it causes devices like bulbs to light up as the moving electrons transfer energy. If the wire is interrupted, the current flow stops, and the bulb goes out. Observing at a microscopic level, one can see electrons colliding with atoms and other electrons as they move through the conductor. This erratic movement leads to what is known as the drift velocity, which is the average speed at which electrons move in response to an electric field.

To understand how much current flows through a material when an electric field is applied, consider a small piece of wire with length L , cross-sectional area A , and a specific number density of electrons n . The total charge q in the wire can be calculated by considering the charge of individual electrons and the number of electrons in a given volume $V=AL$. The current, defined as the flow of charge q per unit time t , depends on the number of charge carriers, their density, and their drift velocity V_d . The current density j , a measure of the current per unit area, is directly proportional to the product of the charge carrier density and the drift velocity. This means that materials with a higher density of carriers n or faster drift speeds V_d will exhibit higher current densities.

$$q = (nV)e = (nAL)e$$

$$V_d = \frac{L}{t} \quad \text{or} \quad t = \frac{L}{V_d}$$

$$i = \frac{q}{t} = \frac{nALe}{L/V_d} = nAeV_d$$

$$V_d = \frac{i}{nAe} = \frac{j}{ne} \quad \left(\text{as } j = \frac{i}{A} \right)$$

$$j = neV_d$$

Drift velocity V_d , though central to understanding current flow, is surprisingly small. Electrons in a conductor move very slowly on average due to constant collisions. For instance, an electron in a wire may take hours or even days to travel a relatively short distance. Despite this slow individual movement, electricity appears to travel almost instantaneously. This phenomenon is

explained by the collective motion of electrons. When a switch is closed, electrons near the switch push those ahead of them, setting off a chain reaction that transmits energy across the entire circuit at nearly the speed of light.

In subsequent lectures, we will explore topics such as resistivity and conductivity, which depend on factors like the material's structure and the type of charge carriers present. While most discussions of conductivity focus on electrons, there are cases, such as in semiconductors, where positive charge carriers (holes) also contribute to current flow. This interplay of positive and negative carriers can be observed more closely when studying phenomena like the Hall effect, which provides insight into the behavior of holes as current carriers. These topics will be discussed in greater detail in future lectures.

LECTURE # 27_ THE MAGNETIC FIELD

Discovery of Magnets

The phenomenon of magnetism was first observed in ancient times. It is said that a shepherd named Magnus discovered stones that attracted pieces of iron. The term "magnet" is derived from his name. Initially, magnets were used for navigation in shipping, but today, they are integral to various technologies.

Applications of Magnetism

Magnets are present in various aspects of daily life and advanced technology. For instance:

- **Magnetic Levitation (Maglev) Trains:** These trains float above tracks, lifted by powerful magnetic forces, reducing friction.
- **Magnetic Resonance Imaging (MRI):** This technology allows detailed imaging of internal organs without physical contact, leveraging magnetic fields to interact with cellular structures.

Properties of Magnets

Magnetic Poles

Magnets have two poles: North and South. **Opposite poles attract, while like poles repel.** This interaction can be demonstrated using bar magnets or other magnetic materials.

Magnetic Fields

When iron filings are scattered around a magnet, they form patterns indicating the magnetic field's strength and direction. The concentration of filings is highest near the poles, where the field is strongest.

Differences Between Magnetism and Electrostatics

Magnetism differs fundamentally from electrostatics:

- Unlike electrostatics, where positive and negative charges exist, magnetostatics (study of stationary magnets) does not have isolated magnetic poles. Instead, every magnet always has a North and South Pole.
- In electrostatics, a stationary charge experiences a force in an electric field, whereas in magnetostatics or magnetism, a force acts on a moving charge in a magnetic field.

Force on a Charge in a Magnetic Field (Magnetic Force)

A stationary charge in a magnetic field experiences no force. When the charge moves, the force F is given by:

$$\vec{F} = q(\vec{v} \times \vec{B})$$

Where q is the charge, $\vec{v}$ is the velocity of the charge, $\vec{B}$ is the magnetic field. $\vec{v} \times \vec{B}$ denotes the cross product, which ensures the force is perpendicular to both $\vec{v}$ and $\vec{B}$, as determined by the **right-hand rule** shown in figure 27.1.

The Right-Hand Rule

To determine the direction of the magnetic force, use the right-hand rule:

1. Forefinger (Pointer Finger): Point your forefinger in the direction of the velocity $\vec{v}$ of the positive charge.
2. Middle Finger: Point your middle finger in the direction of the magnetic field $\vec{B}$.
3. Thumb: Your thumb will point in the direction of the magnetic force $\vec{F}$ acting on the charge.

For a negative charge, like an electron, the force direction will be opposite to that indicated by your thumb.

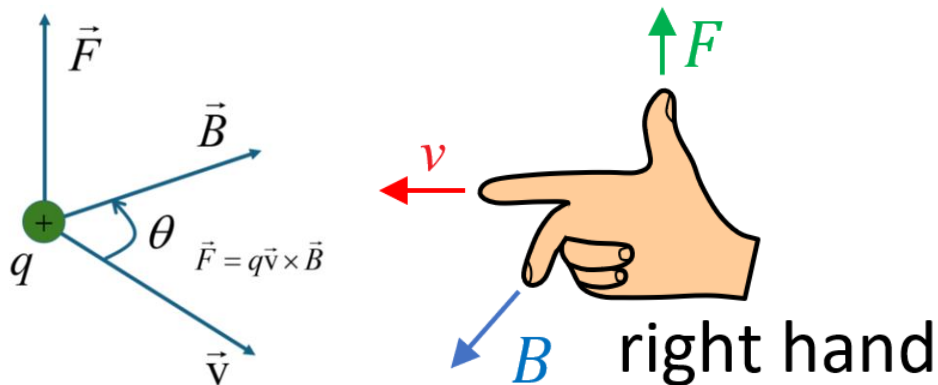


Figure 27. 1. Right hand rule to find the direction of force on a charged particle in a magnetic field.

Magnetic Units and Field Strength

Tesla and Gauss

The unit of magnetic field strength is the Tesla (T). One Tesla is defined as:

$$\vec{F} = q\vec{v} \times \vec{B}$$

$$F = qvB \sin \phi$$

$$B = \frac{F}{qv} \quad (\text{if } \phi = \frac{\pi}{2})$$

$$1 \text{ tesla} = 1 \frac{\text{newton}}{\text{coulomb} \cdot \text{meter/second}}$$

$$1 \text{ tesla} = 1 \frac{\text{newton}}{\text{ampere} \cdot \text{meter}} = 10^4 \text{ gauss}$$

Examples of Magnetic Field Strengths

- | | |
|---------------------------|---------------------|
| 1. Earth's surface | 10^{-4} T |
| 2. Bar magnet | 10^{-2} T |
| 3. Powerful electromagnet | 1 T |
| 4. Superconducting magnet | 5 T |

The Lorentz Force

When both electric and magnetic fields are present, the total force on a charge is the Lorentz Force:

$$\vec{F} = q\vec{E} + q(\vec{v} \times \vec{B}) = q\vec{E} + q\vec{v} \times \vec{B}$$

Here E is the electric field and $\vec{v} \times \vec{B}$ is the magnetic component.

This relationship is pivotal in understanding the interplay of electric and magnetic forces, which can act in different directions.

Magnetic and Electric Forces in Opposing Directions

Consider a scenario where magnetic and electric forces are aligned in opposite directions such that they cancel each other. This setup can be visualized with two parallel plates: one negatively charged and the other positively charged. A positively charged particle placed between the plates experiences a downward electric force due to the electric field.

If a positively charged particle enters the region between these plates from one side, it is pushed downward by the electric field. Now, suppose a magnetic field is introduced perpendicular to the plane of motion. The magnetic field can exert an upward force on the particle, opposing the electric force. By adjusting the potential difference between the plates, the electric field can be fine-tuned so that the electric and magnetic forces exactly cancel each other.

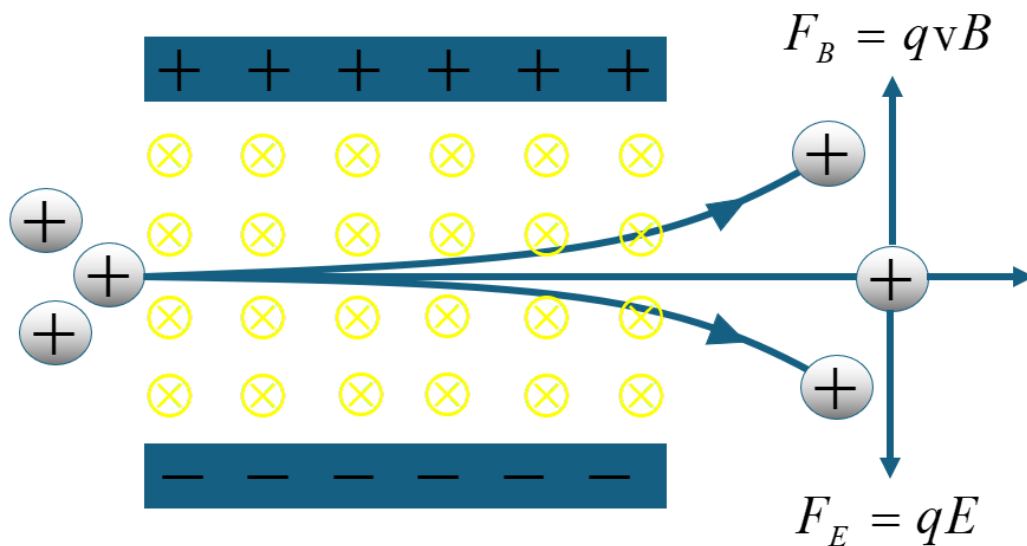


Figure 27. 2. Working mechanism of velocity selector.

The condition for this balance can be mathematically expressed as:

$$qE = qvB \Rightarrow v = \frac{E}{B} \quad \therefore \vec{v} \times \vec{B} = vB \sin \theta = vB \sin 90 = vB$$

This arrangement acts as a **velocity selector**, allowing particles with a specific velocity to pass through undeflected (see figure 27.2).

Work Done by Electric and Magnetic Fields

- The electric field does work on a charged particle by exerting a force that moves the particle along the direction of the field.
- In contrast, the magnetic field does not perform work on a charged particle. The force exerted by a magnetic field is always perpendicular to the velocity of the particle, resulting in no displacement in the direction of the force. Hence, there is no work done by the magnetic force.

Mathematically:

$$\begin{aligned}
 dW &= \vec{F} \cdot d\vec{r} = q(\vec{v} \times \vec{B}) \cdot d\vec{r} & \therefore \text{putting the value of } \vec{F} \\
 &= q(\vec{v} \times \vec{B}) \cdot \frac{d\vec{r}}{dt} dt \\
 &= q(\vec{v} \times \vec{B}) \cdot \vec{v} dt & \therefore \frac{d\vec{r}}{dt} = \vec{v} \\
 &= 0
 \end{aligned}$$

This is because $\vec{v} \times \vec{B}$ is perpendicular to $\vec{v}$, and the dot product of perpendicular vectors is zero.

Motion of Charged Particles in Magnetic Fields

When a charged particle enters a magnetic field, it experiences a force that changes its direction but not its speed. This results in circular motion, with the **magnetic force acting as the centripetal force**. The radius r of this circular path is determined by:

$$\begin{aligned}
 qvB &= m \frac{v^2}{r} \\
 r &= \frac{mv}{qB} \\
 \omega &= \frac{v}{r} = \frac{qB}{m}
 \end{aligned}$$

where m is the mass of the particle, v is its velocity, q is its charge, and B is the magnetic field strength. *Larger values of m or v result in a larger radius, while stronger magnetic fields (higher B) lead to a smaller radius.*

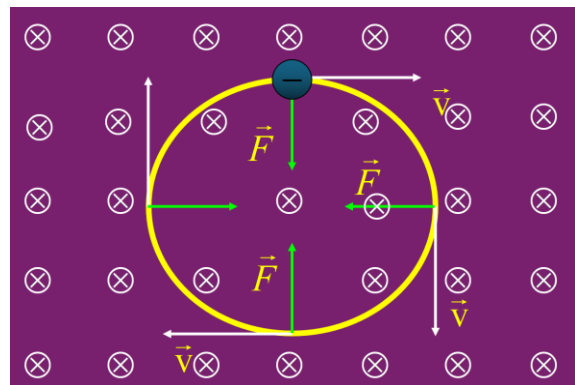


Figure 27. 3. Circulating charges in a magnetic field.

Applications of Magnetic Fields

Mass Spectrometer

A mass spectrometer is a device that uses the principles of electric and magnetic fields to measure the mass of atoms. In a mass spectrometer:

1. Charged particles are first passed through a velocity selector, ensuring they have a specific velocity.
2. These particles enter a region with a magnetic field, where they follow circular paths as shown in figure 27.4.

3. The radius of the path depends on the particle's mass, velocity, charge, and the magnetic field strength. As the particles enter the magnetic field, those with a higher mass-to-charge ratio will have a larger radius of curvature and will be deflected less. Conversely, lighter particles will be deflected more and follow a smaller radius.

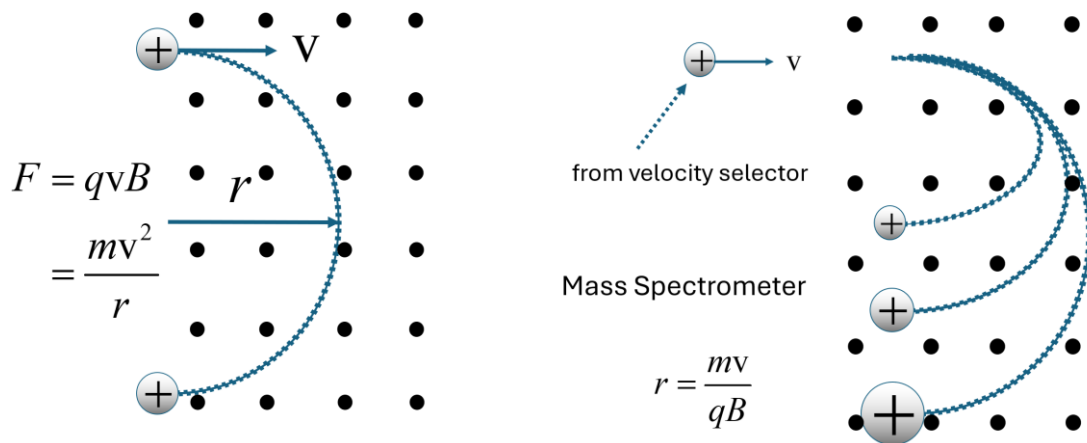


Figure 27. 4. Working mechanism of mass spectrometer.

By measuring the radius of the path, the mass of the particle can be calculated. This technique separates particles like lithium, sodium, oxygen, and lead, enabling precise mass measurements.

The Protective Role of Earth's Magnetic Field

The Earth's magnetic field plays a crucial role in protecting life. Charged particles from the Sun, such as protons and alpha particles, are deflected by the Earth's magnetic field. Without this deflection, these high-energy particles would collide with the Earth, causing significant damage.

Particle Accelerators

Magnetic fields are also used in particle accelerators to study the fundamental properties of matter. In these accelerators, charged particles are made to travel in circular paths within magnetic fields. With each cycle, their energy is increased until they reach speeds close to the speed of light. By colliding these high-energy particles, scientists can investigate the basic constituents of nature. Facilities like CERN in Geneva utilize this principle to conduct cutting-edge research in physics.

Magnetic Force on Current-Carrying Conductors

When many charges move together, as in an electric current, the collective force can be calculated. The motion of electrons within the conductor is responsible for the current, and their interactions with the magnetic field result in the observed force.

When electrons move through a wire, they do not move at a uniform velocity but rather exhibit an average velocity known as **drift velocity** $\vec{v}_d$. This drift velocity is used to calculate the force on a current-carrying wire in a magnetic field.

$$\vec{F} = Ne\vec{v}_d \times \vec{B}$$

The **number density of electrons** in the wire is crucial for this calculation, which is the number of electrons per unit volume ($n = N / V$). The total number of electrons in the wire can be determined by multiplying this number density by the volume of the wire, which is the cross-sectional area multiplied by the length ($N = nV = nAL$).

$$\begin{aligned}\vec{F} &= nALe\vec{v}_d \times \vec{B} \\ nALe\vec{v}_d &= \vec{I} \\ \vec{F} &= \vec{I} \times \vec{B}\end{aligned}$$

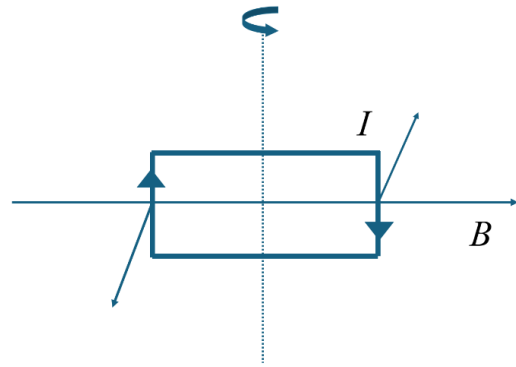
This shows that the force increases as the current and magnetic field strength increase.

When a straight wire carrying current is placed inside a magnetic field, a force is exerted on it. The direction of the force is given by the **cross product** of the current and magnetic field vectors. If the wire is in the shape of a rectangular loop, with current flowing through it, the force acts differently on various parts of the loop:

- The part of the wire that is parallel to the magnetic field experiences no force, as the cross product of the current and magnetic field vectors results in zero force.
- The parts of the wire that are perpendicular to the magnetic field experience forces that act inward or outward, creating a **torque** that causes the loop to rotate.

$$\begin{aligned}
 \tau &= F \frac{a}{2} + F \frac{a}{2} \\
 &= IBb \frac{a}{2} + IBb \frac{a}{2} \\
 &= I \times (\text{area of loop}) \times B \\
 &= \mu B
 \end{aligned}$$

$\mu = IA$ is called the MAGNETIC MOMENT



Magnetic Moment

The **magnetic moment** of a current loop is a vector quantity, and it represents the strength and orientation of the magnetic effect generated by the current. The direction of the magnetic moment is perpendicular to the plane of the loop, and its magnitude is given by the product of the current and the area of the loop.

When a magnetic moment is placed in an external magnetic field, it experiences a torque that tries to align it with the magnetic field. The energy of the system is minimized when the magnetic moment is aligned with the magnetic field. If the magnetic moment is reversed, its energy is maximized.

The behaviour of magnetic moments is critical in various fields such as atomic physics, nuclear physics, and material science, where currents circulate at the atomic or molecular level. The smallest magnetic moment is associated with an electron, which can be considered as the smallest magnet. The electron's magnetic moment arises due to its charge and its rotational motion.

Ampere's Law

Ampere's Law describes the relationship between a current and the magnetic field it produces. Ampere's Law states that the integral of the magnetic field around a closed loop is proportional to the total current enclosed by the loop. The mathematical expression for Ampere's Law is:

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I_{\text{enclosed}}$$

Where B is the magnetic field, ds is a differential element of the area around the loop, I_{enc} is the enclosed current, μ_0 is the permeability of free space.

Ampere's Law is often applied to a hypothetical loop, called an **Amperian loop**, that encloses a current. By integrating the magnetic field along the loop and calculating the enclosed current, we can determine the magnetic field in the region surrounding the current. In the loop shown in figure 27.5, total current $I = i_1 + i_2$. Here i_3 is excluded as it lies outside.

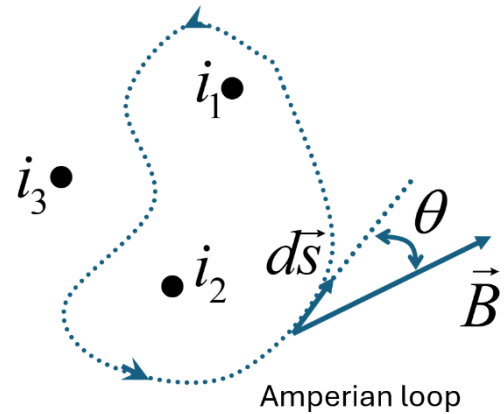


Figure 27. 5. Amperian loop

Application of Ampere's Law

The law is particularly useful for calculating the magnetic field in cases of symmetry, such as around straight wires or circular current loops.

Magnetic Field Around an Infinite Straight Wire

Consider an infinitely long straight wire carrying current I as shown in figure 27.6. To calculate the magnetic field at a point around the wire, we can apply Ampere's Law. The magnetic field produced by the current flows in circular loops around the wire, and at every point on the loop, the magnetic field is tangent to the circle. To calculate the magnetic field at a distance r from the wire, we apply **Ampere's Law**.

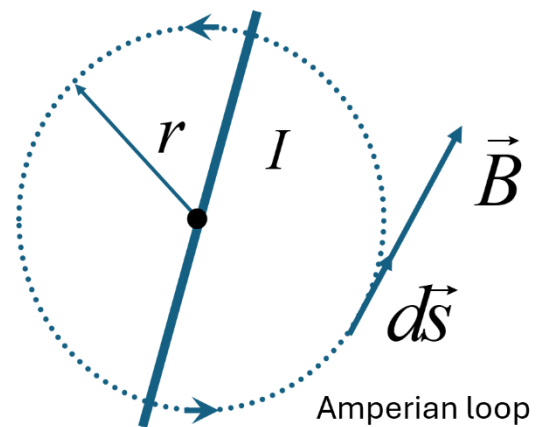


Figure 27. 6. Amperian loop around an infinite straight wire.

Ampere's Law states that the integral of the magnetic field along the loop is equal to the current enclosed by the loop. Mathematically,

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I_{\text{enclosed}} \quad \text{.....(1)}$$

Since the magnetic field and the differential area element ds are parallel, we can write:

$$\begin{aligned} \oint \vec{B} \cdot d\vec{s} &= \int B ds \cos \theta = B \int ds & \therefore \cos \theta &= 1 \\ \oint \vec{B} \cdot d\vec{s} &= B(2\pi r) & \text{.....(2)} & \therefore \int ds = \text{total length of loop} = \text{circumference of circle} \end{aligned}$$

where r is the distance from the wire. Substituting eq (2) in (1), and solving for B , we get:

$$B = \frac{\mu_0 I}{2\pi r}$$

This result shows that the magnetic field strength is directly proportional to the current and inversely proportional to the distance from the wire. As the distance increases, the magnetic field strength decreases.

Magnetic Field Inside a Solenoid

Consider a solenoid, which is a coil of wire with many turns. If current flows through this coil, a magnetic field is generated inside it. The magnetic field inside a solenoid is approximately uniform (see figure 27.7), and it is directed along the axis of the solenoid. To calculate the magnetic field, we use Ampere's Law again. We draw an **Amperian loop** abcd inside the solenoid.

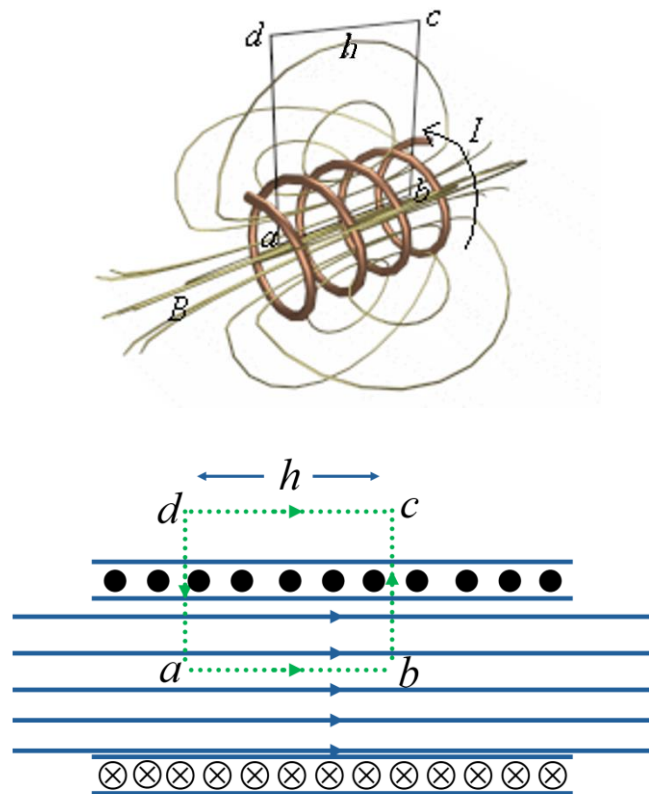


Figure 27. 7. Uniform magnetic field inside a solenoid.

Inside the solenoid, along path ab, the magnetic field is uniform and parallel to the path of integration. The other three sides are outside or perpendicular to the solenoid, where the magnetic field is negligible or perpendicular to the path ($\cos 90 = 0$). Thus, for a solenoid with n turns per unit length and a current I , the magnetic field inside the solenoid is given by:

$$\int \vec{B} \cdot d\vec{s} = \int_a^b \vec{B} \cdot d\vec{s} + \int_b^c \vec{B} \cdot d\vec{s} + \int_c^d \vec{B} \cdot d\vec{s} + \int_d^a \vec{B} \cdot d\vec{s}$$

$$\int \vec{B} \cdot d\vec{s} = \int_a^b \vec{B} \cdot d\vec{s} + 0 + 0 + 0$$

$$I = I_0 nh$$

Applying Ampere's law

$$B\cancel{h} = \mu_0 I_0 n \cancel{h}$$

$$B = \mu_0 I_0 n$$

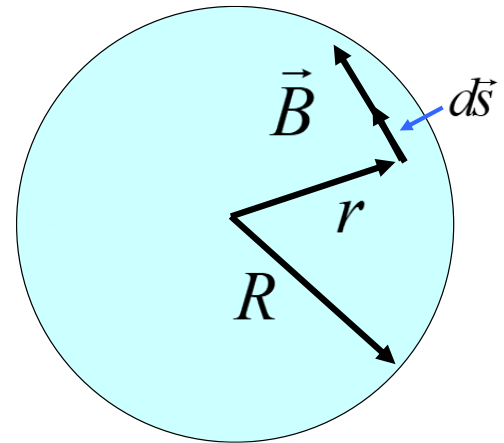
This equation shows that the magnetic field inside the solenoid is directly proportional to the number of turns per unit length and the current.

Magnetic Field Inside an Infinite Wire with Finite Thickness

Consider a thick, infinite wire and calculate the magnetic field inside it. Assume the current density is uniform across the cross-section of the wire, meaning it is the same everywhere within the wire's radius R . The total current I is related to the current density J by:

$$I = J \cdot A_{total}$$

$$J = \frac{I}{A_{total}} = \frac{I}{\pi R^2}$$



Draw an **Amperian loop** inside the wire as shown in figure 27.8. The magnetic field at a distance r inside the wire depends on the amount of current enclosed within that radius.

Figure 27. 8. Amperian loop inside an infinite wire with finite thickness.

For a radius r (where $r < R$), the area of the cross-section enclosed by the Amperian loop is:

$$A_{enclosed} = \pi r^2$$

The current enclosed within this smaller cross-section is proportional to the enclosed area, so:

$$I_{enc} = J \cdot A_{enclosed}$$

Substitute $J = \frac{I}{\pi R^2}$ and $A_{enclosed} = \pi r^2$

$$I_{enc} = \frac{I}{\pi R^2} \cdot \pi r^2 = I \left(\frac{\pi r^2}{\pi R^2} \right)$$

Applying Ampere's law to find the magnetic field at a distance r from the centre of the wire and substituting the value of I_{enc} from the above equation we get:

$$B(2\pi r) = \mu_0 I \left(\frac{\pi r^2}{\pi R^2} \right)$$

$$B = \frac{\mu_0 I r}{2\pi R^2}$$

This result shows that the magnetic field inside the wire increases linearly with the distance r from the centre. At the centre of the wire, the magnetic field is zero, and it increases as we move outward, reaching a maximum when $r = R$ (see graph in fig 27.9).

Magnetic Field Outside the Wire

For a point outside the wire, the magnetic field decreases with the distance from the wire (figure 27.9). At a distance r from the centre of the wire (where $r > R$), the magnetic field is calculated using the total current I flowing through the wire, as no further current is enclosed in the loop.

Applying Ampere's Law for $r > R$:

$$B \cdot 2\pi r = \mu_0 I$$

Solving for B , we get:

$$B = \frac{\mu_0 I}{2\pi r}$$

This is the same formula as for the infinite straight wire, which shows that the magnetic field outside the wire decreases inversely with the distance from the wire.

Magnitude of Magnetic Fields

The magnetic field produced by a current-carrying wire depends on the geometry of the wire:

- For an **infinite straight wire**, the magnetic field decreases inversely with the distance from the wire.
- For a **solenoid**, the magnetic field is uniform inside the solenoid and proportional to the number of turns per unit length and the current.

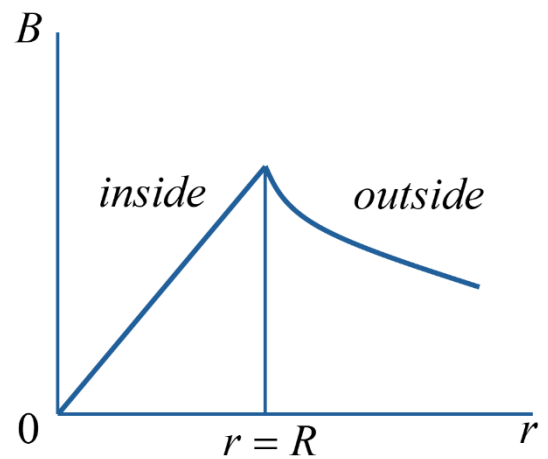


Figure 27. 9. Graph showing the variation of magnetic field B with radius r inside and outside the thick wire.

- For a **thick wire**, the magnetic field inside increases linearly with the distance from the centre, while outside the wire, the field decreases inversely with the distance.

Electron as a Magnet

Electrons themselves behave as tiny magnets due to their rotational motion (spin), and this is the basis for the smallest magnetic moment in nature. On the other hand, objects like **pulsars**, which rotate rapidly, generate extremely powerful magnetic fields.

Lecture 28

Electromagnetic Induction

Introduction to Electromagnetism

When an electric current flows through a conductor, a magnetic field is created around it. Similarly, when a charge moves, it generates electricity and a magnetic field simultaneously. This interconnection between electricity and magnetism forms the basis of electromagnetism. Now, an important question arises: Can electricity be generated from a magnetic field?

The answer is yes, and this principle forms the foundation for devices like generators. Without this understanding, it would have been impossible to create generators or electrical appliances. Hence, this relationship is not just a scientific curiosity but a cornerstone of modern life.

Basic Properties of Magnets

In our previous lectures, we discussed magnets and their poles. Magnets have two poles: north (Shumaal) and south (Junoob). Opposite poles attract each other, while like poles repel. If you examine a magnet closely, you'll observe that its internal structure is arranged in a specific order: north-south, north-south, north-south. This regular alignment generates the magnet's overall magnetic field.

In non-magnetic materials, however, the tiny magnetic domains are disordered, cancelling each other out and resulting in no net magnetism. For instance, in a piece of iron that isn't magnetized, the magnetic domains are scattered in various directions.

Magnetic Lines of Force

The magnetic field around a magnet is represented by lines of force. These lines originate from the north pole, curve around the magnet, and re-enter at the south pole, forming closed loops. Unlike the electric field lines in electrostatics, magnetic field lines have no starting or ending points. This is because there is no equivalent of a "magnetic charge" (like electric charges). Magnetic field lines circulate continuously, without a source or sink. This fundamental difference between electricity and magnetism is a key concept in electromagnetism.

Properties of Magnetic Lines of Force

1. **Closed Loops:** Magnetic lines of force form continuous loops without a source or endpoint.
2. **Non-Intersecting Lines:** Magnetic lines of force never cross each other. To understand why, imagine holding a compass near a magnet. As you move the compass along the magnetic field, its needle always points in one direction, indicating the local direction of the magnetic field. If two lines of force were to intersect, the compass needle would point in two different directions at the same point, which is impossible.

Definition of Magnetic Flux

Magnetic flux is defined as the product of the perpendicular magnetic field and the area through which it passes. This definition holds when the magnetic field is constant and the area is fixed, with the magnetic field being perpendicular to the surface.

$$\phi_B = B.A = BA \cos\theta$$

Where,

- Φ_B is the magnetic flux.
- B is the magnetic field
- A is the area
- θ the angle at which the field lines pass through the given surface area

Magnetic Flux Unit

Magnetic flux is usually measured with a flux meter. The SI and CGS unit of magnetic flux is given below:

- The SI unit of magnetic flux is **Weber (Wb)**.
- The fundamental unit is **Volt-seconds**.
- The CGS unit is **Maxwell**.

Concept of Flux through a Small Area

Consider a small area, denoted as dA , through which a magnetic field passes. If the magnetic field is perpendicular (normal) to this area, the flux through this small piece is the magnetic field B multiplied by the area dA .

Summing Contributions for Total Flux

To calculate the total flux through a surface, the area can be divided into smaller pieces. The flux through each surface element is the magnetic field perpendicular to the surface element multiplied by its area. The total flux is the sum of these contributions, represented by the formula:

$$\phi_B = B_1 \cdot dA_1 + B_2 \cdot dA_2 + B_3 \cdot dA_3 = \sum B_i \cdot dA_i$$

where θ is the angle between the magnetic field and the normal to the surface. When the field is perpendicular, $\theta = 0$, so the flux is simply $B \cdot dA$. If the field is slanted, θ increases, and the flux reduces. When the angle reaches 90° , the flux is zero. The net flux on a closed surface is always zero.

Differences Between Electrostatics and Magnetism

In electrostatics, a closed surface can have a net flux due to the charge enclosed within it, according to Gauss's Law. However, in magnetism, no net flux can emerge from a closed surface, as there are no magnetic charges. This is why a sphere around a current-carrying wire does not have any magnetic flux passing through it. The net flux on a closed surface is always zero.

$$\phi_B = \int \vec{B} \cdot d\vec{A} = 0$$

A simple experiment, conducted by Michael Faraday around 300 years ago, illustrates the relationship between electricity and magnetism. The experiment involves a coil with wire wrapped around it, connected to a meter to measure electromotive force (voltage). When a magnet is moved through the coil, the meter registers a voltage. This experiment demonstrates the fundamental connection between electricity and magnetism.

Demonstrating Electromagnetic Induction

In this experiment, when a magnet is moved through the coil, an EMF is generated, causing the meter's needle to move. As the magnet moves, the voltage is generated, appearing and disappearing in cycles. This effect occurs because the changing magnetic field induces a current in the coil.

The same effect can be achieved by keeping the magnet stationary and moving the coil instead. This experiment, shown in animation, demonstrates that the movement of a magnet through the coil induces electricity, generating a current. This process is known as **induced current**, and it occurs due to the change in the magnetic field.

Electromotive Force (EMF)

Induced current arises from a change in magnetic flux. Although the term "electromotive force" suggests a force, it is not a force in the traditional sense. Rather, it refers to the energy required to move a unit charge through a circuit. The units of EMF are energy, not force.

Faraday's Law of Induction

Faraday's Law explains that the induced electromotive force is proportional to the rate of change of the magnetic flux:

$$\mathcal{E} = -\frac{d\phi_B}{dt}$$

This means that when the magnetic flux changes, EMF is generated. The faster the change in flux, the greater the induced EMF. The negative sign in the equation indicates the direction of the induced current, which is explained by Lenz's Law.

Methods of Changing Magnetic Flux

Three ways to change the magnetic flux and induce EMF are:

1. **Movement of the Magnet:** When the magnet moves through the coil, the flux changes due to the movement of the magnetic field lines.

2. **Changing the Area of the Coil:** If the area of the coil changes (e.g., it becomes larger or smaller), the flux changes, even if the magnet remains fixed.
3. **Rotation of the Magnet:** If the magnet rotates, the flux through the coil changes, even if the magnet does not move closer or farther from the coil.

These three scenarios all lead to a change in flux, inducing EMF in the coil.

Practical Applications

The principle of electromagnetic induction is fundamental to modern technology. Electric generators operate by rotating coils within magnetic fields to produce electricity. Transformers, electric motors, and wireless charging systems are all based on similar principles.

Problem 1: Calculating Induced EMF in a Collapsing Loop

Scenario:

A functional loop of radius **12 cm** is in a uniform magnetic field of strength **0.15 T**. The loop is initially open but is closed at points A and B, collapsing the area to zero within **0.20 seconds**. We need to calculate the induced EMF.

Solution:

Initial Magnetic Flux ($\phi_{initial}$):

The magnetic flux is the product of the magnetic field (B) and the area (A):

$$\phi = B \times A = B \times (\pi r^2)$$

$$\text{Substituting } B = 0.15 \text{ T and } r = 0.12 \text{ m:}$$

$$A = \pi (0.12)^2 \approx 0.0452 \text{ m}^2$$

$$\phi_{initial} = 0.15 \times 0.0452 \approx 0.00678 \text{ Wb (Weber)}$$

Final Magnetic Flux (ϕ_{final}):

When the loop is closed, the area becomes zero. Hence:

$$\phi_{final} = 0$$

Rate of Change of Flux:

The change in flux is given by:

$$\Delta\phi = \phi_{\text{final}} - \phi_{\text{initial}}$$

Substituting the values :

$$\Delta\phi = 0 - 0.00678 = -0.00678 \text{ Wb}$$

Over a time interval $\Delta t = 0.20 \text{ s}$, the rate of change of flux is :

$$\Delta\phi / \Delta t = -0.00678 / 0.20 \approx -0.034 \text{ Wb/s}$$

Induced EMF:

Using Faraday's law:

$$EMF = -\Delta\phi / \Delta t = 0.0339 \text{ V}$$

Thus, the induced EMF is approximately 0.034 V .

Problem 2: EMF with Changing Magnetic Field Strength**Scenario:**

A circular wire loop with a radius of **0.3 m** is placed in a magnetic field. Initially, the field strength is **0.3 T**, perpendicular to the loop. The field changes to **-0.2 T** in **1.5 seconds**. Find the induced EMF.

Solution:**Initial and Final Flux:**

$$\phi = B \cdot A$$

$$A = \pi r^2 = \pi (0.3)^2 \approx 0.2827 \text{ m}^2$$

$$\phi_{\text{initial}} = 0.3 \cdot 0.2827 \approx 0.0848 \text{ Wb}$$

$$\phi_{\text{final}} = -0.2 \cdot 0.2827 \approx -0.0565 \text{ Wb}$$

Change in Flux:

$$\Delta\phi = \phi_{\text{final}} - \phi_{\text{initial}} = -0.0565 - 0.0848 = -0.1413 \text{ Wb}$$

Induced EMF:

$$EMF = -\Delta\phi / \Delta t = -(-0.1413) / 1.5 \approx 0.0942 \text{ V}$$

The induced EMF is approximately 0.095 V.

Flux as a Function of Time

In the graph below, the flux is at its maximum at time $t = 0$, then decreases to zero after about one second, reverses to become negative, and continues to oscillate between positive and negative values. This periodic behavior repeats over time.

Identifying Positive Induced Electromotive Force (EMF)

In the graph, there are three intervals: A, B, and C. The question is: in which interval is the induced electromotive force (EMF) positive?

- In **interval B**, the flux does not change, so the rate of change of flux is zero, meaning the induced EMF is also zero.
- In **interval C**, the flux is increasing, which means the induced EMF is at its maximum and negative.
- In **interval A**, the flux is decreasing. Since the rate of change of flux is negative, the induced EMF is positive.

Thus, the induced EMF is positive in **interval A**.

Induced Electric Field

Electromotive force (EMF) is related to the electric field by the equation:

$$\mathcal{E} = \int \mathbf{E} \cdot d\mathbf{s} = -\frac{d\phi_B}{dt}$$

Faraday's Law states that the induced EMF is the negative rate of change of magnetic flux. The question arises: what is the induced electric field? The electric field is responsible for driving the current, and the induced EMF is the result of the changing magnetic flux.

Example: Moving Rod in a Magnetic Field

Consider a scenario where a metal rod is placed between two parallel conducting rails, forming a complete circuit. As the rod moves, the magnetic flux changes. This causes an induced EMF, which drives current through the circuit. When the current flows, it lights up a bulb.

In this case, the change in flux occurs due to the rod's motion. The induced EMF is proportional to the rate at which the flux changes, which depends on the speed of the rod and the area of the circuit.

Electromagnetic Induction and Current Flow

When a rod moves through a magnetic field, the area of the rod relative to the magnetic field changes. According to Faraday's Law, this change in area results in a change in magnetic flux, which induces an electromotive force (EMF). This EMF pushes the electrons inside the wire, causing current to flow and light up a bulb. The question now arises: How much current is induced, and how can it be calculated?

Using Faraday's Law

To calculate the induced current, we first calculate the flux and then determine how fast it changes. In a typical setup, a loop is being pulled through a magnetic field with speed v . The magnetic field is confined to a region, and as the loop moves, it cuts the magnetic flux lines, resulting in a change in flux.

The induced EMF and the resulting current can be calculated using the formula derived from Faraday's Law:

$$\varepsilon = -\frac{d\varphi_B}{dt} = -\frac{d(BDx)}{dt} = BDv$$

where:

- φ_B is the magnetic flux
- B is the magnetic field strength
- D is the width of the loop
- v is the velocity at which the loop moves

The current I induced in the loop is then given by:

$$I = \frac{\varepsilon}{R} = \frac{BDv}{R}$$

Force and Power Dissipation

The force acting on the loop due to the induced current is calculated by:

$$\vec{F} = I\vec{L} \times \vec{B}$$

which, for a rectangular loop, results in:

$$F_1 = IBD = \frac{B^2 D^2 v}{R^2}$$

The power dissipated due to this force is:

$$P = F_1 v = \frac{B^2 D^2 v^2}{R^2}$$

Alternatively, the power dissipated can also be expressed as:

$$P = I^2 R = \left(\frac{BDv}{R} \right)^2 R$$

where both expressions yield the same result.

Energy Conversion

The mechanical energy used to pull the loop is converted into electrical energy due to the induced EMF in the magnetic field. This process demonstrates the relationship between magnetic fields, induced currents, and energy dissipation, as governed by Faraday's Law of Induction.

Applications of Magnetic Induction

Magnetic induction has numerous applications in the world around us. Let's explore a few examples through an experiment and practical devices.

Experiment with Two Coils

In the experiment, two coils are involved: a primary coil and a secondary coil. When current flows through the primary coil, a magnetic field is generated, which causes a change in the magnetic flux. This change induces an electromotive force (EMF) in the secondary coil. The EMF is measured across the coil using a meter, demonstrating how a changing flux induces an EMF, as stated in Faraday's Law.

Telephone Cards with Magnetic Stripes

A practical example of magnetic induction is the magnetic stripe on telephone cards. The magnetization of the stripe varies across different areas. As the card moves through a reader, the changing magnetic flux induces an EMF in the reader, which reads the information encoded on the card, such as call duration or remaining balance.

VHS Cassettes and Magnetic Tapes

Another example is the VHS cassette, which contains information recorded on a magnetic tape. The playback head of a VCR detects changes in the magnetic flux as it moves over the tape, inducing an EMF. This allows the VCR to retrieve the recorded information, such as video and audio signals.

Magnetic Induction in Rotating Magnets

In a rotating device, such as a motor or generator, a rotating magnet causes changes in the magnetic flux. This induces an EMF, which powers the device. A battery inside the device supplies current to a coil, and the rotation of the magnet continuously changes the flux, inducing a current. This process does not violate conservation of energy, as energy is supplied by the battery.

These examples illustrate the widespread use of magnetic induction in daily life, from telephone cards to VCRs and motors. The key principle in all of these applications is Faraday's

Law, which states that a change in magnetic flux induces an electromotive force, powering various devices.

Lenz's Law and the Direction of Induced Current

After understanding the applications of Faraday's Law, one crucial question remains: in which direction will the induced current flow? This was first addressed by Lenz's Law, which states that:

"Whenever a current is induced, its magnetic field opposes the agent that caused the induction."

In other words, the direction of the induced magnetic field always works against the cause of its induction.

Example of Lenz's Law

In the following animation, you can observe the behaviour of a magnet as it moves towards a coil. When the magnet approaches the coil, a current is induced, and the magnetic field generated by the current opposes the approach of the magnet. When the magnet moves in the opposite direction, the current reverses, and the induced magnetic field opposes this motion as well.

Ring Example with a Magnet

Consider a ring made of wire. When a magnet is brought closer to the ring, an induced current flows, creating a magnetic field. As the magnet gets closer, the magnetic field generated by the induced current increases, making it more difficult to bring the magnet near the ring. This opposing force arises due to the induced current's magnetic field.

Superconducting Wire and High Induced Currents

If the ring were made of a superconducting material, there would be no resistance to the current. Even a small induced electromotive force would produce a very high current, creating a powerful magnetic field that strongly resists the magnet's approach. This shows that Lenz's Law not only governs the direction of induced currents but also explains energy conservation—the induced field always works to prevent further changes.

Energy Conservation and Lenz's Law

The direction of the induced current is governed by Lenz's Law, which ensures energy conservation. The magnetic field generated by the induced current always opposes the change in the magnetic flux that caused it. This principle can be observed in various practical situations, highlighting the fundamental nature of Lenz's Law.

Eddy Currents and Energy Loss

In some cases, induced currents result in energy loss rather than benefits. For instance, in a motor, when the armature rotates, induced currents generate heat in the resistance, leading to energy dissipation. This is referred to as **eddy currents**. Eddy currents are also generated when magnets induce currents in conductors, resulting in energy loss.

Example of Eddy Currents in Motion

Consider a system with two magnets. If one magnet rotates within a conductor, an electromotive force (EMF) is induced. This induced current causes energy loss in the form of heat, as the conductor resists the flow of current. This energy loss due to the induced currents is an example of eddy currents.

Practical Uses of Eddy Currents

While eddy currents often result in unwanted energy loss, they can also be harnessed for useful applications. For example, when detecting metals underground, eddy currents can help identify the presence of metal objects. A magnetic field is applied, and the resulting current is measured to determine if the material is metal. Eddy currents help distinguish between metal and non-metallic substances like wood or stone.

Eddy Currents in Metal Detection

In a mine detector, a pulse of current generates a magnetic field. As this magnetic field changes, an induced current is created in nearby conductive materials, such as metal. According to Faraday's Law, the rate of change in the magnetic flux induces a current. The induced current generates a magnetic field that opposes the applied magnetic field, causing a decrease in the total magnetic field. This difference allows for the detection of metal objects underground.

The Relationship Between Electricity and Magnetism

This lecture highlights the deep and fundamental relationship between magnetic fields and electricity. A changing magnetic field induces a current, while a steady current generates a magnetic field. Understanding these interactions is crucial for various technological applications, from metal detection to electrical engineering.

The Role of Steady Currents

A steady current in a wire generates a magnetic field, but this magnetic field does not affect a stationary charge. This is due to the fact that the magnetic force on a stationary charge is zero, as the velocity vector of the charge is zero. Therefore, a magnetic field alone does not exert force on a charge at rest.

A Thought-Provoking Question

At this point, a thought-provoking question arises: Imagine a wire with positive and negative charges moving through it. If you are standing next to the wire, when the current starts to flow, you would not feel the magnetic field since the charge in your hand is at rest. However, if you were moving with the charge, you would observe the magnetic force acting on you, leading to a contradiction.

The Relativity of Electric and Magnetic Fields

This paradox is a serious question that was first addressed by Einstein. The magnetic field appears differently depending on the frame of reference. Similarly, the electric field also transforms between frames of reference. This transformation of electric and magnetic fields is a key topic in the study of Relativity. While the question will not be answered here, it is worth exploring when studying the theory of relativity in detail.

Lecture 29

Alternating Current

Introduction:

You must have opened or closed a switch many times and whenever you open or close a switch, the circuit is complete. Current flows in this circuit and generally in our homes and in factories, this current is alternating current or AC. The theme of today's lecture is Alternating Current and Alternating Current Circuits.

Alternating current is very useful for us because we can produce it very easily but what I mean by AC current is that it is the current that first goes into one direction then into the other direction and it keeps oscillating or alternating, so if you Look at the electron inside of a wire, sometimes it goes in one direction and sometimes it goes in another direction but on the average, it stays at one place.

First of all, it comes to the question of how AC can be produced.

In the last lecture, we discussed about the Faraday's Law of Induction, based on which “if a magnetic field or say a magnetic flux gets changed, then as a result, an electromotive force is generated”. A force is generated which is called EMF and EMF is equal to minus the rate of change of flux ($EMF = -d\phi/dt$). Using this principle, all the generators in the world produce AC or any current, though it is a big generator of tarbela dam or a small electric motor or a motor or a generator which is attached to your bicycle, which is called a dynamo. The basic principle of all this is that electromotive force is induced by changing magnetic fields or a magnetic flux.

Let us calculate if a coil is inside a magnetic field. If it rotates, how much electromotive force is produced by it?

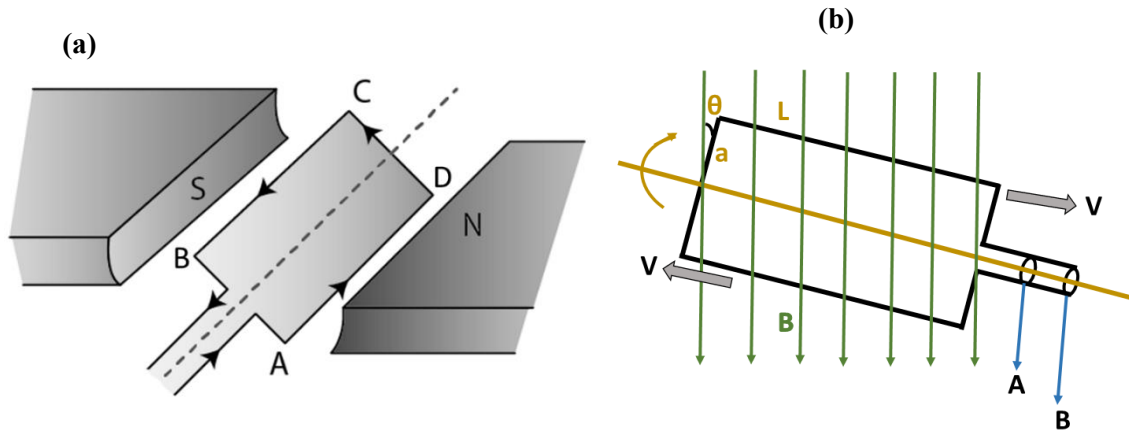


Fig. 29.1: A simple AC generator (a) It represents the rotation of a rectangular coil ABCD between two magnets (b) Representation of the magnetic flux through the rectangular coil having length L.

Here you see that a coil is rotating between two magnets. The length of this coil is L and width is A . Here the EMF is produced only in the parts BC and DA. However, sides AB and CD, being parallel to the magnetic field lines, do not cut across the magnetic field as the coil rotates. Therefore, there is no change in magnetic flux through these sides, resulting in no induced EMF along AB and CD.

Both the parts, BC and DA are equal in length and an equal electromotive force is also produced in them because the electromotive force is the rate of change of magnetic flux, hence the electromotive force in BC and DA is,

$$\mathcal{E}_{BC} = \mathcal{E}_{DA} = Blv_{\perp}$$

$$\mathcal{E}_{total} = 2Blv_{\perp} = 2Blv\sin\theta$$

If the loop is rotating with angular velocity ω , then

$$\mathcal{E} = 2Blv\sin\omega t$$

$$\because v = r\omega = a/2 \cdot \omega$$

$$\mathcal{E} = 2Bl\left(\frac{a}{2}\omega\right)\sin\omega t$$

When the loop is rotating with an angular velocity ω , the linear velocity v of any point on the loop can be expressed in terms of the distance from the axis of rotation ($a/2$) and the angular velocity (ω).

The linear velocity v of a point rotating at a distance r from the axis of rotation with angular velocity ω is given by $v = r\omega$.

In the case of a rectangular loop, if we consider a point at a distance $a/2$ from the center of rotation, its linear velocity v can be expressed as:

$$V = a/2 \cdot \omega$$

Here, $a/2$ represents the distance from the center of rotation to the midpoint of one side of the rectangular loop. Multiply this by ω , the angular velocity, to get the linear velocity of a point on the loop.

Here you see the $\sin\omega t$ appears as if a sine wave is produced as a result of this rotation and this is the reason why it is so easy to produce alternating current. Only if there is a coil in between two magnets, it will be easy to generate AC.

Now let us want to understand AC better. The first thing is that if the current flows in one direction for a period of time, then it flows in the opposite direction for the same period of time as shown in fig 29.2. Therefore, the average value of the current or voltage is zero.

$$\begin{aligned}\mathcal{E} &= \mathcal{E}_m \sin\omega t \\ \mathcal{E}_{average} &= \langle \mathcal{E} \rangle = 0 \quad \dots\dots(1)\end{aligned}$$

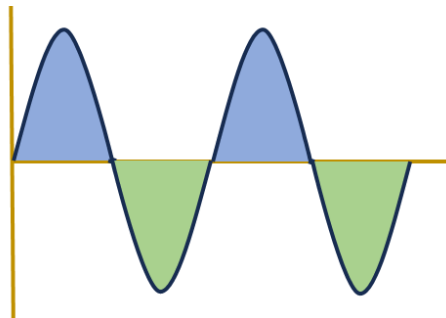


Fig. 29.2: Representation of a sin wave.

According to eq. (1) as much time as the sin wave remains positive, the same time it remains negative; hence the average value of this EMF is zero. But if the average is zero then this does not mean that there is no electricity or that current does not flow.

Remember that power is proportional to v squared divided by r ($P = v^2/r$). It means that you square the electromotive force and then divide it by the resistance r . The basic thing is that a resistor does not care in which direction the current is going; the power dissipation will be in the same direction.

Now the question is that, how can we measure the AC?

To measure a DC, one can use a common meter because its needle stops at one place but if you want to see AC then we have to measure, how many times the needle will have to move back and forth. To measure the AC, we use Oscilloscope and AC signal is identified from two aspects, one is “what is its amplitude”, or “what is its maximum voltage or maximum current”. And second is how fast it goes up and down or in other words “what is its frequency”, or “how many cycles does it complete within a second”. A signal generator is attached with an oscilloscope and the signal generator produces AC voltage and that voltage is going to oscilloscope which then shows the AC signal. We can also decrease or increase its amplitude and frequency from the signal generator to adjust AC signal.

We can measure AC current or voltages either by their maximum current or maximum voltage or there is another quantity called root mean square (RMS) value as shown in fig. 29.3. In RMS, we first square the current or voltage and then take its average and its average will not be zero, because whenever you square a quantity, it becomes positive again.

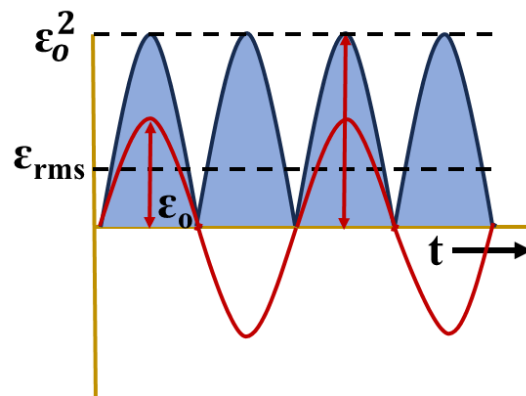


Fig. 29.3: Representation of a root mean square and average of the sin wave.

See here we want to find the average of sine square ωt which I have defined as,

$$\langle \sin^2 \omega t \rangle = \frac{1}{T} \int_0^T dt \sin^2 \omega t$$

$$\therefore \omega = \frac{2\pi}{T}$$

$\therefore$ Using the identity, $\sin^2 x = \frac{1 - \cos 2x}{2}$, if $x = \omega t$ then,

$$\langle \sin^2 \omega t \rangle = \frac{1}{T} \int_0^T dt \left(\frac{1 - \cos 2\omega t}{2} \right)$$

$$\langle \sin^2 \omega t \rangle = \frac{1}{2T} \left(\int_0^T dt - \int_0^T \cos(2\omega t) dt \right)$$

Integrating "dt" over the interval $[0, T]$ gives T:

$$\langle \sin^2 \omega t \rangle = \frac{1}{2T} \left(T - \int_0^T \cos(2\omega t) dt \right)$$

The integral of $\cos(2\omega t)$ over one period is 0, since it completes one full cycle:

$$\langle \sin^2 \omega t \rangle = \frac{1}{2T} (T - 0) = \frac{1}{2}$$

Here, there is only one term left, and this tells us that the average value of $\sin^2 \omega t$ is half. So, whenever you see a quantity which is proportional to $\sin \omega t$ or $\cos \omega t$, its average will definitely be zero and the average of its square will be half.

If it is for voltage (ε) and current (I) then the root mean square of them will be,

$$\varepsilon_{rms} = \sqrt{\frac{\varepsilon_m^2}{2}} = \frac{\varepsilon_m}{\sqrt{2}} = 0.707 \varepsilon_m$$

$$I_{rms} = \sqrt{\frac{I_m^2}{2}} = \frac{I_m}{\sqrt{2}} = 0.707 I_m$$

ε_m , I_m are the peak value of the voltage and current.

Here, one has talked about sin and cosine but alternating current wave can be triangular or square wave. However here also the average will be zero but not the root mean square value. And the root mean square value will not be $1/\sqrt{2}$. We will have to calculate this by squaring it, then integrate it from zero to t, and divide by $1/t$. However, in this lecture we will deal only with sine and cosine functions because they can be very easily produced. A very important thing about AC is that we can increase or decrease its voltage as per our wish and for this we only need a transformer.

Transformer:

A transformer has two coils which are called primary and secondary, then the voltage that is put inside the primary coil gets transformed into AC voltage and then reaches inside the secondary coil. There is a core in the transformer which is usually made of iron or some magnetic material as shown in fig. 29.4. The primary coil that is wrapped around the core is responsible to generate a magnetic field inside this core. This field reverses its direction after some time. If it completes 50 cycles within a second, then the magnetic field first occurs in one direction and then after one second it will change its direction. In this way, a changing magnetic flux is generated inside it.

Also, in secondary coil, a changing magnetic field is passing through the secondary coil and due to this, a current and emf is induced inside it according to the faraday's law.

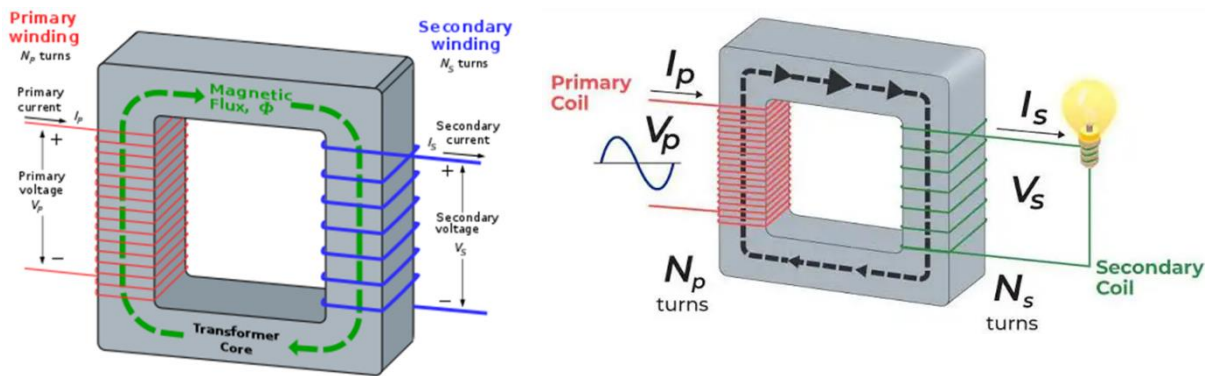


Fig. 29.4: Schematic representation of a transformer.

Flux in transformer:

To calculate the flux in the core, we have:

ϕ = Flux in core

$$\text{EMF (Primary coil)} = \varepsilon_p = -N_p \frac{\Delta\phi}{\Delta t}$$

$$\text{EMF (Secondary coil)} = \varepsilon_s = -N_s \frac{\Delta\phi}{\Delta t}$$

$$\frac{\varepsilon_p}{\varepsilon_s} = \frac{-N_p \frac{\Delta\phi}{\Delta t}}{-N_s \frac{\Delta\phi}{\Delta t}} = \frac{N_p}{N_s}$$

According to the energy conservation,

Primary power = Secondary power

$$\varepsilon_p I_p = \varepsilon_s I_s$$

$$\frac{I_p}{I_s} = \frac{\varepsilon_s}{\varepsilon_p} = \frac{N_s}{N_p}$$

Well, now you must have seen that wapda has very big transformers and the power transmission lines as shown in fig. 29.5 which can be up to 512 kilovolts where electricity is generated.

Electricity generation, transmission, and distribution

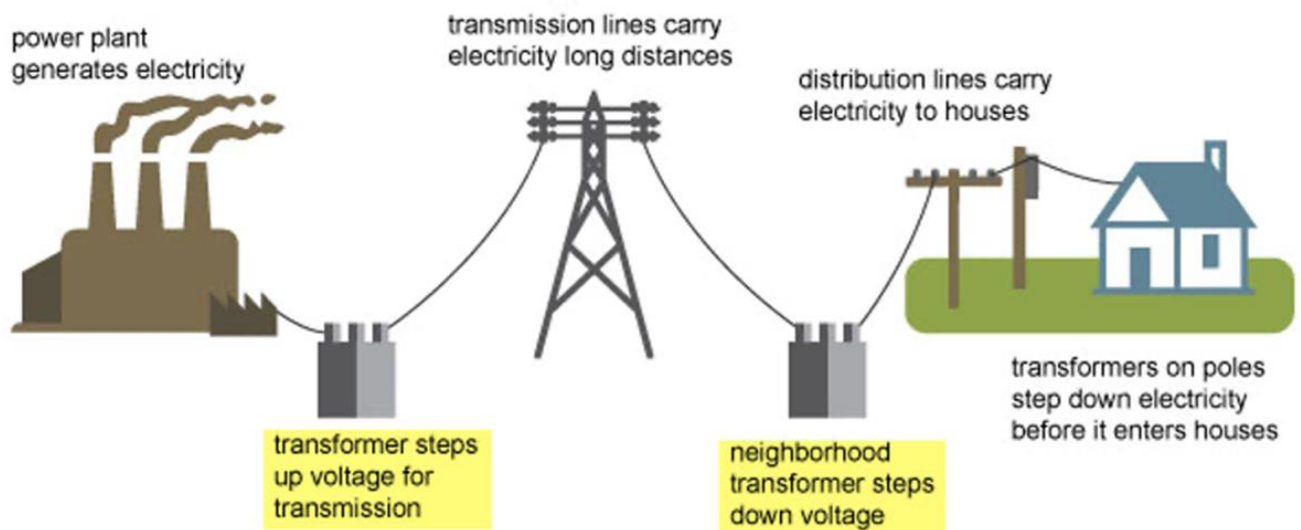


Fig. 29.5: Schematic representation of an electricity transfer from power plant to cities.

In tarbela/mangla dam or in any power station, the voltage is first increased through transformers, then when this electricity reaches the cities, it is decreased through step down

transformers. Why does this happen? Why are we not kept this to the same level? And another question is that why we don't use DC?

The reason for this is that power loss in electrical transmission lines is directly proportional to the square of the current (I^2) and the resistance (R) of the transmission lines. By increasing the voltage and thereby decreasing the current, the power loss in the transmission lines can be minimized. This is described by the formula,

$$P_{\text{loss}} = I^2 R.$$

By reducing the current, the amount of power lost as heat due to resistance in the wires decreases significantly but the product of both the voltage and current remains constant ($VI = \text{constant}$).

Lower power loss means more of the generated electricity reaches its destination. This is particularly important for large-scale power stations like Tarbela and Mangla dams, where huge amounts of electricity are generated and distributed over extensive networks.

Higher voltages used in long-distance transmission lines reduce the need for thicker, more expensive cables, making the system more economically viable. Additionally, lower currents associated with higher voltages are safer for workers and the public, reducing the risk of electrical hazards. In this way the electricity generated at the source in a generator or in a power station reaches the city. And perhaps there would have been a loss of 2, 4 or maximum 5%.

If we had used DC, then about 90% of the electricity would have been wasted. Transformer work on the principle of electromagnetic induction, which relies on the changing magnetic field produced by alternating current (AC). DC (direct current) does not produce a changing magnetic field, so traditional transformers do not work with DC.

Self-inductance:

Self-inductance is a property of every coil and every wire. We define it in such a way that when any current passes through that coil or that wire, as a result a magnetic field (B) is generated which is proportional to the current (I). So, B is proportional to I . This proportional constant will be more or less depending on the number of turns and on the thickness of this coil etc. Phenomenon of self-inductance is shown in fig 29.6.

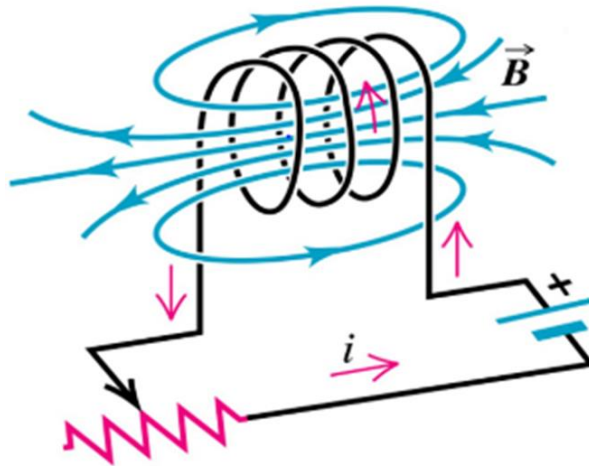


Fig. 29.6: Constructional representation of a self-inductance of a solenoid.

The definition of inductance is,

For any geometry of loop, the magnetic flux through the loop produced by the current in the loop is proportional to the current. The inductance L is the constant of proportionality. So,

$$\Phi_B = LI$$

The unit of inductance is called henry:

$$1\text{H} = 1\text{T}\cdot\text{m}^2/\text{A}$$

$$1\text{H} = 1(\text{T}\cdot\text{m}^2/\text{s}) (\text{s}/\text{A})$$

$$1\text{H} = 1\text{V}\cdot\text{s}/\text{A}$$

Here note that the inductance, like capacitance is a geometric quantity, which means that its size depends on, how many turns a coil have, its thickness, and the area of cross section.

Calculation of inductance for a solenoid:

Consider a solenoid having N number of turns of wire in the solenoid, A is the cross-sectional area of the solenoid (in square meters, m^2), and l is the length of the solenoid (in meters, m).

Magnetic field inside a solenoid is given as,

$$B = \mu_o n I$$

$$\therefore n = N / l$$

$$B = \mu_o \frac{N}{l} I$$

Magnetic flux for a solenoid having N number of turns is:

$$N\phi_B = NBA = (nl)BA$$

$$N\phi_B = (nl)\mu_o n I A = \mu_o n^2 l I A$$

Self-inductance for N number of turns is given as,

$$\phi = N I$$

$$L = \frac{N\phi_B}{I} = \frac{\mu_o n^2 l I A}{I} = \mu_o n^2 l A$$

L is the self – inductance of the solenoid (in Henries, H)

μ_o is the permeability of free space, approximately $4\pi \times 10^{-7} \text{ H / m}$

Let's solve a problem here.

Problem:

A coil whose radius is 5 cm and the length is 10 cm and there are 3500 turns on it. Find the self-inductance of the coil.

Solution:

Here,

$$\text{number of turns} = N = 3500$$

$$\text{radius} = r = 5 \text{ cm} = 0.05 \text{ m}$$

$$\text{length} = l = 10 \text{ cm} = 0.10 \text{ m}$$

$$\text{As, } A = \pi r^2$$

$$L = \frac{\mu_o N^2 A}{l} = \frac{\mu_o N^2 \pi r^2}{l}$$

$$L = 4\pi \times 10^{-7} \frac{\text{T.m}^2}{\text{A}} \cdot 3500^2 \cdot \frac{\pi(0.05 \text{ m})^2}{0.10 \text{ m}}$$

$$L = 1.21 \frac{\text{T.m}^2}{\text{A}} = 1.21 \text{ H}$$

Faraday's Law and Inductance:

Self-inductance is that when current passes through this coil, a magnetic flux is generated and from Faraday's law when this flux changes, an EMF is induced as a result.

$$\begin{aligned}\mathcal{E} &= -\frac{d\phi_B}{dt} \\ \therefore \phi_B &= LI \\ \mathcal{E} &= -\frac{d(LI)}{dt} \\ \mathcal{E} &= -L\frac{dI}{dt}\end{aligned}$$

This equation shows the induced electromotive force is equal to the negative of the inductance to the rate of change of current. Now we want to calculate the energy required to reach to that factor of current.

Energy Stored in a Magnetic Field:

Suppose we have an inductor, so when we connected it to the battery, a current flowed in it. When the current flowed, it generates a magnetic field. The magnetic field prevented this current from exceeding, so we had to work out how much energy is there to fill the magnetic energy inside it. Now let us calculate it.

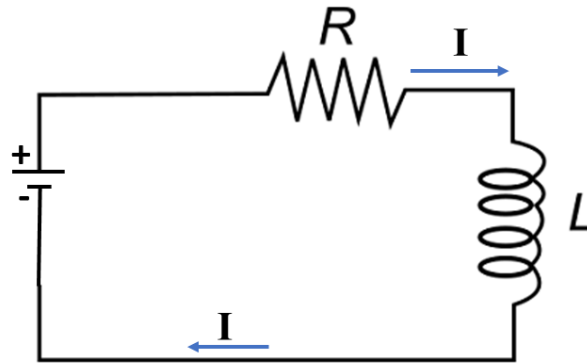


Fig. 29.7: Constructional representation of a resistor with inductor to a battery.

Now imagine that we connect a battery with a resistor and an inductor and close the switch, then current start flowing in it as shown in fig 29.7. Now as soon as the current passes through it, the factor ' LdI/dt ' become non-zero, which means that there will be an emf in the opposite direction, a voltage will be generated which will try to ensure that current does not flow in it. Mathematically, it is represented as:

$$\varepsilon = IR + L \frac{dI}{dt}$$

$$\varepsilon I = I^2 R + LI \frac{dI}{dt}$$

Where,

εI = Power expended by battery

$I^2 R$ = Power dissipated in resistor

$LI \frac{dI}{dt}$ = Work done on inductor

The factor LdI/dt shows the battery did some work for the movement of current and the amount of that work is εI , and $I^2 R$ power is lost in the resistor so whatever is left remains in the form of magnetic energy in this inductor. Just as we had charged the capacitor by connecting it with the battery, here we have charged the inductor to achieve the energy balance. Mathematically, it is given as:

$$\frac{dU_B}{dt} = LI \frac{dI}{dt}$$

$$\int_0^{U_B} dU_B = L \int_0^I IdI$$

$$U_B = \frac{1}{2} LI^2$$

As energy stored in a capacitor is $U_E = 1/2 CV^2$

Now this is exactly the same formula as we drew once before for the capacitor. For capacitor, all energy is stored in the electric field and for the inductor, all the energy is stored in magnetic field.

Now what will happen when the magnetic field becomes zero?

It will be happened only in such a situation, if you connect this inductor with a resistor then a current will flow and all the energy that was present inside the magnetic field will be lost in a resistor.

Energy Density of Magnetic Field:

$$U_B = \frac{1}{2} LI^2 \quad \dots\dots\dots(1)$$

As,

$$L = \mu_0 n^2 l A \text{ and } B = \mu_0 n I \Rightarrow I = B / \mu_0 n$$

Use solenoid results in equation (1):

$$U_B = \frac{1}{2} (\mu_0 n^2 l A) (B / \mu_0 n)^2$$

$$\therefore l A = V(\text{volume})$$

$$U_B = \frac{1}{2} (\mu_0 n^2 V) (B^2 / \mu_0^2 n^2)$$

$$\frac{U_B}{V} = \frac{B^2}{2\mu_0}, \quad \therefore u_B = \frac{U_B}{V}$$

$$u_B = \frac{B^2}{2\mu_0}$$

Units of energy density:

Energy density = Magnetic energy per unit volume (J/m^3)

$$\text{Energy density} = B^2 / 2\mu_0$$

Unit conversion for 1 tesla field:

$$\text{Energy density} = 1\text{T}^2 / 8\pi \times 10^{-7} \text{ T.m/A}$$

$$\text{Energy density} = 3.98 \times 10^5 \text{ T.A/m}$$

$$\text{Energy density} = 3.98 \times 10^5 \text{ T.Am/m}^2$$

$$\text{Energy density} = 3.98 \times 10^5 \text{ N/m}^2 = 3.98 \times 10^5 \text{ J/m}^3$$

Problem:

Let's solve a problem in which both a resistor and an inductor are used. And it has a battery which is connected with an inductor and a resistor. As you know that whenever we connect it, a back emf will be generated which will prevent the current from increasing but eventually the current will reach its maximum value. And this maximum value is the voltage of the battery or its EMF divided by r . We would like to solve its equation now,

$$\text{Voltage across resistor} = V_R = IR$$

$$\text{Voltage across inductor} = V_L = L \frac{dI}{dt}$$

$$\text{Electromotive force} = V_L + V_R$$

$$\mathcal{E} = L \frac{dI}{dt} + IR$$

The solution of the equation is:

$$I(t) = \frac{\mathcal{E}}{R} (1 - e^{-t/\tau})$$

$$\frac{dI}{dt} = \frac{\mathcal{E}}{R} \frac{1}{\tau} e^{-t/\tau}$$

$$\tau = \frac{L}{R} = \text{Time constant}$$

Time constant tells the time after which the current has reached or is close to its maximum value. Let's prove how time constant is equal to second by following:

$$[\tau] = \frac{[L]}{[R]} = \frac{\text{henry}}{\text{ohm}} = \frac{\text{volt.second/ampere}}{\text{ohm}}$$

$$[\tau] = \left(\frac{\text{volt}}{\text{ampere.ohm}} \right) \text{second}$$

$$\therefore \frac{\text{volt}}{\text{ampere.ohm}} = \frac{IR}{IR} = 1$$

$$[\tau] = \text{second} = t$$

$$I = \frac{\mathcal{E}}{R} (1 - e^{-t/\tau}) = \frac{\mathcal{E}}{R} (1 - e^{-t/t})$$

$$I = \frac{\mathcal{E}}{R} (1 - e^{-1}) = \frac{\mathcal{E}}{R} \left(1 - \frac{1}{2.718} \right)$$

The natural exponential function $e = 2.718$

$$I = \frac{\mathcal{E}}{R} (1 - 0.37)$$

$$I = \frac{\mathcal{E}}{R} 0.63$$

Conclusion:

In an RL circuit, at $t = \tau$, the current through the inductor has reached 63% of its final steady-state value $\mathcal{E}/R$. This means that after one-time constant τ , the inductor current has risen to 63% of the value.

Similarly, we have solved the case of a resistor and a capacitor. For a capacitor,

$$C = \frac{Q}{V}$$

And energy stored in a capacitor is given as:

$$U = \frac{Q^2}{2C} = \frac{(CV)^2}{2C} = \frac{C^2 V^2}{2C}$$

$$U = \frac{1}{2} CV^2$$

When we connect the capacitor with the resistor and the battery, the current flows for some time and then it ends. After how much time does it end, this is based on the time constant which is there in the resistor and the capacitor, and this time constant is equal to RC .

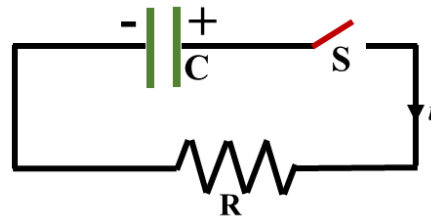


Fig. 29.8: Representation of a circuit diagram of a resistor and a capacitor to a battery.

Let us solve the equation for capacitor here:

$$\text{Voltage across resistor} = V_R = iR$$

$$\text{Voltage across capacitor} = V_C = \frac{q}{C}$$

$$\text{Electromotive force} = V_C + V_R$$

$$\varepsilon = \frac{1}{C} \frac{dq}{dt} + \frac{di}{dt} R$$

$$\frac{1}{C} \frac{dq}{dt} + \frac{di}{dt} R = 0$$

As there is no battery attached to it hence, $\varepsilon = 0$

$$\text{and } \frac{dq}{dt} = i$$

$$\frac{di}{dt} = -\frac{1}{RC} i$$

The solution of the equation is:

$$i = i_0 (e^{-t/\tau}) = i_0 (e^{-t/RC})$$

$$\therefore \tau = RC = \text{Time constant}$$

Now let us understand that if only capacitors and inductors are present and if there is no resistor, then what will happen? This is a very interesting thing to know.

As, power is spent inside the resistor, power is never wasted in capacitor, it is exactly like a tank, you can fill water inside it but it itself does not absorb water from it and the same thing applies to an inductor. You fill magnetic energy inside the inductor as when current passes through it, a magnetic field is produced. As magnetic field is energy.

When an inductor is connected to a capacitor, they form an LC circuit which exhibits oscillatory behavior. Here's a summary of the process:

Initial State: The capacitor is fully charged, storing energy in its electric field, while the inductor has no current flowing through it, storing zero energy in its magnetic field. The energy is entirely within the electric field of the capacitor.

Initiation of Current Flow: When the circuit is closed, current begins to flow. As current flows, the magnetic field inside the inductor starts to increase, while the electric field across the capacitor starts to decrease.

Equalization of Energies: As current continues to flow, the magnetic energy stored in the inductor increases, while the electric energy stored in the capacitor decreases. Eventually, the magnetic energy becomes equal to the initial electric energy.

Reversal of Energies: Further flow of current causes the magnetic energy to exceed the initial electric energy. At this point, the capacitor is discharged, and all the energy is stored in the magnetic field of the inductor.

Oscillatory Behavior: The cycle repeats as the stored energy oscillates between the electric field of the capacitor and the magnetic field of the inductor. The total energy, however, remains constant, as it is conserved in the ideal case with no resistive losses.

Electromagnetic oscillations:

$$U = U_B + U_E$$

$$U = \frac{1}{2}LI^2 + \frac{1}{2}\frac{q^2}{C}$$

$$U = \text{constant so } dU/dt = 0$$

$$\frac{dU}{dt} = \frac{d}{dt}\left(\frac{1}{2}LI^2 + \frac{1}{2}\frac{q^2}{C}\right)$$

$$0 = \frac{d}{dt}\left(\frac{1}{2}LI^2 + \frac{1}{2}\frac{q^2}{C}\right)$$

$$I = \frac{dq}{dt}, \quad \frac{dI}{dt} = \frac{d^2q}{dt^2}$$

$$\frac{d^2q}{dt^2} + \frac{1}{LC}q = 0$$

$$\therefore \text{Oscillation frequency} = \omega^2 = \frac{1}{LC}$$

$$\frac{d^2q}{dt^2} + \omega^2q = 0$$

The solution of the equation is:

$$q = q_m \cos \omega t$$

The frequency of oscillation determines how quickly the energy exchanges between the electric and magnetic fields in the LC circuit. Higher frequencies correspond to faster oscillations, while lower frequencies correspond to slower oscillations.

RLC driven by an alternating voltage:

When analysing circuits with alternating current (AC) sources, such as sinusoidal voltage $V = V_0 \sin(\omega t)$, the behaviour of resistors, capacitors, and inductors differs due to their inherent electrical properties. Let's delve into these differences:

Behaviour of Resistors (R):

For resistors, the voltage V and the current I are in phase. This means that both the voltage across and the current through a resistor reach their maximum and minimum values at the same time during each cycle of the AC waveform. The relationship between voltage, current, and resistance for a resistor is described by Ohm's Law:

$$V = I \cdot R$$

Behaviour of Capacitors (C):

For capacitors, the relationship between voltage and current is characterized by a phase difference of 90° . When the voltage across a capacitor reaches its maximum (or minimum), the current through the capacitor is zero, and vice versa. This phase difference leads the current and voltage lag the current by 90° . The relationship between voltage and current for a capacitor in an AC circuit is given by:

$$I = C dV/dt$$

Behaviour of Inductors (L):

For inductors, the relationship between voltage and current also exhibits a 90° phase difference, but in this case, the current lags the voltage. When the voltage across an inductor reaches its maximum (or minimum), the current through the inductor is zero, and vice versa. This phase difference causes the current to lag the voltage by 90° as shown in fig 29.9. The relationship between voltage and current for an inductor in an AC circuit is given by:

$$V = L di/dt$$

An inductive element:

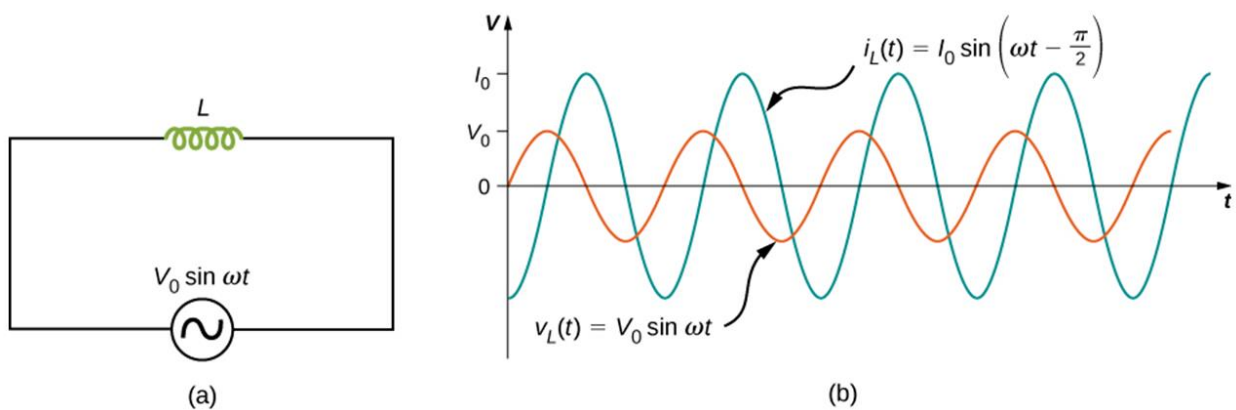


Fig. 29.9: (a) It represents an AC inductive circuit (b) Representation of the waveform where voltage leads the current by 90 degrees.

$$L \frac{dI}{dt} = V_o \sin \omega t$$

$$\frac{dI}{dt} = \frac{V_o}{L} \sin \omega t$$

On integrating both sides:

$$I = -\frac{V_o}{\omega L} \cos \omega t$$

$$I = \frac{V_o}{\omega L} \sin(\omega t - \frac{\pi}{2})$$

This equation describes the current (I) flowing through an inductor in an AC (alternating current) circuit. Where:

- I represent the instantaneous current flowing through the inductor.
- V_o represent the peak voltage of the AC source.
- ω is the angular frequency of the AC source, measured in radians per second ($\omega = 2\pi f$, where f is the frequency in Hertz).
- L is the inductance of the inductor, measured in Henry.
- $\cos(\omega t)$ represent the time-varying behavior of the current, where t is time.

Now, let's discuss the physical significance of this equation:

Current Variation with Time:

The term $\cos(\omega t)$ represents a sinusoidal variation in current with time. As time progresses, the current oscillates sinusoidally between positive and negative values.

The cosine function describes the oscillation pattern, indicating that the current reaches its peak at different times depending on the phase angle ωt .

Peak Current:

The peak current is given by $V_o/\omega L$. This term signifies that the peak current is directly proportional to the peak voltage (V_o) of the AC source and inversely proportional to both the angular frequency (ω) and the inductance (L) of the inductor.

A higher peak voltage or lower inductance would result in a higher peak current, assuming the angular frequency remains constant.

Similarly, a higher angular frequency or higher inductance would result in a lower peak current, assuming the peak voltage remains constant.

Phase Relationship:

The phase angle ωt determines the phase relationship between the current and the voltage in the circuit.

For a purely inductive circuit, the current lags the voltage by a phase angle of 90° (or $\pi/2$ radians). This lag is due to the behavior of inductors, which oppose changes in current.

The cosine function captures this phase relationship, indicating the time-varying behavior of the current with respect to the voltage.

In conclusion, this equation describes how the current flowing through an inductor in an AC circuit varies sinusoidally with time, depending on the peak voltage of the source, the angular frequency, and the inductance of the inductor. It also indicates the phase relationship between the current and voltage in the circuit.

A capacitive element:

If a capacitor is connected to an alternating source, then current will also flow in it. Here we get current from the charge. The behavior of a capacitor with AC is shown in fig 29.10.

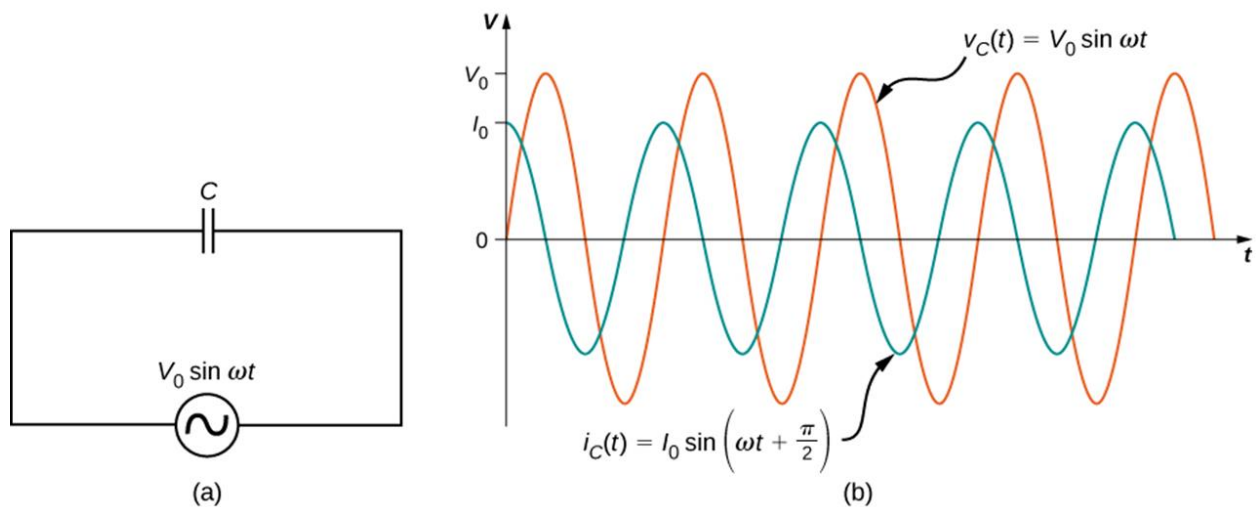


Fig. 29.10: (a) It represents an AC capacitive circuit (b) Representation of the waveform where voltage lags the current by 90 degrees.

What is the charge for a capacitor? Current is the rate of change of charge.

$$q = CV_o \sin \omega t$$

$$I = \frac{dq}{dt} = \frac{d}{dt}(CV_o \sin \omega t)$$

On differentiating:

$$I = \omega CV_o \cos \omega t$$

$$I = \omega CV_o \sin\left(\frac{\pi}{2} - \omega t\right)$$

The equations are related to the behavior of a capacitor in an AC (alternating current) circuit. Where:

- q represents the charge stored in the capacitor at any given time t .
- ω_o is the angular frequency associated with the capacitor's behavior. It depends on the capacitance (C) and inductance (L) of the circuit ($\omega_o = 1/LC$).
- $\sin(\omega t)$ represents a sinusoidal variation of charge with time. As time progresses, the charge oscillates sinusoidally between positive and negative values, following a sine wave pattern.
- The maximum charge ($q_{\max}$) is reached when $\sin(\omega t) = 1$ or -1 , and it is proportional to ω_o . The larger the capacitance (C), the larger the maximum charge.
- I represent the current flowing through the circuit at any given time t .
- dq/dt represent the rate of change of charge with respect to time, which is the definition of current.
- ω is the angular frequency of the AC source.
- C is the capacitance of the capacitor.
- V_o is the peak voltage of the AC source.
- $\cos(\omega t)$ represents the variation of current with time. It's a cosine function because the current is leading the voltage by 90° due to the behavior of capacitors.
- $\sin(90 - \omega t)$ also represents the variation of current with time, but it's expressed in terms of a sine function. This equation emphasizes the phase relationship between current and voltage: the current leads the voltage by 90° .

In conclusion, these equations describe the behavior of a capacitor in an AC circuit. The charge stored in the capacitor oscillates sinusoidally with time, and the current flowing through the circuit follows a sinusoidal pattern as well. The phase relationship between current and voltage is such that the current leads the voltage by 90° .

Comparison between RLC results:

$$I = \frac{V_o}{R} \sin \omega t \quad \text{For resistor}$$

$$I = \frac{V_o}{\omega L} \sin\left(\omega t - \frac{\pi}{2}\right) \quad \text{For inductor}$$

$$I = \frac{V_o}{1/\omega C} \sin\left(\frac{\pi}{2} - \omega t\right) \quad \text{For capacitor}$$

That's why:

$$Z_L = \omega L \quad \text{is called inductive reactance}$$

$$Z_C = 1/\omega C \quad \text{is called capacitive reactance}$$

Here, we came to know that how R, L and C respond when an alternating voltage applied to them. One special thing about resistor, capacitor and inductor is that the energy inside the inductor and capacitor remains filled in them and does not go away. They are storages and storehouses for energy. If you keep energy in them, the energy does not lose. But resistor is not like this. When current passes through a resistor, power is then lost to heat.

Lecture#30

Electromagnetic waves

The purpose of this lecture is to present you the great achievement of James Clark Maxwell's which he had accomplished in the year 1865. The reason why this achievement is so famous is because Maxwell unifies the two different forces i.e., electric and magnetic force.

From the previous studies, one can define the **electric field** as, “the force F experienced by the charge q ”. Since the force is a vector quantity hence electric field is also a vector quantity. The electric force is given as:

$$\vec{F} = q\vec{E}$$

The more powerful the electric field, the greater its strength, the force will also be greater accordingly, so this is the identity of the electric force. Here, an important thing is that it does not matter whether the charge is standing or at rest or whether it is moving, the force will always be in the direction of the electric field.

The magnetic force is completely different from this.

The **magnetic field** is the region surrounding the particle in which the magnetic force is felt. **Magnetic force** refers to the force exerted between magnets or between a magnet and a moving charged particle. The force on a moving charged particle in a magnetic field is given by,

$$\vec{F} = q(\vec{v} \times \vec{B})$$

In this formula, B is the magnetic field and see that $(\vec{v} \times \vec{B})$ means that the force that acts on it is perpendicular to both $\vec{v}$ and $\vec{B}$.

Direction of the magnetic force:

The direction of the magnetic force can be determined using the **right-hand rule**: if you point the fingers of your right hand in the direction of $\vec{v}$ and curl them towards $\vec{B}$, your thumb points in the direction of the force on a positive charge.

Lorentz force:

The Lorentz force is the force exerted on a charged particle moving through electric and magnetic fields. It is a fundamental concept in electromagnetism and is described by the Lorentz force law. The total Lorentz force $\vec{F}$ experienced by a particle with charge q in the presence of an electric field $\vec{E}$ and a magnetic field $\vec{B}$ is given by:

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$$

Now the question arises here is that, where does the electric field come from? What produces the electric field? And the answer is that “the charge” and from the charge the electric field lines are produced. If you place a positive charge, then that charge is the source of the electric field, and the direction of the field lines will be outward. And if the charge is negative then the electric field lines will be inward.

Basic Equations of Electromagnetism:

The fundamental equations of electromagnetism are following:

1. Gauss' law of electricity

$$\int \vec{E} \cdot d\vec{A} = \frac{q}{\epsilon_0}$$

Where E is the electric field, ϵ_0 is the permittivity of free space (8.854×10^{-12} F/m (farads per meter)).

Illustration of an electric flux through a closed surface is shown in fig.30.1.

2. Gauss' law of magnetism

$$\int \vec{B} \cdot d\vec{A} = 0$$

Gauss's law for magnetism states that the net magnetic flux through any closed surface is zero as shown in fig. 30.2. And hence gauss's law for electric fields and gauss's law for magnetic fields are completely different from each other.

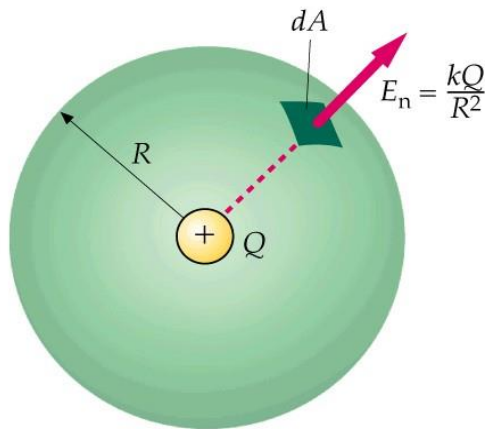


Fig. 30.1: Illustration of an electric flux through a closed surface (Gaussian surface), with field lines emanating from a positive charge inside the surface.

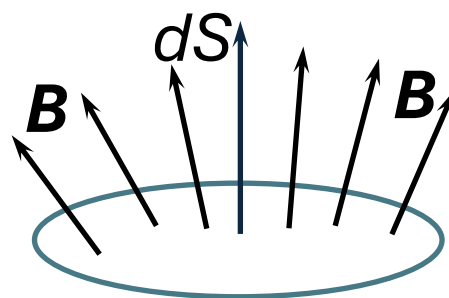


Fig. 30.2: An illustration of magnetic field lines passing through a closed surface (ds). Unlike electric field lines, magnetic field lines form closed loops, indicating that there are no magnetic monopoles.

3. Faraday's law of induction

$$\int \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt}$$

$$\Phi_B = \oint \vec{B} \cdot d\vec{S}$$

This is Faraday's Law which tells us that when a magnetic flux changes with time, as a result, an electromotive force comes into existence and that motive force then produce the current in a wire or in a circuit.

4. Ampere's law

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I$$

Ampère's Law relates the circulating magnetic field in a closed loop to the electric current passing through the loop. Its states that the integral of the magnetic field B around a closed loop is proportional to the total current I passing through the loop as shown in fig. 30.3.

From here it is clear that the source of the magnetic field is electric current, just as a charge produces an electric field, similarly current produces a magnetic field.

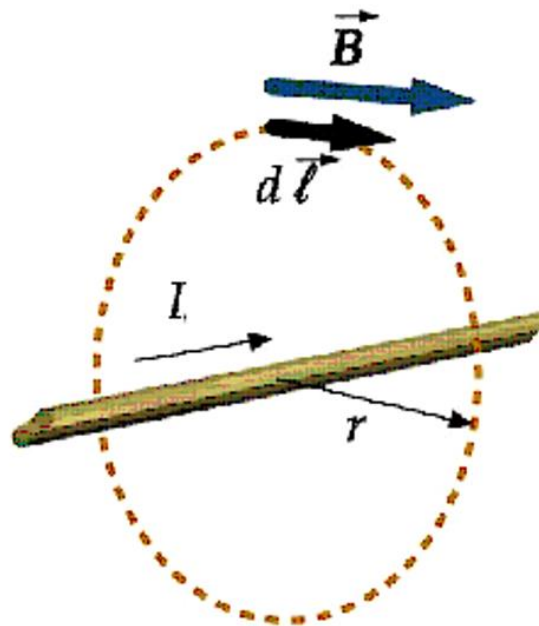


Fig. 30.3: Illustration of a circulating magnetic field around a boundary path produced by an electric current or changing electric flux through a closed surface.

Maxwell's Displacement Current:

250 years ago, James Clark Maxwell Started paying attention to these four laws. Maxwell was a resident of Scotland, and he has interest in all fields of physics but was very interested in electricity and magnetism. He saw that there was a problem in Ampere's law and then he created an imaginary example to point out the mistake. He used two plates of capacitors and connected these plates to an alternate current source. Now if it is an AC source then when the electric field changes between these plates, a current will definitely flow as a result. Then he thought to make different surfaces to see how Ampere's law applies for capacitors as shown in fig. 4.

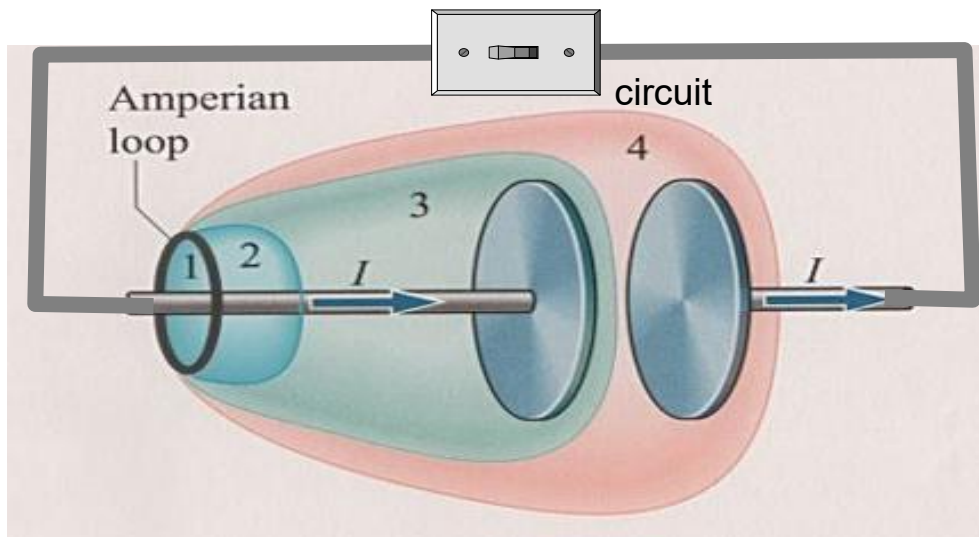


Fig. 30.4: Illustration of Ampère's Law with Maxwell's addition. This figure shows a closed loop encircling a current-carrying wire and a capacitor connected to an alternating current (AC) source. As the electric field between the capacitor plates changes, Maxwell realized that a displacement current $\epsilon_0 d\Phi_E/dt$ must be added to account for the changing electric field.

From the fig. 4 we can see that there are four different surfaces in it, which is named as one, two, three and four. There is only one emperian loop in it. On this loop, we will calculate the integral according to Ampere's law as:

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I$$

So now if you choose surface one, two and four then since only one current can pass through this wire, so here the enclosed current will be I. Now suppose if we will choose the number three surface instead of one, two and four then the enclosed current is zero and for this the integral is equal to zero.

$$\oint \vec{B} \cdot d\vec{s} = 0$$

Hence, James Clerk Maxwell realized around 1865 that the equations of electricity & magnetism were inconsistent.

$$\left[\int \vec{B} \cdot d\vec{s} \right]_{(1,2,4)} \neq \left[\int \vec{B} \cdot d\vec{s} \right]_3$$

Then he thought, there must be another effective current which can produce magnetic field, and he called it a **displacement current**. Maxwell generalized Ampere's law like this, and he said that there is such a thing as a displacement current which is added on to the other current and its generalization is:

$$\int \vec{B} \cdot d\vec{s} = \mu_0 (I + I_D)$$

$$I_D = \epsilon_0 \frac{d\Phi_E}{dt}$$

The current (rate of charges) that flows in the circuit is:

$$I = \frac{dQ}{dt}$$

But the charge on the capacitor plate is: $Q = \epsilon_0 EA$

Hence,

$$I = \frac{d}{dt} (\epsilon_0 EA)$$

$$I = \epsilon_0 \frac{d(EA)}{dt}$$

$$I = \epsilon_0 \frac{d\phi_E}{dt} \equiv I_D$$

In other words, we can also say that if Maxwell didn't add the displacement current I_D , then the current would not have been conserved which means that the charges would have been lost while passing through a wire and they would have disappeared somewhere, and conservation of charge will become violate. Thus, in order to maintain the conservation of charge, Maxwell was forced to invent this additional term, and this is a great miracle of mathematical physics. The Maxwell equations of electromagnetism are now completed, which will not change anymore as these have been tested many times and now there is not even a single flaw in it. So, now the four Maxwell equations are:

1. $\oint \vec{E} \cdot d\vec{S} = \frac{Q}{\epsilon_0}$
2. $\oint \vec{B} \cdot d\vec{S} = 0$
3. $\oint \vec{E} \cdot d\vec{\ell} = -\frac{d\phi_E}{dt}$
4. $\oint \vec{B} \cdot d\vec{\ell} = \mu_0 \left(I + \epsilon_0 \frac{d\phi_E}{dt} \right)$

So, Maxwell actually discovered that when an electric field changes, its effect is exactly the same as that of a current. The flow of current produces a magnetic field and similarly, due to the changing electric field, a magnetic field is generated. Hence, whether there is a current of an electron or a displacement current (remember that it is proportional to the rate of change of electric flux), the effect of both is the same. So, whenever the field changes its value, a magnetic field is produced as a result.

This brings to light a very important new fact that electromagnetic waves that can pass through space, can exist as a result of these equations. And the basic reason for the existence of electromagnetic waves is that the changing electric flux produces a magnetic field.

Up till now, according to Ampere's law, we saw that the magnetic field was produced by the current. But now, the current may not be there, we may be in free space where there are no electrons or nothing, but if there is a changing electric field in the free space then it results a magnetic field.

Introduction to electromagnetic waves:

The basic idea behind electromagnetic waves is similar to other types of waves. Understanding these waves requires familiarity with concepts like wavelength, frequency, and amplitude also shown in fig. 5.

Wavelength (λ):

Wavelength is the distance between two successive points that are in phase on a wave. This means it is the distance over which the wave's shape repeats.

Frequency (f):

The number of cycles of displacement that pass a point in one second. It is measured in hertz (Hz). Frequency represents how many times the wave oscillates per second.

Relationship Between Wavelength, Frequency, and Speed:

Wave Speed (v):

The product of wavelength (λ) and frequency (f) is equal to the speed of wave propagation.

$$V = \lambda \cdot f$$

Amplitude (A):

The maximum magnitude of displacement in a wave. Higher amplitude means greater energy in the wave. Amplitude indicates how much the medium is displaced from its rest position.

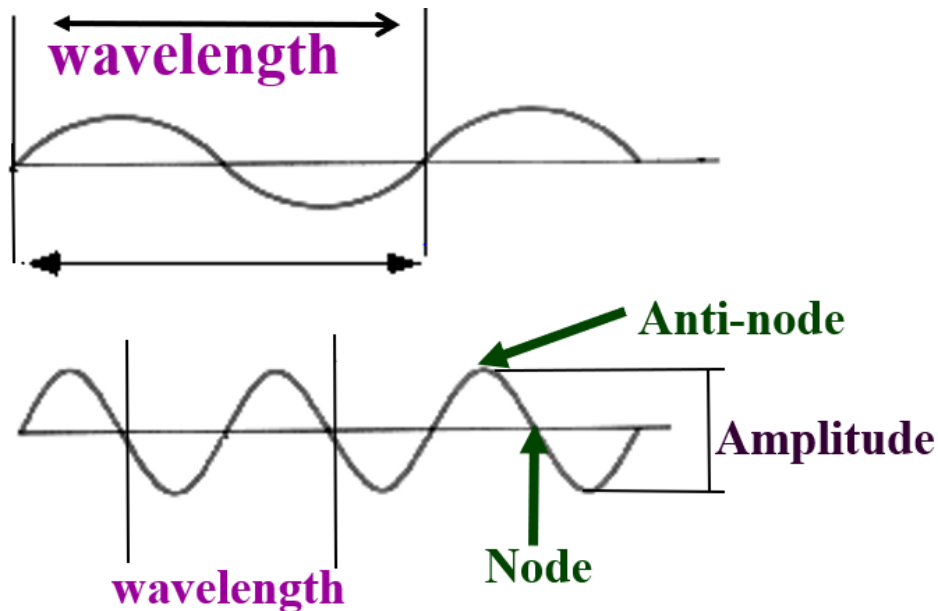


Fig. 30.5: Illustration of a standing wave with labelled features: amplitude, nodes, anti-nodes, and wavelength. The amplitude is the maximum displacement from the rest position. Nodes are points where there is no displacement, while anti-nodes are points of maximum displacement. The wavelength λ is the distance between two consecutive nodes or anti-nodes.

Electromagnetic Waves:

These waves have no restrictions on frequency or amplitude. The electromagnetic spectrum ranges from zero frequency to extremely high frequencies, although generating or detecting zero or infinite frequencies is not feasible.

Example of Electromagnetic Waves:

Red Light: It has a wavelength of approximately 700 nanometers (nm) as shown in fig. 6.

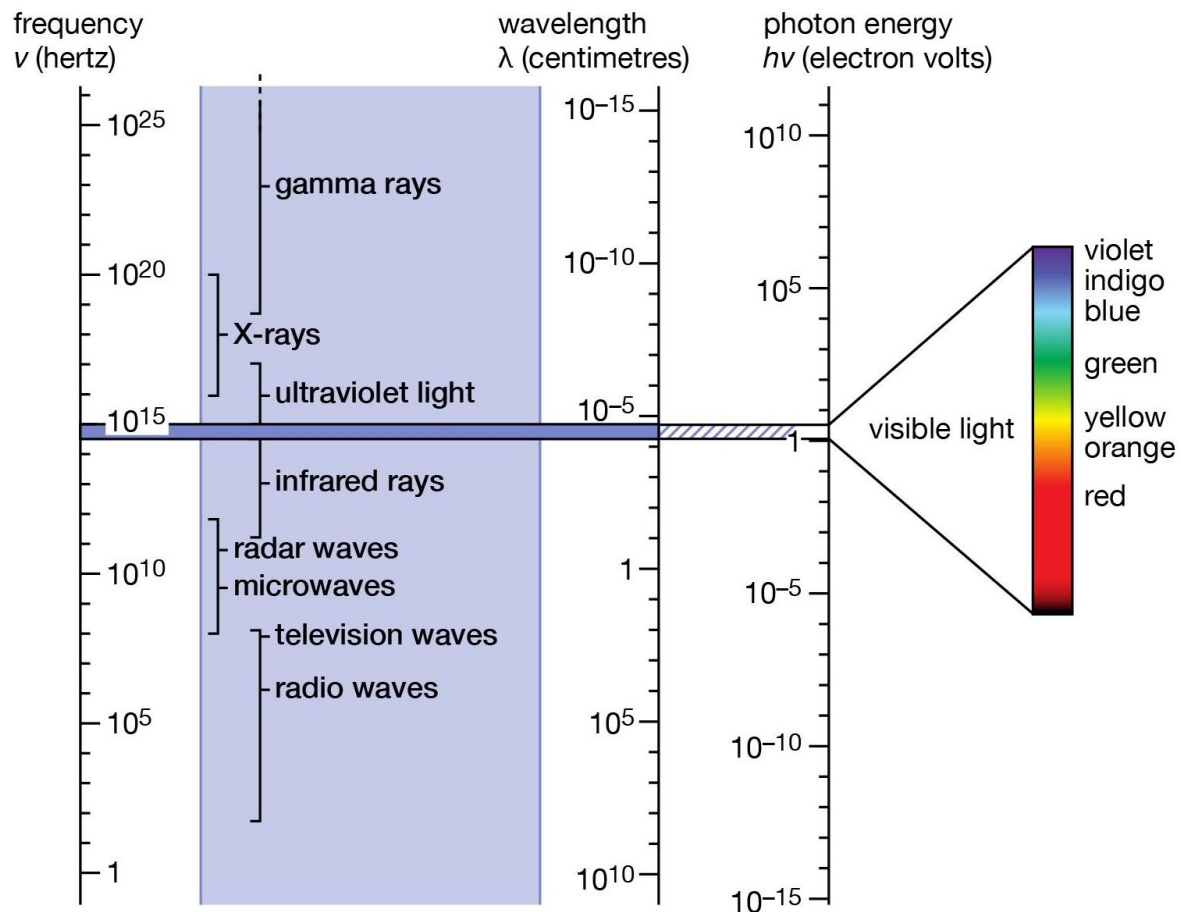


Fig. 30.6: A representation of the electromagnetic spectrum, showing the range of all possible electromagnetic radiations. It spans from low-frequency radio waves to high-frequency gamma rays, including microwaves, infrared, visible light, ultraviolet, and X-rays.

Speed of Light

Light travels at a speed of 3×10^8 meters per second (or 3×10^{10} centimeters per second).

Example Calculation:

Suppose a red-light having wavelength 7×10^{-7} meters (700 nm). Calculate its frequency.

Calculation:

Speed of light (c) = 3×10^8 meters per second

Wavelength (λ) = 7×10^{-7} meters

$$V = \lambda \cdot f$$

$$f = V / \lambda$$

$$f = 3 \times 10^8 / 7 \times 10^{-7} \text{ m} = 4.29 \times 10^{14} \text{ Hz}$$

Result: The frequency of red light is approximately 4.29×10^{14} hertz (Hz). 1 Hertz (Hz) equals one cycle per second. Thus, 10^{14} Hz means 10^{14} cycles per second.

Electromagnetic Spectrum:

Range: The electromagnetic spectrum is extremely wide, with no upper limit on frequency. The lower limit approaches zero but can never actually reach zero.

Examples:

1. **Radio Waves:** Frequencies range from 10^4 to 10^{10} cycles per second.
2. **Amplitude Modulation (AM) Radio waves:** Frequencies range from 10^6 to 10^7 Hz.
3. Radio Pakistan operates around 800 kilocycles (800 kHz).

Frequency Modulation (FM) Radio:

FM radio is another type of radio transmission, differing from AM radio in terms of frequency modulation.

Frequency: FM radio operates at around 100 megahertz (MHz), which equals 100 million cycles per second.

Television Frequencies Range: Television frequencies range from 10^8 to 10^9 hertz (Hz).

Microwave Ovens: Common household microwave ovens heat food using microwaves without the need for fire.

Frequency: Microwaves have a frequency of approximately 10^{12} Hz.

Wavelength: The wavelength of microwaves is about 5 centimeters; hence they are also called centimeter waves.

Cosmology: Microwaves are significant in cosmology and astrophysics, particularly in the study of black body radiation, such as the 3-degree black body radiation from the Big Bang.

Light Spectrum:

Visible Light: Light follows microwaves in the electromagnetic spectrum. For example, red light has a frequency of about 10^{14} Hz.

Ultraviolet Light: Beyond visible light is ultraviolet (UV) light.

X-Rays: X-rays have frequencies higher than UV light.

Speed of Light in Different Media:

Vacuum: The speed of light in a vacuum is constant for all frequencies.

Materials: When light passes through a medium, different frequencies travel at different speeds. This means some frequencies pass through the medium faster than others.

Traveling Electromagnetic Fields:

Traveling electromagnetic fields refer to the propagation of electromagnetic waves through space. These fields consist of oscillating electric and magnetic fields that move together at the speed of light.

Electromagnetic Wave Generation:

Electromagnetic waves are generated because a changing electric field creates a magnetic field and vice versa. To understand this in more detail, consider an example of a long wire with a current passing through it.

Current and Magnetic Field:

When a current passes through the wire, it is due to a potential difference, causing charges to move, which generates the current. As a result of the current, a magnetic field is produced around the wire as shown in fig. 30.7. According to Ampere's Law, the strength of this magnetic field is determined by the integral of the magnetic field over a closed loop around the wire.

Ampere's Law is given by the integral form $\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I$, where $\mathbf{B}$ is the magnetic field, $d\mathbf{l}$ is the differential length element of the loop, and I is the current through the loop.

Application: For a long straight wire, the magnetic field B at a distance r from the wire is:

$$B = \mu_0 I / 2\pi r.$$

An electric field is also present due to the potential difference along the wire. Current cannot flow unless there is an electric field driving the charges from a higher potential to a lower potential. When the current in the wire changes direction, both the electric and magnetic fields change direction. The electric field and magnetic field are perpendicular to each other and change rapidly over time as the current alternates.

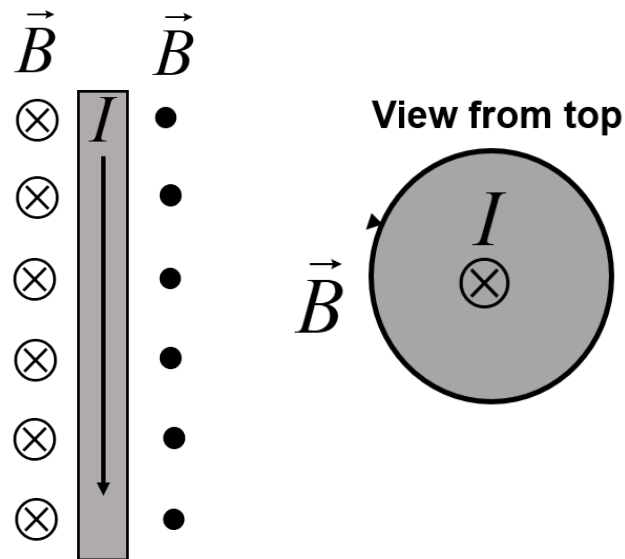


Fig. 30.7: Illustration of the magnetic field B produced around a long straight wire due to an electric current I . The dot convention indicates the current is considered to flow into the wire (entering the wire) while the cross convention indicates that the current is flowing out of the wire (leaving the wire).

Faraday's Law and Maxwell's Displacement Current:

Faraday's Law: A time-varying magnetic field induces an electromotive force (EMF) and, consequently, an electric field.

Maxwell's Addition: A changing electric field induces a magnetic field, a concept known as the Maxwell Displacement Current.

Electromagnetic Waves Propagation: The changing electric field creates a changing magnetic field, and the changing magnetic field creates a changing electric field. This continuous process allows an electromagnetic wave to propagate through space.

Field directions in an electromagnetic wave:

An electromagnetic wave consists of an electric field component and a magnetic field component. Both fields are perpendicular to each other and to the direction of wave propagation.

Visualization

Imagine a positively charged particle placed in the path of the electromagnetic wave:

Electric Field Direction: Along the z-axis.

Magnetic Field Direction: Along the y-axis.

Wave Propagation: Along the x-axis, all three vectors (electric field, magnetic field, and direction of propagation) are perpendicular to each other as shown in fig. 30.8. If the direction of propagation changes, the electric and magnetic fields adjust accordingly to remain perpendicular.

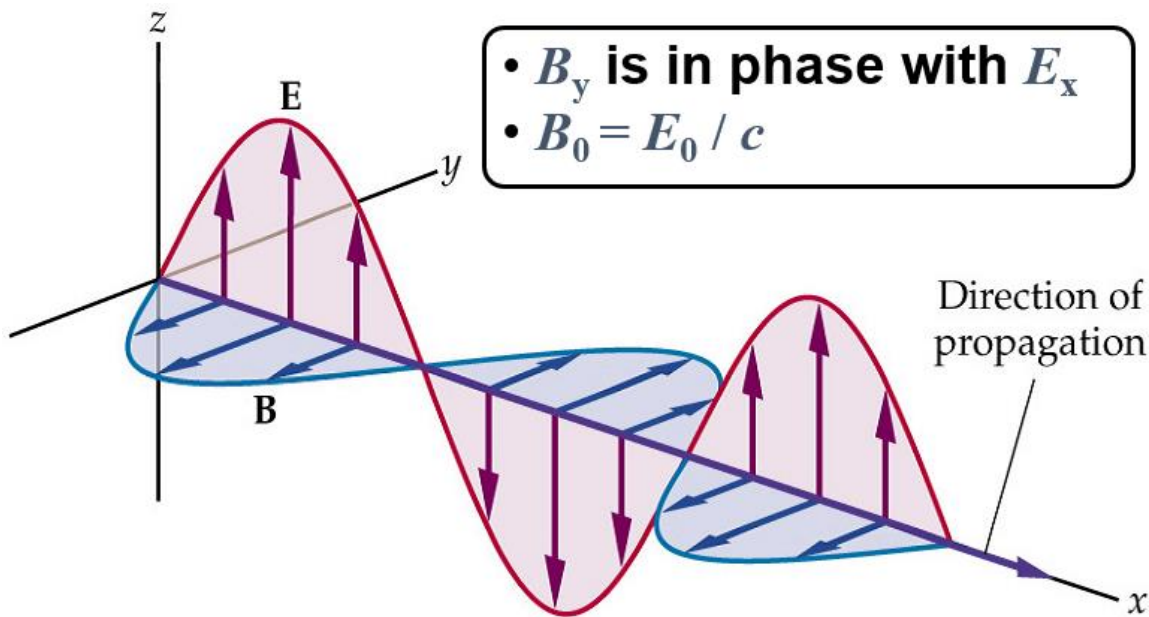


Fig. 30.8: Illustration of the orientation of an electromagnetic wave with a positively charged particle in its path. The electric field (z-axis), magnetic field (Y-axis), and wave propagation direction (x-axis) are shown as mutually perpendicular vectors.

Plane Harmonic Wave:

The electric field can be represented sinusoidally, varying with time and distance as shown in fig. 30.9. The wave moves in the z-direction at the speed of light. The wave equation is typically expressed as,

$$E_x = E_0 \sin(kz - \omega t)$$

$$\text{where, } \omega = kc, E_0 = cB_0$$

$$B_y = B_0 \sin(kz - \omega t)$$

Note: the direction of propagation is given by,

$$\hat{s} = \hat{e} \times \hat{b}$$

where $(\hat{e}, \hat{b})$ are the unit vectors in the (E, B) directions.

Where k is the wave number and ω is the angular frequency.

To observe the wave's movement, one must travel at the speed of light. Gauss's Law indicates that both electric and magnetic flux through a closed surface is zero because there are no net charges inside the wave. Historically, the concept of 'ether' was proposed as a medium for wave propagation, but it has been disproven.

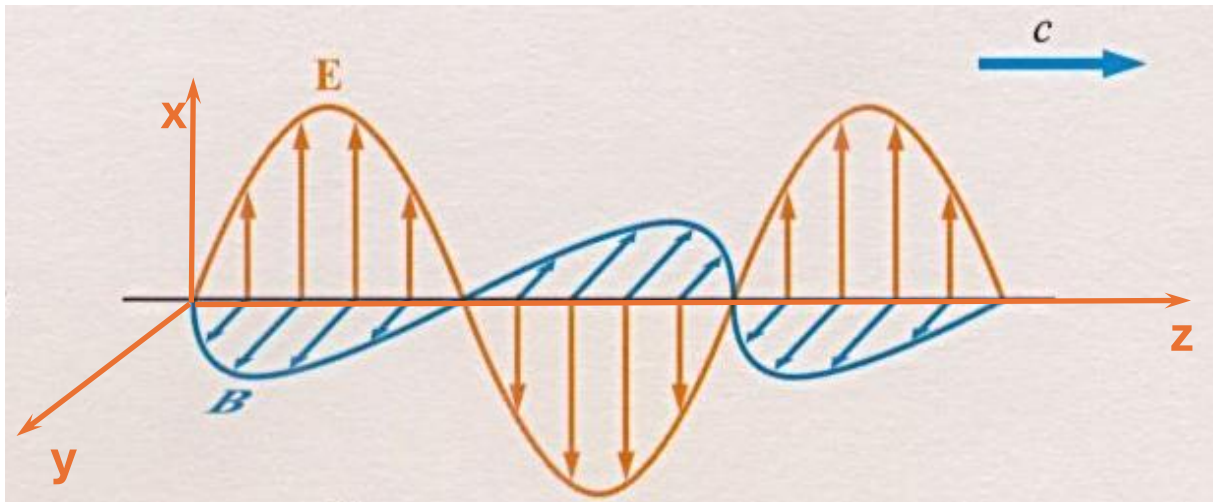


Fig. 30.9: Representation of the sinusoidal variation of the electric field over time and distance for an electromagnetic wave propagating in the z-direction at the speed of light. The electric field (x-axis), magnetic field (Y-axis), and wave propagation direction (z-axis) are shown as mutually perpendicular vectors.

Production of Electromagnetic waves:

Wave Generation Mechanism:

- Electromagnetic waves are generated where charges accelerate, and currents change.
- A typical setup involves a transmitter that sends a current through a transformer to an antenna.
- The antenna contains electrons that move back and forth, creating an oscillating current.
- This oscillating current generates electromagnetic waves that radiate outward as shown in fig. 30.10.

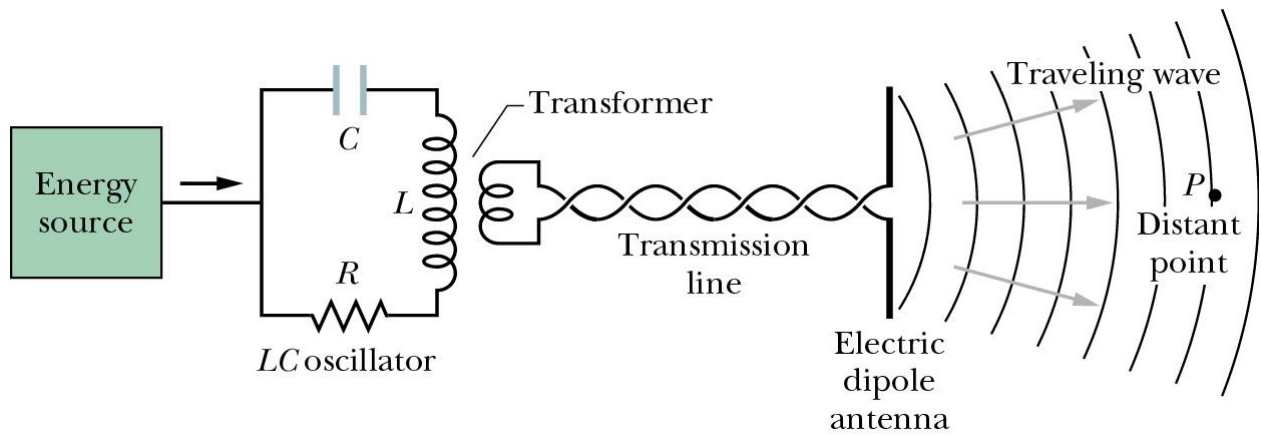


Fig. 30.10: Illustration of the production mechanism of electromagnetic waves: electromagnetic waves are generated when charges accelerate, or currents change. Typically, a transmitter sends a current through a transformer to an antenna. Electrons in the antenna move back and forth due to the oscillating current. This oscillation of electrons creates electromagnetic waves that propagate outward from the antenna.

Types of Antennas and Their Function

Loop Antenna:

- It consists of a ring where current circulates rapidly as shown in fig. 30.11.
- Changing current produces a changing magnetic field, inducing an electromotive force (EMF).
- This process creates an oscillating electric field, generating electromagnetic waves.

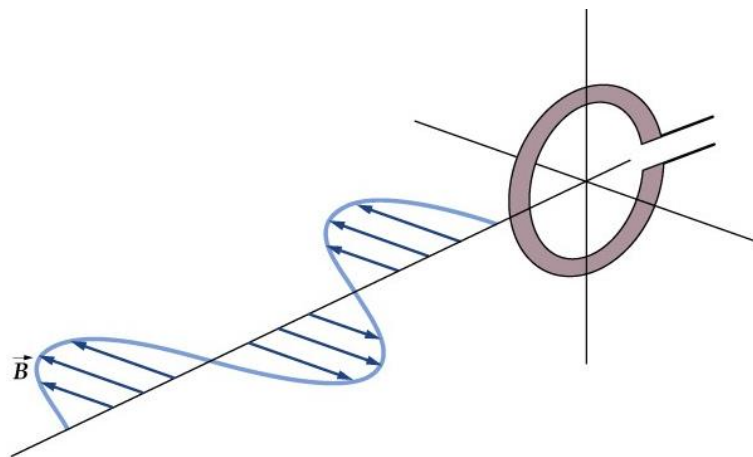


Fig. 30.11: Illustration of a circular loop with circulating current. Changing current induces EMF via magnetic field change, generating oscillating electric field and electromagnetic waves.

Dipole Antenna:

- Composed of two wires with an alternating current source as shown in fig. 30.12.
- Creates an electric field that changes direction with the current cycle.
- Changing electric fields induce magnetic fields, producing electromagnetic waves.

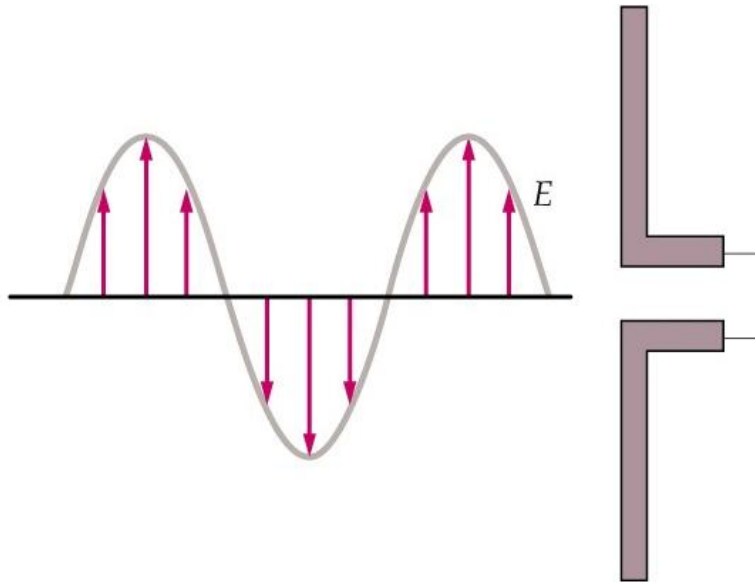


Fig. 30.12. Representation of a dipole antenna, generating changing electric fields inducing magnetic fields, and emitting electromagnetic waves.

Electromagnetic Wave Radiation:

- Waves radiate outward, forming plane waves at a distance.
- The dipole antenna's radiation pattern is strongest perpendicular to the wires and zero along the axis.
- Power radiated is proportional to $\sin^2(\theta)$, where θ is the angle from the axis.
- Intensity decreases with distance, following an inverse square law ($1/r^2$) as shown in fig. 13.
- $$I(\theta) \propto \frac{\sin^2 \theta}{r^2}$$
- Electromagnetic waves from the dipole antenna are polarized along the direction of the electric field.

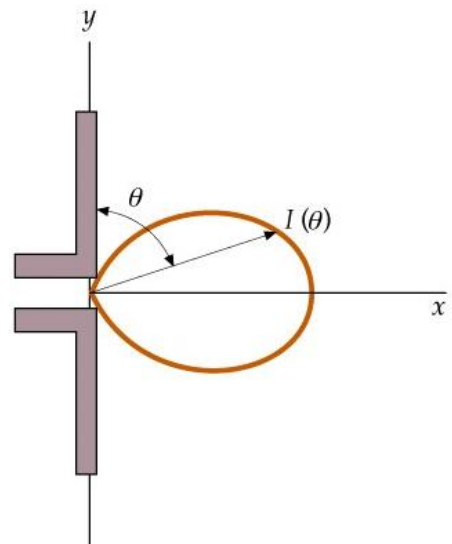


Fig. 30.13: Representation of the electromagnetic waves radiated from a dipole antenna propagate outward, forming plane waves at a distance. The antenna exhibits a radiation pattern where the strongest emission occurs perpendicular to its wires, with zero radiation along the axis.

Practical Implementation of Antennas

Transmitter and Receiver:

- A transmitter generates electromagnetic waves using an antenna.
- A receiver antenna, often on a vehicle or another device, captures these waves, completing the communication process.

Function of Transmitter:

- Inside the transmitter, electronic components like transistors, capacitors, inductors, and resistors, powered by a battery.
- The battery supplies energy, converted into electromagnetic radiation when signals are sent out.
- Moving the lever changes the current, creating a changing magnetic field, which generates a changing electric field, resulting in electromagnetic radiation.

Example of radio frequency (RF) Generator:

- An RF generator can produce different frequencies, controlled by many dials.
- It can generate frequencies ranging from kilocycles to megacycles.
- A simple antenna can be used to emit and receive these frequencies.

Types of Modulation

Amplitude Modulation (AM):

- Amplitude Modulation (AM) uses changes in the signal *strength* to convey information.
- The amplitude of the electromagnetic waves varies according to the sound signal's amplitude.
- The electromagnetic wave (carrier) is modulated by the sound (signal), changing its amplitude to match the sound's intensity as shown in fig. 30.14.

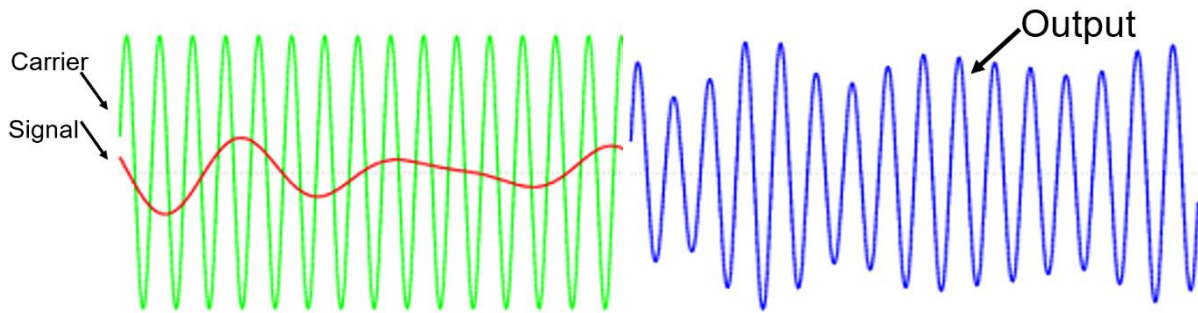


Fig. 30.14: Illustration of an Amplitude Modulation (AM) technique in signal processing and transmission. The waveform on the left depicting the unmodulated carrier wave and the waveform on the right representing the AM signal, where the amplitude of the carrier wave varies in accordance with the amplitude of the modulating signal, while the frequency remains constant.

Frequency Modulation (FM):

- Frequency Modulation (FM) uses changes in the wave's *frequency* to convey information.
- The frequency of the electromagnetic wave varies while the amplitude remains constant as shown in fig. 30.15.
- FM is more complex to produce and detect compared to AM, but it provides better sound quality.

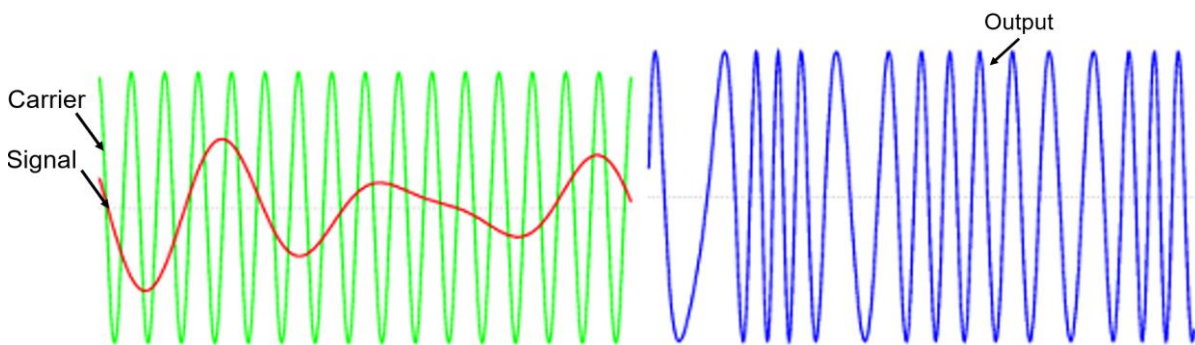


Fig. 30.15: Figure illustrates Frequency Modulation. The waveform on the left-hand side represents the unmodulated carrier wave with a constant frequency. The waveform on the right-hand side depicts output of the FM signal, where the carrier frequency varies according to the amplitude of the modulating signal, while its amplitude remains constant.

Detection of Modulated Signals

AM Detection:

- AM signals are easier to detect using simple circuits.

- The received AC current is passed through a diode or diode bridge and smoothed with a capacitor to extract the original sound signal as shown in fig. 30.16.

FM Detection:

- FM signals require more complex circuitry for frequency demodulation, which adjusts the frequency according to the sound signal's intensity.

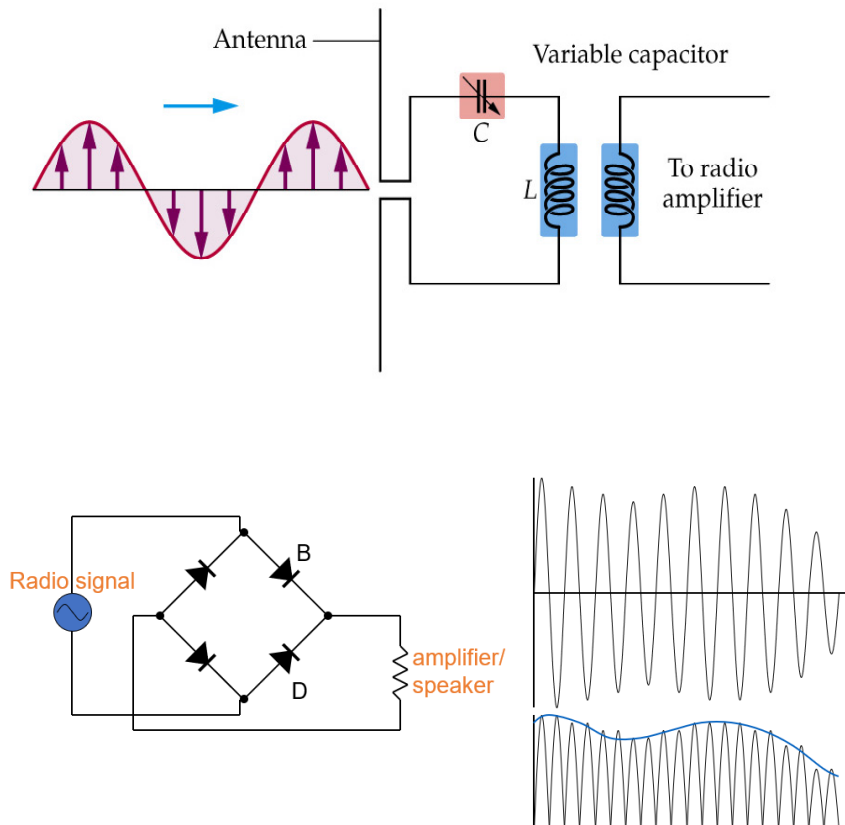


Fig. 30.16: Illustration of AM detection with a circuit involving a diode or diode bridge and capacitor to recover the original signal.

Polarization:

Electromagnetic Wave Polarization:

Electromagnetic Waves from an antenna are polarized along the antenna's direction. And other electromagnetic waves, such as sunlight, can have different polarizations.

Unpolarized Radiation:

When electromagnetic waves reach the ground, their polarization can be in any direction, resulting in unpolarized radiation. For a wave traveling in the z direction, if it is polarized along one direction, then the electric field is defined, and the magnetic field is perpendicular to it.

Polarized Waves:

The direction of the electric field determines the wave's polarization. Polarized radiation can be achieved from specific sources like antennas, while natural sources like the sun or light bulbs emit unpolarized radiation. Polarizers can select a particular polarization from unpolarized light by allowing only the light with an electric field aligned in a certain direction to pass through.

A typical plane electromagnetic wave equation is given by,

$$\begin{aligned} E_x &= E_0 \sin(kz - \omega t), & E_y &= 0, & E_z &= 0 \\ B_x &= 0, & B_y &= B_0 \sin(kz - \omega t), & B_z &= 0 \\ \therefore B_0 &= \frac{E_0}{c} \end{aligned}$$

This wave is an example of a *linearly* polarized wave. The direction of polarization is the direction of the oscillation of the electric field vector.

If a wave traveling in the z -direction and polarized at an arbitrary angle θ in the x - y plane, then the electric field vector be written as;

$$\begin{aligned} E_x &= E_0 \cos \theta \cdot \sin(kz - \omega t) \\ E_y &= E_0 \sin \theta \cdot \sin(kz - \omega t) \\ E_z &= 0 \end{aligned}$$

Linear Polarization:

Absorption/reflection of vector component of wave $\perp$ to polarizer can produce linear polarization as shown in fig. 30.17.

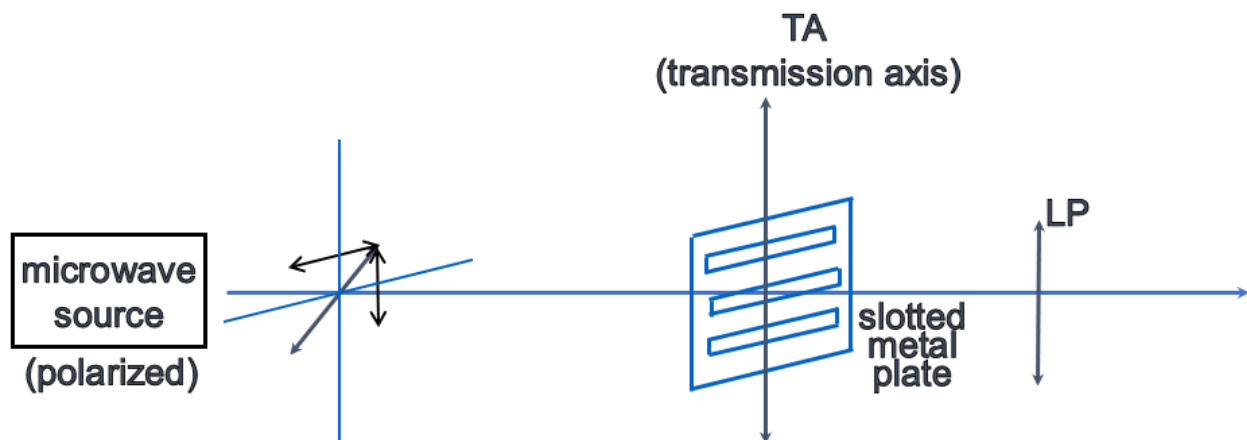


Fig. 30.17: This figure depicts the concept of light polarization. On the left, unpolarized light is represented, showing light waves vibrating in multiple planes perpendicular to the direction of propagation. On the right, polarized light is shown, where light waves are restricted to vibrate in a single plane.

- The E-field component parallel to the slots is absorbed and/or reflected.
- The E-field component perpendicular to the slots is transmitted.
- Absorption produces linearly polarized waves but also reduces intensity.

Mechanism of Polarizers:

- Simple polarizers can be made from finely scratched iron sheets or long molecules that absorb electric fields aligned with them, blocking perpendicular fields.
- Polarized sunglasses (ID glasses) reduce glare by selecting a specific polarization of light.

Historical Aspect:

- James Clark Maxwell's mathematical work over 250 years ago, led to the understanding of electromagnetic waves.
- Experimental confirmation of these waves came 20-22 years later through Hertz's experiments.
- Later, the identification of the three basic forces of nature included electromagnetism, tying in with the weak force.

Recognition:

- Professor Abdus Salam, a Pakistani physicist, contributed significantly to this field and was honoured for his work on the death anniversary of James Clark Maxwell in Scotland.

Lecture#31

The Reality of Light

Understanding the nature of light has been a long journey, requiring the collective efforts of thousands of people over centuries. Only about 250 years ago, we began to understand its true nature. Today, we know that light is actually electromagnetic waves, produced when a charge moves. This movement generates a current, which in turn produces these waves. The importance of light in our lives drives our desire to know everything about it.

The Speed of Light

One of the first questions people asked about light was about its speed. Early studies attempted to measure this, with Galileo being one of the pioneers in this field.

Galileo's Experiment

Galileo conducted an experiment in 1667 where he asked one of his assistants to stand on a mountain with a lantern. He stood on another mountain with his lantern. The idea was that when Galileo uncovered his lantern, the assistant would uncover his as soon as he saw the light, and they would measure the time difference. However, this method proved to be impractical due to the incredibly high speed of light.

Roemer's Discovery

- In 1675, the Danish astronomer Ole Roemer made a significant contribution to understanding the speed of light.
- He observed the moons of Jupiter and noticed that the eclipses of these moons occurred at different times depending on the Earth's position relative to Jupiter as shown in fig.31.,1.
- When the Earth was closer to Jupiter, the eclipses happened sooner, and when it was further, the eclipses were delayed by about 16 minutes and 6 seconds (16.6 min. late).
- Roemer used this time difference to estimate the speed of light (estimated $c = 2.3 \times 10^8$ m/s), concluding it takes this amount of time for light to travel the additional distance across the Earth's orbit.

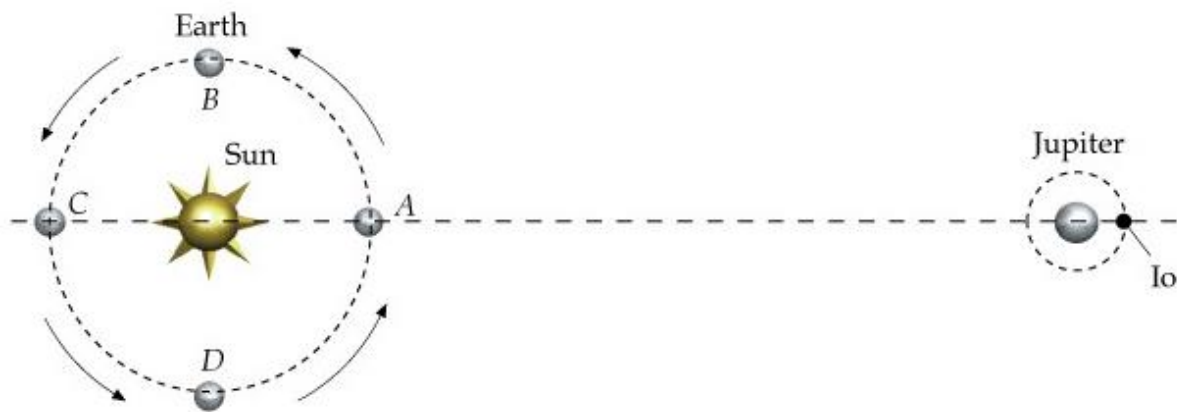


Fig. 31.1: Ole Roemer's observations of Jupiter's moons' eclipses in 1675 showed time variations based on Earth's position, leading to an estimate of the speed of light at 2.3×10^8

Fizeau's Experiment

Overview

Hippolyte Fizeau, a French scientist, conducted a pioneering experiment in 1849 to measure the speed of light. He used a rotating cogwheel apparatus as shown in the fig. 31.2.

1. A beam of light was directed through the gaps between the teeth of a rotating wheel.
2. The light travelled a known distance L to a distant mirror.
3. The mirror reflected the light back through the rotating wheel.
4. By adjusting the rotational speed of the wheel, Fizeau determined the speed of light with remarkable accuracy.

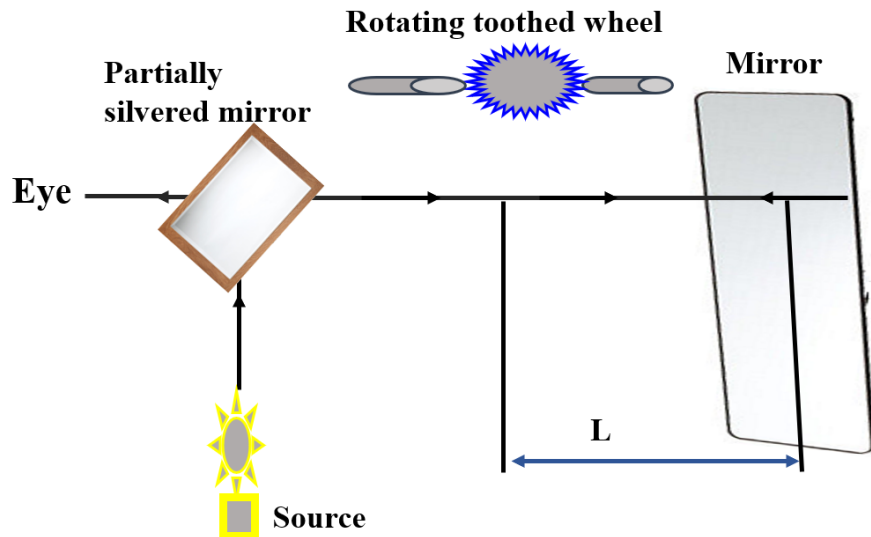


Fig. 31.2: Fizeau's 1849 experiment used a rotating cogwheel and a distant mirror to measure the speed of light by timing the light's round trip and adjusting the wheel's rotation speed.

Derivation of the Formula to calculate the speed of light:

The light beam travels a total distance of $2L$ (to the mirror and back).

The time taken for the light to travel this distance is given by:

$$t = \frac{2L}{C}$$

The cogwheel has N teeth and an equal number of gaps.

The angular distance between consecutive teeth (or gaps) is:

$$\theta = \frac{2\pi}{2N} = \frac{\pi}{N}$$

The wheel rotates with an angular velocity ω (radians per second).

In time t , the wheel rotates through an angle ωt .

For the returning light beam to be observed, it must pass through the next gap.

Therefore, the angle rotated by the wheel in time should equal to:

$$\theta = \omega t$$

$$\text{Using, } t = \frac{2L}{C}$$

$$\theta = \omega \left(\frac{2L}{C} \right)$$

$$\frac{2L}{C} = \frac{\theta}{\omega}$$

$$C = \frac{2L\omega}{\theta} \approx 3 \times 10^8 \text{ m/s}$$

Where,

L : Distance to the mirror.

c : Speed of light.

θ : Angular distance between the teeth of the cogwheel (in radians).

ω : Angular velocity of the rotating wheel.

Foucault's Experiment

In 1860, Léon Foucault improved upon Fizeau's work using a rotating mirror system. This method involved:

1. A light source directed towards an eight-sided rotating mirror as shown in the fig.31.3.
2. The light reflected from the mirror to a fixed mirror and back.
3. The rotation speed was adjusted so the light returned to the observer through the same path.
4. He found no signal unless mirror rotates $1/8$ turn in ΔT
5. This experiment further confirmed the high but finite speed of light, $c = 2.98 \cdot 10^8$ m/s

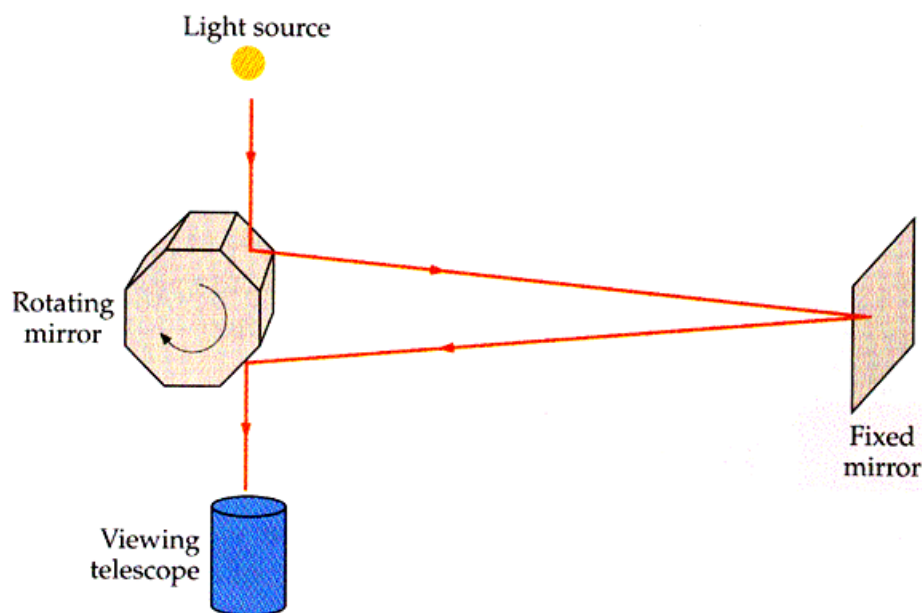


Fig. 31.3: Foucault's 1860 experiment used an eight-sided rotating mirror and a fixed mirror to measure the speed of light, refining the estimate to 2.98×10^8 m/s by adjusting the rotation speed for precise light reflection.

The Nature of Light and Electromagnetic Waves

All these experiments show that light travels incredibly fast around 300,000 kilometres per second, completing multiple orbits (7-8) around the Earth in one second. This explains why we

see lightning before hearing thunder since sound waves travel much slower, at 1100 meters per second.

Light travels much faster than sound. For example:

1. Thunder and lightning start at the same time, but we will see the lightning first.
2. When a starting pistol is fired, we see the smoke first and then hear the bang.

The Visible Spectrum

Light and sound are waves, but light is part of a vast electromagnetic spectrum. The visible range for humans is from about 400 nanometers (deep blue) to 700 nanometers (deep red) as shown in fig.4. Different light sources emit varying wavelengths within this range.

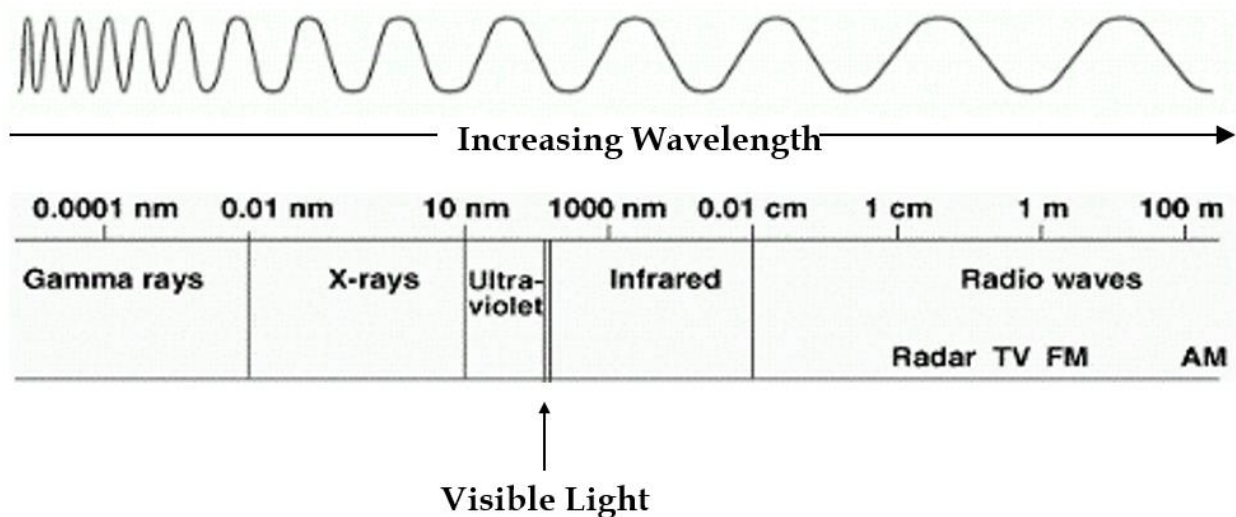


Fig. 31.4: The visible spectrum, ranging from 400 nm (deep blue) to 700 nm (deep red), represents the portion of the electromagnetic spectrum perceivable by the human eye.

White Light and Color Perception

White light contains all colours in the visible spectrum. It is perceived as white when all the frequencies are approximately equal. Combining different colors can create other colors:

- Blue and red make turquoise.
- Blue and green make light blue.
- Red and green make yellow.

- Combining red, blue, and green in equal amounts produces white as shown in fig. 5.

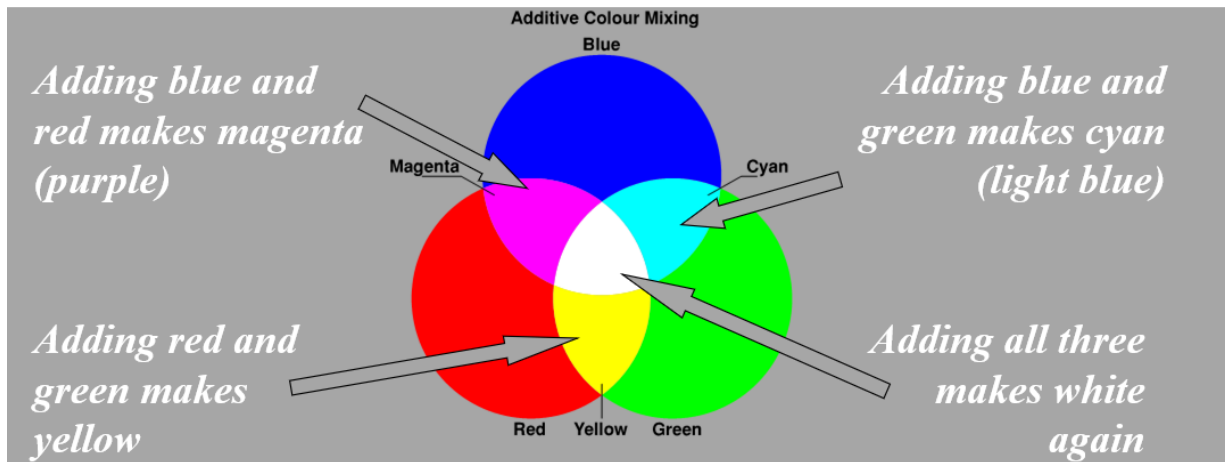
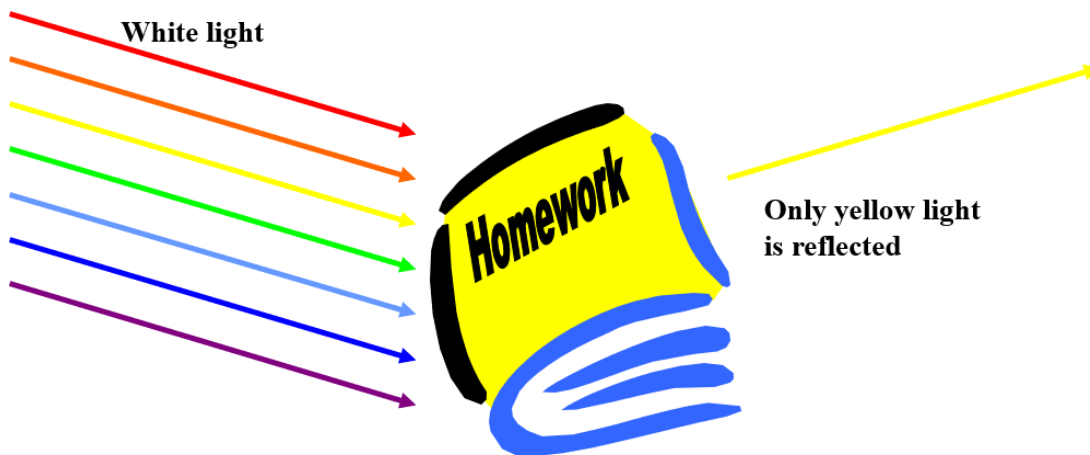


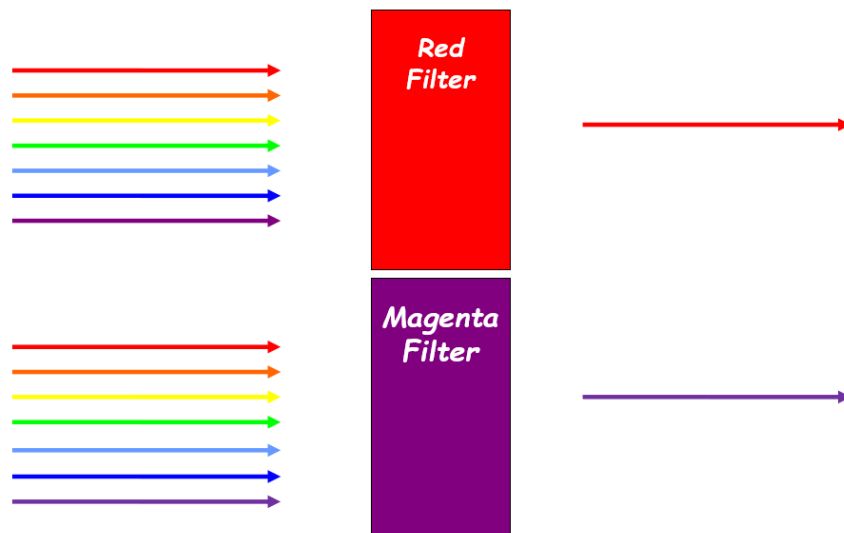
Fig. 31.5: White light, containing all colors of the visible spectrum, can be split into individual colors. Combining different colors like blue and red, blue and green, and red and green can create turquoise, light blue, and yellow, respectively, while equal amounts of red, blue, and green produce white.

Objects appear coloured based on the light they reflect. For example, a yellow object reflects yellow light, while a white object reflects all colours.



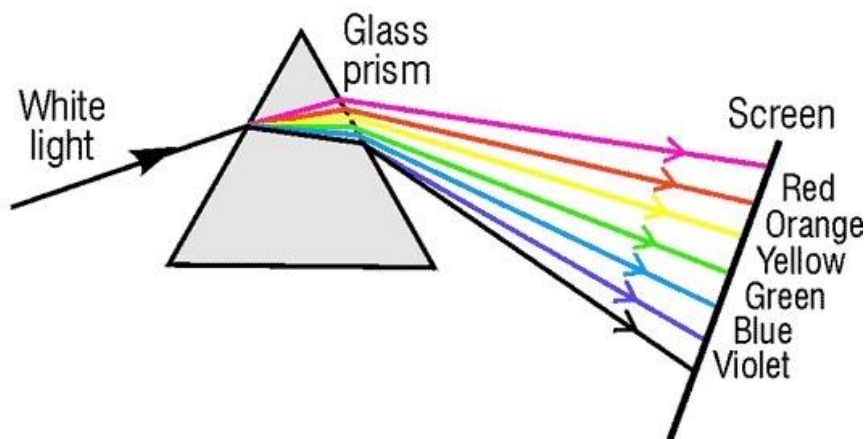
Filters and Color Selection

Colour filters can select specific colours from white light by absorbing other colours. For example, a red filter allows only red light to pass through.



Light Dispersion

White light is not a single colour, when white light passes through a prism, it disperses into its component colours due to different wavelengths traveling at different speeds through the medium. This results in a spectrum of colours from blue to red.



Exploring Light: Reflection and Refraction

Understanding the behaviour of light through simple experiments can illuminate the principles of reflection and refraction.

Reflection of Light

Experiment with a Laser and Mirror

Using a laser, we can conduct experiments to explore how light reflects off surfaces. When a laser beam hits a mirror:

1. Part of the light passes through the mirror.
2. Another part of the light is reflected.

Key Concepts of Reflection

- **Incident Ray:** The incoming light ray that strikes the mirror.
- **Reflected Ray:** The light ray that bounces off the mirror.
- **Angle of Incidence:** The angle between the incident ray and the normal (a line perpendicular to the surface).
- **Angle of Reflection:** The angle between the reflected ray and the normal.

In a smooth mirror, the angle of incidence is equal to the angle of reflection as shown in fig. 31.6.

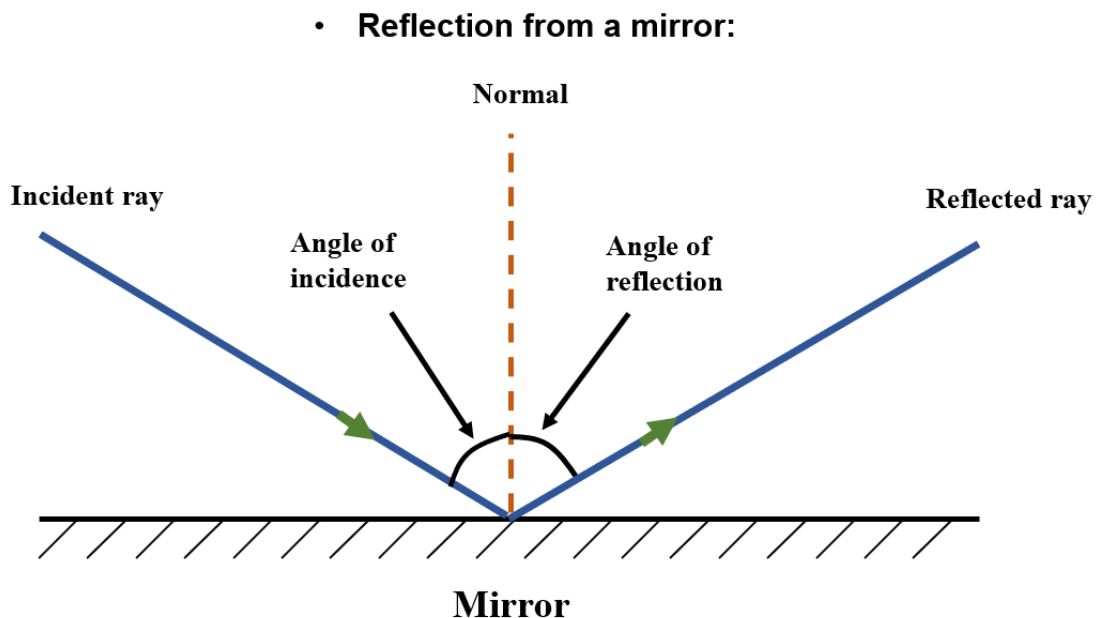


Fig. 31.6: demonstrating the law of reflection in a smooth mirror, where the angle of incidence is equal to the angle of reflection.

Diffuse Reflection

When light hits a rough surface, such as a coat, it scatters in many directions. Each small element of the surface reflects light according to the law of reflection, but the overall effect is diffuse reflection, where light is reflected in multiple directions as shown in fig. 31.7.

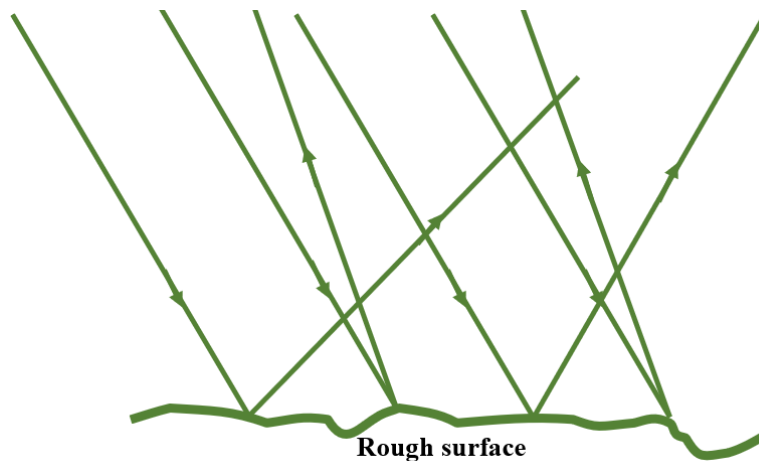


Fig. 31.7: Diffuse reflection occurs when light strikes a rough surface, causing it to scatter in various directions due to the irregularities on the surface.

Refraction of Light

Experiment with a Laser and Glass

When a laser beam passes through a glass surface:

1. The light enters the glass and bends towards the normal.
2. As it exits the glass and re-enters the air, it bends away from the normal.

Key Concepts of Refraction

- **Refraction:** The bending of light as it passes from one medium to another.
- **Normal:** A line perpendicular to the surface at the point of incidence.
- **Angle of Incidence:** The angle between the incident ray and the normal.
- **Angle of Refraction:** The angle between the refracted ray and the normal.

When light travels from air to glass, it bends towards the normal as shown in fig. 8. Conversely, when it exits the glass and enters air, it bends away from the normal. The incident light and transmitted light are parallel but displaced.

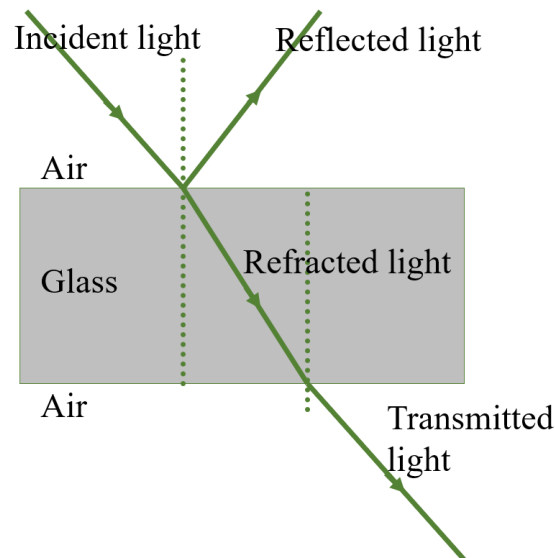


Fig. 31.8: Refraction of light at the interface between air and glass, demonstrating bending towards the normal upon entering glass and away from the normal upon exiting, with incident and transmitted light paths shown as parallel but displaced.

Understanding Refractive Index and Light Behavior

Definition of Refractive Index

The refractive index (n) is defined as the speed of light in a vacuum (c) divided by the speed of light in a medium (v):

$$\text{Refractive index} = \frac{\text{Speed of light in vacuum}}{\text{Speed of light in material}}$$

$$n = \frac{c}{v}$$

When light enters a medium, it usually slows down due to interactions with the medium's particles, causing a change in speed.

Reasons for Light Slowing Down

When light enters a medium, the electrons within the medium interact with the electromagnetic field of the light, causing them to vibrate and emit electromagnetic radiation. This process results in the formation of a new wave that moves forward but at a slower speed than the original wave.

Refractive Index of Common Materials:

- **Air:** 1.0003
- **Water:** 1.33
- **Alcohol:** 1.36
- **Oil:** 1.5
- **Diamond:** 2.42

Light Refraction Experiment with Water

1. Shine a laser through an empty glass.
2. Gradually fill the glass with water.
3. Observe the laser beam bending towards the normal as water is added to the glass due to the higher refractive index of water (1.33).

Effects of Refraction

Objects submerged in water appear bent or displaced due to the change in light direction as it passes from water (higher refractive index) to air (lower refractive index).

Fermat's Principle

Historical Context

Fermat's Principle, which is fundamental in the study of optics, states that light takes the path that requires the least time when traveling from one point to another. This principle was also mentioned by Ibn Al-Haytham (Ibne Haim) in his significant work "Kitab al-Manazir" (The Book of Optics), though the mathematics behind it was not fully developed during his time. Fermat's Principle provides an easier and more straight forward understanding of the behavior of light compared to the methods available in earlier periods.

Fermat's Principle Explained Light in Free Space:

In free space or a vacuum, light travels in straight lines. According to Fermat's Principle, this is because the straight-line path is the shortest and therefore the quickest route for light to travel between two points.

To mathematically represent this, consider two points A and B in free space as shown in fig. 31.9. The time t taken for light to travel this distance L is given by:

$$t = L / c$$

where c is the speed of light in a vacuum. Since the distance L in a straight line is the shortest possible distance, the time t is minimized.

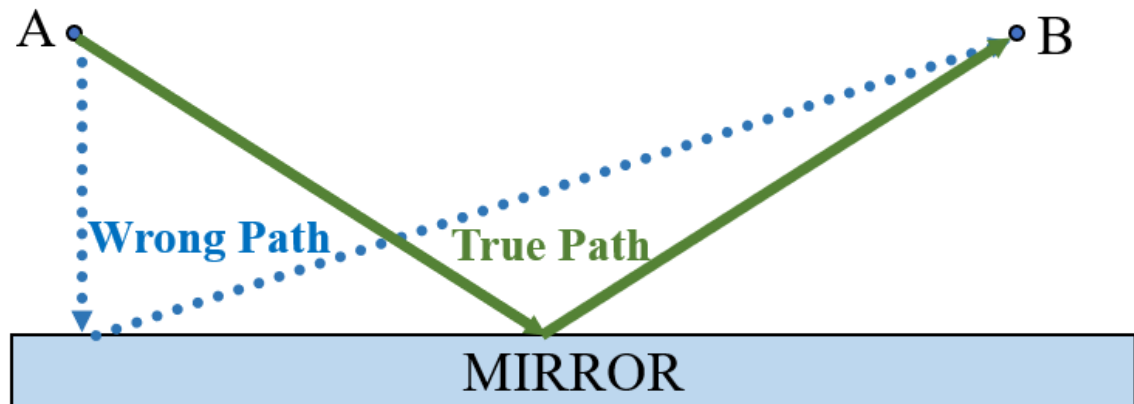
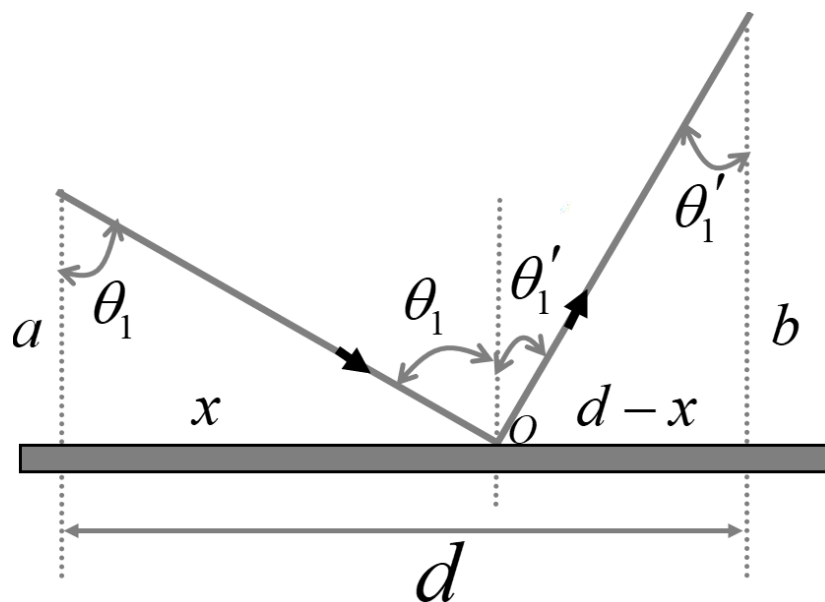


Fig. 31.9: Fermat's Principle dictates that light travels in straight lines in free space or a vacuum, following the path that minimizes the time between two points A and B.

A light ray traveling from one fixed point to another fixed point follows a path such that, compared with nearby paths, the time required is minimum.

For a case of Reflection

Consider a light wave traveling from point A to point B, reflecting off a mirror at point O. Fermat's Principle can be used to determine the path of the light.



The total distance traveled by the light is,

$$L = \sqrt{a^2 + x^2} + \sqrt{b^2 + (d - x)^2}$$

$$t = L / c$$

Taking derivative w.r.t 'x',

$$\frac{dt}{dx} = \frac{1}{c} \frac{dL}{dx}$$

where:

- a is the vertical distance from A to the mirror.
- b is the vertical distance from B to the mirror.
- x is the horizontal distance from A to O.
- d is the total horizontal distance between A and B.

To minimize the travel time, take the derivative of t with respect to x and set it to zero.

$$\frac{dt}{dx} = 0$$

$$\begin{aligned} 0 &= \frac{1}{c} \frac{dL}{dx} = \frac{1}{c} \frac{d}{dx} (\sqrt{a^2 + x^2} + \sqrt{b^2 + (d - x)^2}) \\ &= \frac{1}{2c} (a^2 + x^2)^{-1/2} (2x) + \frac{1}{2c} [b^2 + (d - x)^2]^{1/2} (2)(d - x)(-1) \end{aligned}$$

On simplifying,

$$\frac{x}{\sqrt{a^2 + x^2}} = \frac{d - x}{\sqrt{b^2 + (d - x)^2}}$$

$$\sin \theta_1 = \sin \theta_1'$$

$$\theta_1 = \theta_1'$$

This results in the equality of angles of incidence and reflection, confirming that the angle of incidence θ_1 is equal to the angle of reflection θ_2 .

Refraction and Snell's Law

When light travels from one medium to another, its speed changes due to different refractive indices. This change causes the light to bend, a phenomenon known as refraction.

Derivation Using Fermat's Principle

Consider a light ray traveling from point A in medium 1 (with refractive index n_1) to point C in medium 2 (with refractive index n_2). The light refracts at the boundary between the two mediums at point O.

Let:

- θ_1 be the angle of incidence.
- θ_2 be the angle of refraction.
- x be the horizontal distance from A to O.
- L_1 be the distance from A to O.
- L_2 be the distance from O to C.

Using Fermat's Principle, the total time t taken by light to travel from A to C is the sum of the time taken in each medium:

$$t = \frac{L_1}{v_1} + \frac{L_2}{v_2}$$

where v_1 and v_2 are the speeds of light in mediums 1 and 2, respectively. The speeds of light in the mediums are related to the refractive indices by:

$$n = \frac{c}{v}, \Rightarrow v_1 = \frac{c}{n_1}, \Rightarrow v_2 = \frac{c}{n_2}$$

$$t = \frac{n_1 L_1 + n_2 L_2}{c} = \frac{L}{c}$$

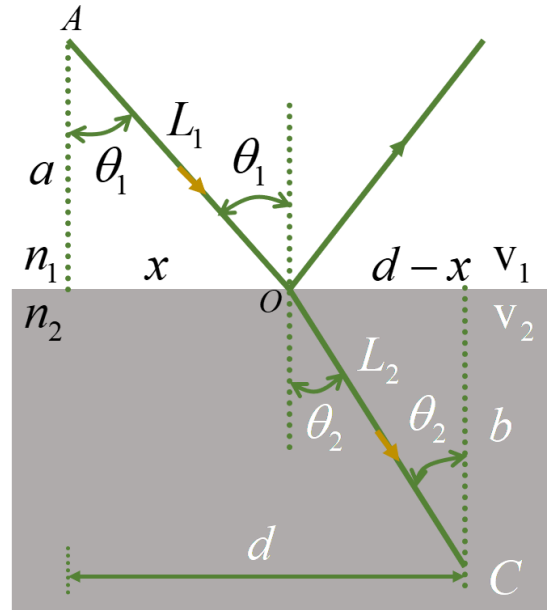
$$L = n_1 L_1 + n_2 L_2 \quad (\text{Optical path length})$$

$$L = n_1 \sqrt{a^2 + x^2} + n_2 \sqrt{b^2 + (d-x)^2}$$

$$0 = \frac{dt}{dx} = \frac{1}{c} \frac{dL}{dx} = \frac{n_1}{2c} (a^2 + x^2)^{-1/2} (2x) + \frac{n_2}{2c} [b^2 + (d-x)^2]^{1/2} (2)(d-x)(-1)$$

$$n_1 \frac{x}{\sqrt{a^2 + x^2}} = n_2 \frac{d-x}{\sqrt{b^2 + (d-x)^2}}$$

$$n_1 \sin \theta_1 = n_2 \sin \theta_2 \quad (\text{called snell's law of refraction})$$



Total Internal Reflection

- When light travels from a medium with a higher refractive index to one with a lower refractive index, it can be totally internally reflected if the angle of incidence exceeds a certain critical angle.

- If the angle of incidence is greater than θ_c , total internal reflection occurs, and the light does not pass into the second medium.

Critical Angle and Total Internal Reflection

- Consider a laser emitting light at different angles. When the angle of incidence reaches the critical angle (θ_c), the light travels along the surface of the medium. If the angle of incidence exceeds θ_c , total internal reflection occurs, and the light remains within the medium as shown in fig. 10.
- One can calculate the critical angle by following:

$$n_1 \sin \theta_c = n_2 \sin 90^\circ$$

$$\theta_c = \sin^{-1} \frac{n_2}{n_1}$$

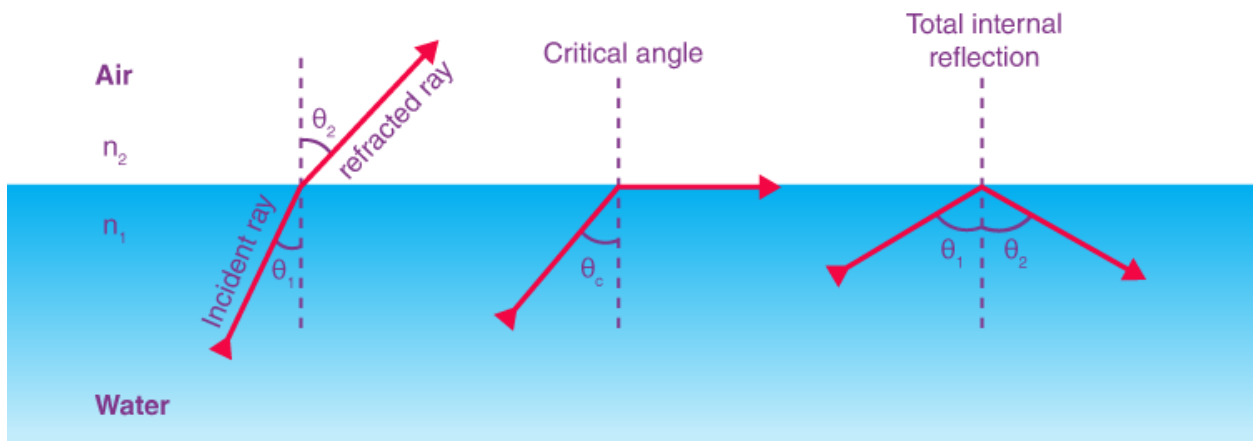


Fig. 31.10: Total internal reflection occurs when light within a medium strikes the boundary at an angle exceeding the critical angle θ_c , causing it to reflect back into the medium rather than refracting out.

Importance of Total Internal Reflection

- **Critical Angle:** The angle beyond which light is totally internally reflected within the medium.
- **Applications:**
 - Optical fibers for communication.
 - Diamond brilliance due to a low critical angle and high refractive index.

Optical Fibers:

Optical fibers use total internal reflection to transmit light signals over long distances with minimal loss. An optical fiber consists of a core with a high refractive index surrounded by cladding with a lower refractive index. Light entering the fiber is reflected internally, ensuring it travels through the core without escaping.

For effective transmission, the optical fiber must be very thin, and the refractive index must be controlled to ensure total internal reflection.

Optical Fiber Communication and the Nature of Light

Structure of Optical Fiber:

- **Core:** Thin glass thread, thinner than a human hair, with a high refractive index as shown in fig. 31.11.
- **Cladding:** Surrounds the core and has a lower refractive index to ensure total internal reflection.
- **Protective Covering:** Encases the cladding for protection.

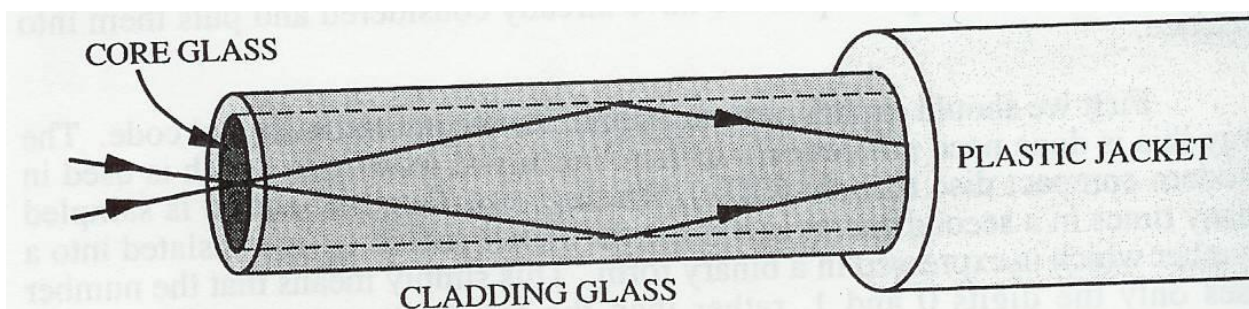


Fig. 31.11: Representation of an optical fibre, comprising a core with a high refractive index, surrounded by cladding with a lower refractive index, and a protective covering, essential for efficient light transmission through total internal reflection.

Optical Fiber in Communication

1. Historical Context:

- 100 years ago, cables could only support 50-60 telephone calls simultaneously.
- Modern optical fibers can handle millions of calls due to total internal reflection.

2. Mechanism:

- Light travels through the core, reflecting internally due to the difference in refractive indices.

- This prevents signal loss, even if the fiber is bent or under slight pressure.

3. Advantages:

- High capacity for data transmission.
- Minimal signal loss.

The Nature of Light: Particle or Wave?

Historical Debate

- **Newton's Particle Theory:** Light is made of particles.
- **Wave Theory:** Light behaves as waves.

Modern Understanding

- Light exhibits both particle and wave properties, known as wave-particle duality.
- A *wave* is a moving pattern that is revealed by its interaction with particles

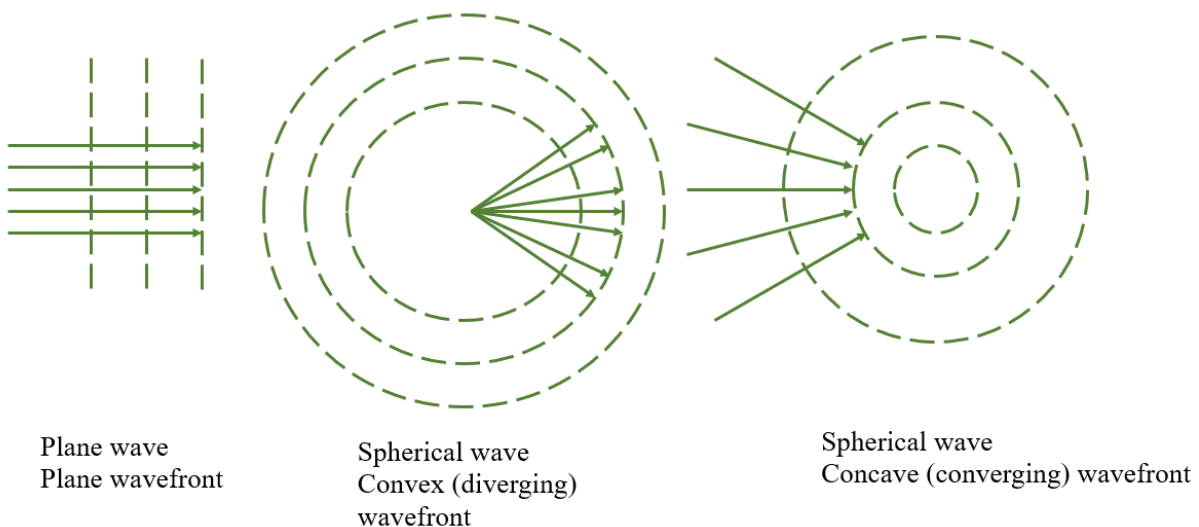
Concept of Wavefronts

1. Wavefronts:

- **Circular Wavefront:** Originates from a point source, spreads out in circles.
- **Plane Wavefront:** Forms when circular wavefronts travel far and flatten.

2. Interaction with Obstacles:

- When waves hit an obstacle, they can change direction and create new wavefronts.
- This results in regions of light and dark, demonstrating wave behavior.



Huygens' Principle

- **Basic Concept:**
 - When light encounters an obstacle, each point on the wavefront acts as a source of new wavelets.
 - These wavelets spread out and form new wavefronts as shown in fig. 31.12.
- **Implications:**
 - Light can bend around corners.
 - This bending and spreading of light waves are due to their wave nature.

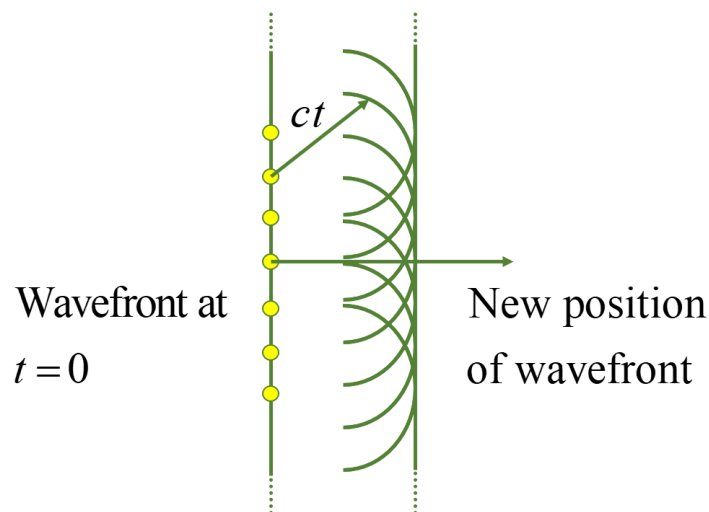


Fig. 31.12: Huygens' Principle illustrates how each point on a wavefront acts as a source of secondary wavelets, which propagate and form new wavefronts, allowing light to bend around corners and demonstrating its wave nature.

Light as Electromagnetic Radiation

- **Generation of Light:**
 - Light is generated by accelerating charges.
 - Examples: Heated filament in a light bulb, or electrons moving in atoms.
 - Power radiated by a single charge is:

$$P = \frac{2}{3} \frac{e^2 a^2}{c^3}$$

- Electrons in atoms emit radiation when they accelerate.
- Classical physics could not explain why electrons don't lose energy and spiral into the nucleus.

- **Quantum Mechanics:**

- Introduced to explain atomic behavior and energy quantization.

- **Electromagnetic Waves:**

- Composed of electric and magnetic fields oscillating perpendicular to each other.
- Energy carried by electromagnetic waves depends on the amplitude of these fields.
- Electromagnetic waves transport linear momentum as well as energy
- For complete absorption of energy U ,

$$p = \frac{U}{c}$$

- For complete reflection of energy U ,

$$p = \frac{2U}{c}$$

- Radiation pressures can be determined experimentally

$$\frac{E_{\max} B_{\max}}{2\mu_0 c} = \frac{E_{\max}^2}{2\mu_0 c^2} = \frac{B_{\max}^2}{2\mu_0}$$

Colors and Temperature

- **Color of Light:**

- The color of light from an object indicates its temperature.
- Hotter objects emit light of shorter wavelengths (e.g., blue stars).
- Cooler objects emit light of longer wavelengths (e.g., our sun appears yellow).

Radiation Pressure

- **Concept:**

- The electric field of an electromagnetic wave that strikes a surface act on an electron, giving it a velocity v as shown in fig. 31.13. The magnetic field then exerts a force on the moving charge in the direction of propagation of the incident light. That is the origin of the “light pressure”.
- Light exerts pressure on objects when it is absorbed or reflected.
- This radiation pressure is typically very small and not noticeable in everyday life.

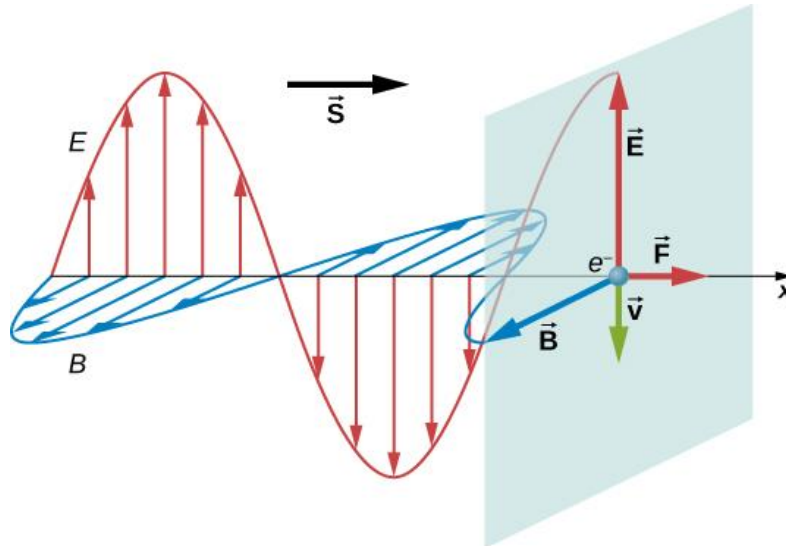


Fig. 31.13: Radiation pressure arises from the momentum carried by electromagnetic waves. When light is absorbed or reflected, it exerts a small but measurable pressure on surfaces, stemming from the interaction of the electric and magnetic fields with the surface of electrons.

- **Solar Radiation Pressure:**

- Within the sun, radiation pressure counteracts gravitational pull, maintaining equilibrium.

Radiant Flux:

- Total amount of energy passing through a surface or region of space *per unit time*
 - Typically denoted by Φ (Phi)
 - Also called *power*
 - Measured in watts (W) or joules/second (J/s)
 - These are metric quantities (rather than e.g., calories/second)
- Typically used for a light's total output.

Lecture#32

Introduction:

In this lecture, we will focus on the interaction between matter and light. Understanding this interaction helps us answer questions such as:

- Why do objects have different colors?
- Why is the sky blue?
- How does light pass through, refract, or reflect off objects?

Spectroscopy:

To study light properly, we use a spectrograph. The process involves:

- Light passing through a slit.
- Entering a tube where the waves become parallel.
- Entering a dispersive element (prism/mirror).
- Different colors separating and reaching an imaging lens.
- Recording the light on a photographic film or another device as shown in fig. 32.1.

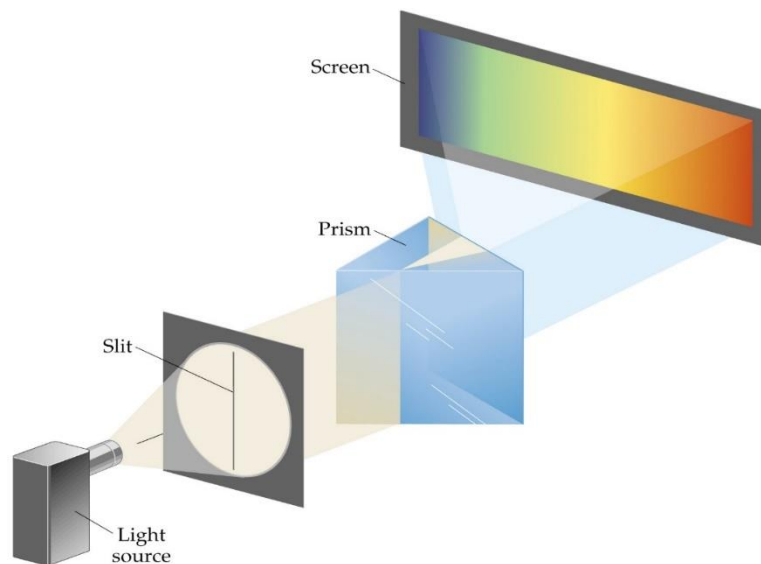


Fig. 32.1: Representation of a spectrograph, where light passes through a slit, is collimated, dispersed into colors by a prism, and appeared on a screen.

Light as an Electromagnetic Wave:

Light is an electromagnetic wave that does not need a medium to travel but can also pass through various mediums. Each source of light emits different frequencies and wavelengths. For example:

Red light: Frequency of 4.29×10^{14} Hz, wavelength of approximately 700 nm.

Blue light: Frequency of 7.50×10^{14} Hz, wavelength of approximately 400 nm.

Basic Spectrograph:

A basic spectrograph is an essential tool used to study the interaction between matter and light, particularly how light is emitted, absorbed, transmitted, and reflected by different substances.

Basic Spectrograph Components and Function

A spectrograph is an instrument designed to disperse light into its constituent wavelengths, creating a spectrum that can be analysed as shown in fig. 32.2. Here's how it works:

- **Light Source:** The journey begins with a light source, such as a torch. The light emitted by the torch contains a wide range of frequencies and wavelengths.
- **Slit:** The light enters the spectrograph through a narrow slit. This helps to produce a precise, collimated beam of light.
- **Collimating Tube:** The light beam passes through a collimating tube, which makes the light rays parallel.
- **Dispersive Element:** The parallel light rays then hit a dispersive element, such as a prism or a diffraction grating. This element separates the light into its component colors (wavelengths).
- **Imaging Lens:** After dispersion, the separated light passes through an imaging lens, which focuses the different colors.
- **Recording Device:** Finally, the focused light reaches a recording device, such as a photographic film or a digital sensor. This device captures the spectrum, allowing for detailed analysis.

Function of the Spectrograph

- **Separation of Light:** The spectrograph separates light into its component wavelengths. This separation is crucial for studying the properties of the light.
- **Recording Spectra:** The recording device captures the spectrum, which can then be analysed to determine the light's characteristics and the properties of the source emitting it.

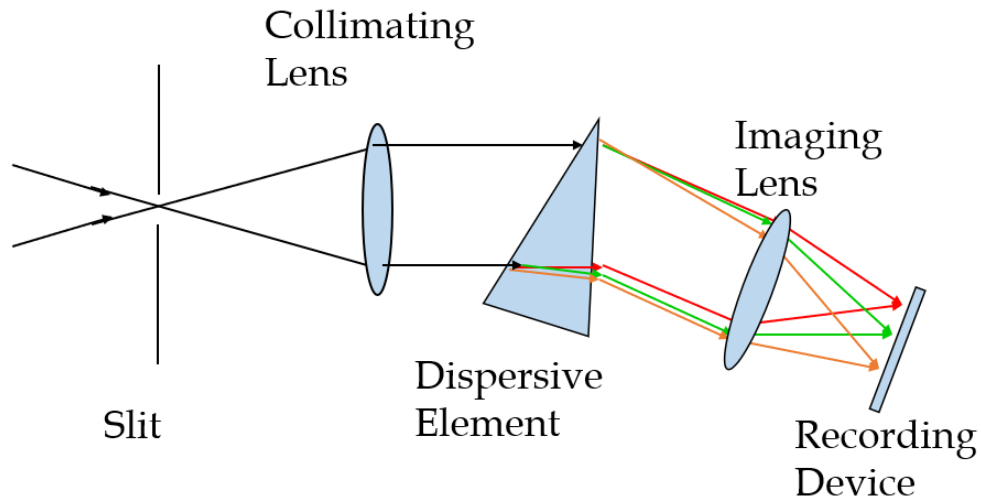


Fig. 32.2: Basic components of a spectrograph, illustrating the process from light entry through a slit to spectrum and recorded by a detector.

Eye Sensitivity to Color:

Our eyes are most sensitive to green light but can see from deep blue to deep red. Sensitivity varies with wavelength, peaking in the red range as shown in fig. 32.3.

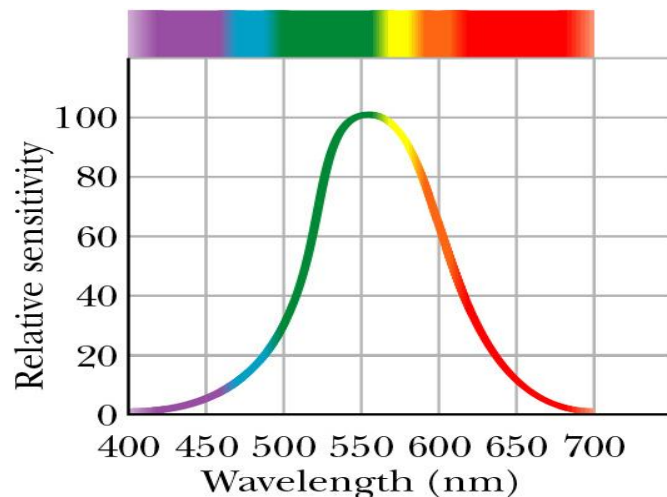


Fig. 32.3: A plot between wavelength and relative eye sensitivity to color, illustrates the peak sensitivity in the red range and visible spectrum extending from deep blue to deep red.

Light and Matter Interaction:

The interaction between light and matter can be divided into four parts:

- 1) **Emission:** How light is emitted from matter.
- 2) **Absorption:** How light is absorbed by matter.
- 3) **Transmission:** How light passes through matter.
- 4) **Reflection:** How light is reflected by matter.

Emission:

When a filament in a bulb is heated by an electric current, it emits light. The process involves:

- Atoms in the filament moving and accelerating electrons, causing electromagnetic radiation.
- The temperature of a filament in a common light bulb is around 100 degrees centigrade, emitting white light. As it cools, it emits red light, and if cooled further, only heat without visible light.

Black Body Radiation:

Blackbody radiation is a term used to describe the relationship between an object's temperature, and the wavelength of electromagnetic radiation it emits. A black body is an idealized object that absorbs all electromagnetic radiation it comes in contact with.

In a welding shop, as metal is heated, its color changes due to black body radiation. The sequence of colors is:

- Dark red
- Red
- Yellow
- Pure white

When metal cools, it reverses this color sequence. Black body radiation occurs because the atoms and molecules inside the heated object move, causing the emission of light.

Box Example

Imagine a closed box with a small hole. When the box is heated, light waves inside become excited and exit through the hole, demonstrating black body radiation. This process occurs due to the movement of atoms, molecules, and electrons within the heated substance.

Black body radiation is produced by the movement and acceleration of electrons within atoms and molecules, emitting electromagnetic radiation. The color of this radiation varies with temperature.

Temperature and Wavelength

The relative brightness of radiation changes with temperature as shown in fig. 32.4. For example:

- At 8000 K, radiation peaks around 300 nm, appearing blue.
- At 6000 K, radiation peaks around 500 nm.
- At 4000 K, radiation peaks around 700 nm, appearing reddish.

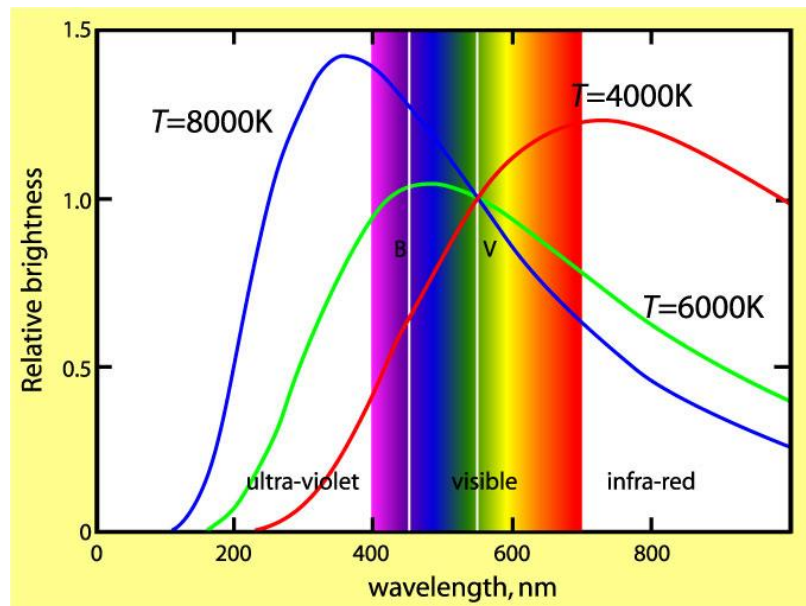


Fig. 32.4: A plot between wavelength and relative brightness illustrates the relative brightness of radiation at different temperatures, with peaks at 300 nm (blue) for 8000 K, 500 nm for 6000 K, and 700 nm (reddish) for 4000 K.

Wien's Law:

Wien's Law relates the temperature of an object to the wavelength at which it emits the maximum radiation: $\lambda_{\text{max}} \times T = 2.90 \times 10^{-3} \text{ m K}$ as shown in fig. 5. This law applies when a substance is in thermal equilibrium.

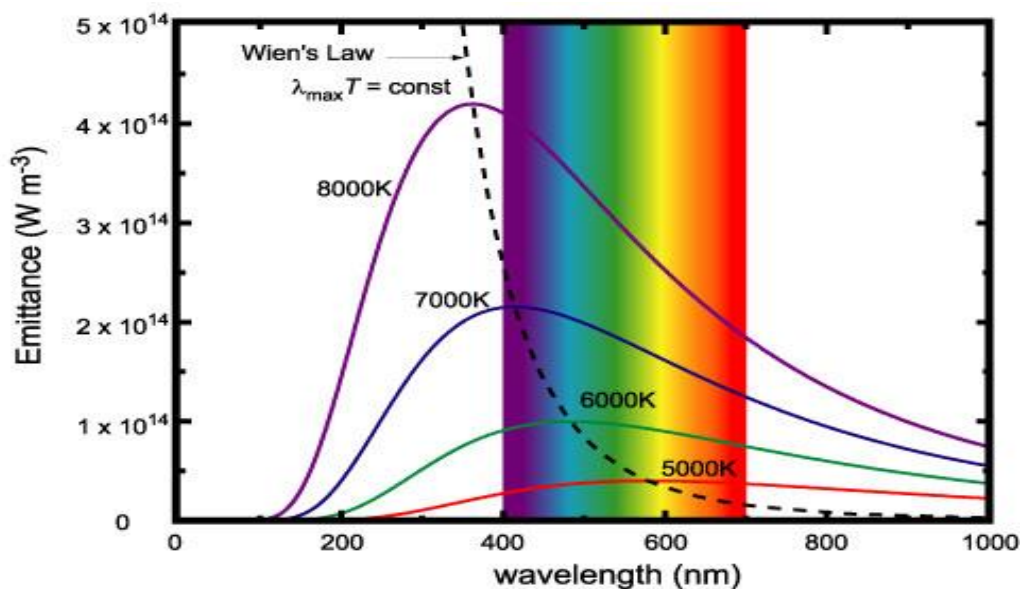


Fig. 5: Wien's Law, illustrating the relationship between an object's temperature and the wavelength of maximum radiation emission, given by $\lambda_{\text{max}} \times T = 2.90 \times 10^{-3} \text{ m K}$, applicable to substances in thermal equilibrium.

When can you use Wien's Law?

Application of Wien's Law:

- Only for objects that emit light not for those that reflect light.
- Light emitted by hot, solid objects obey Wien's Law.
- The Sun and other stars obey Wien's Law since the gases they are composed of remain at a high density (at least up to the outermost layers of the star).

Wien's Law can be used to determine the temperature of the Sun and stars, which are in thermal equilibrium. For example:

- Stars that appear blue have temperatures around 15000 K.
- The Sun's surface temperature is around 5800 K.
- Cooler stars appear red as shown in fig. 32.6.

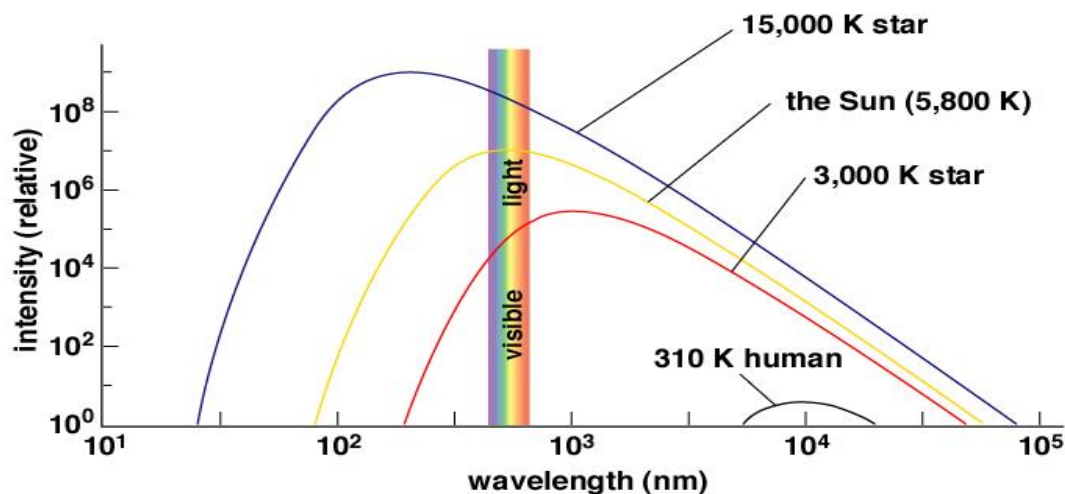


Fig. 32.6: A plot between the wavelength and intensity to illustrate the temperature of stars, with blue stars at approximately 15000 K, the Sun at 5800 K, and cooler stars appearing red.

Stefan-Boltzmann Law:

The Stefan-Boltzmann Law calculates the power radiated per unit area of a black body given as:

$$P = \sigma T^4$$

where σ is the Stefan-Boltzmann constant ($5.67 \times 10^{-8} \text{ W/m}^2\text{K}^4$). All heated matter radiates energy, and hotter objects radiate more energy.

Problem Example:

A planet is at distance R from the sun. The sun has temperature T_0 and radius R_{sun} . Find the temperature T of the planet.

Solution:**Calculating the Temperature of a Planet:**

Consider a planet orbiting a star (like the Sun). The planet's temperature can be determined by equating the energy it receives from the Sun to the energy it radiates.

According to Stefan-Boltzmann Law: Energy radiated $P = \sigma T^4$, where T is the planet's temperature.

Energy received from the sun:

The sun's energy distributed over a sphere hence,

$$\text{Energy received} = \frac{\sigma T_{\text{sun}}^4 \times 4\pi R_{\text{sun}}^2}{4\pi R^2}$$

R is the distance from the sun, R_{sun} is the radius of the sun, and T_{sun} is the sun's temperature.

In thermal equilibrium, heat received per unit area is equal to power radiated per unit area.

$$\sigma T^4 = \frac{\sigma T_{\text{sun}}^4 \times 4\pi R_{\text{sun}}^2}{4\pi R^2}$$

Solving for T :

$$\Rightarrow T = T_{\text{sun}} \sqrt{\frac{R_{\text{sun}}}{R}}$$

Conclusion:

- Planets closer to the Sun will be hotter.
- The temperature decreases with the square root of the distance from the Sun.

Emission Spectrum:

When atoms are heated, they emit light at specific wavelengths, creating an emission spectrum unique to each element. This spectrum serves as the element's identity.

Example of Emission Spectrum

- **Copper Sulphate:** When heated, it emits a green color.
- **Sodium Chloride (Common Salt):** Emits a red color when heated.
- **Hydrogen:** Emits three distinct lines in its spectrum, known as H-alpha, H-beta, and H-gamma, corresponding to wavelengths of 6565, 4863, and 4342 Å respectively.

- **Helium:** Emits multiple lines, differing from hydrogen due to having two protons as shown in fig. 7.

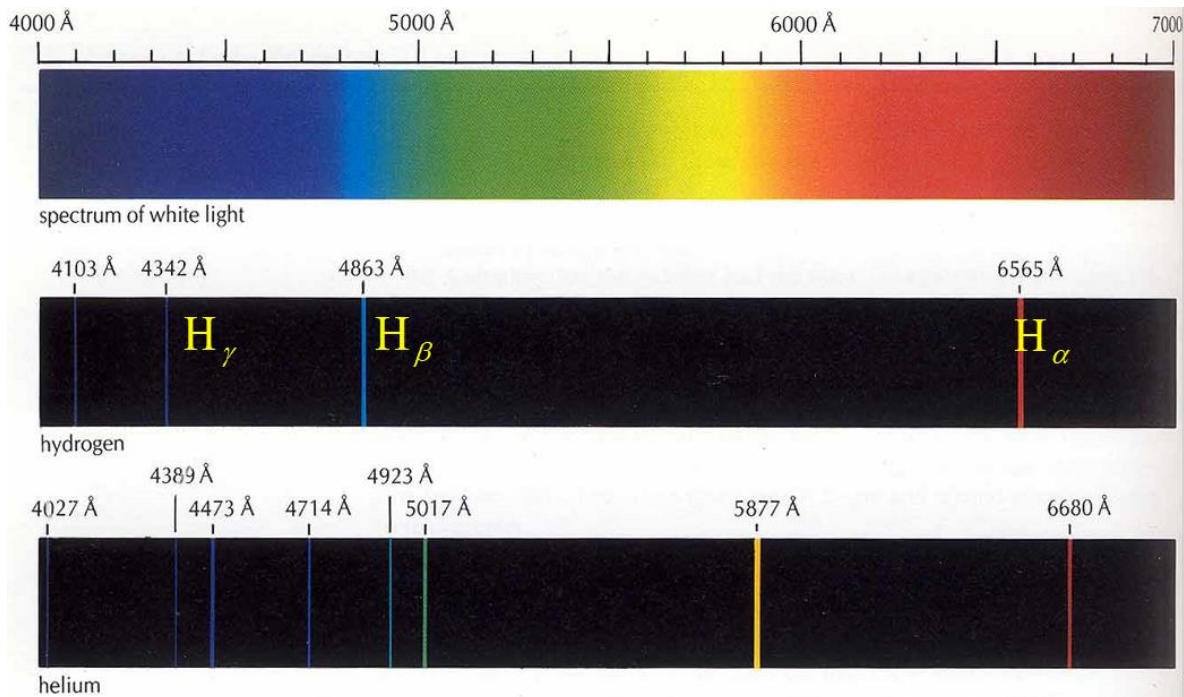


Fig. 32.7: Representation of emission spectra for various elements: Copper Sulphate emits green light, Sodium Chloride emits red, Hydrogen shows H-alpha, H-beta, and H-gamma lines at 6565 Å, 4863 Å, and 4342 Å respectively, and Helium emits multiple distinct lines.

Practical Application

Using a spectrograph, scientists can identify elements by their emission spectra, similar to identifying a person by their fingerprint.

Spectra:

- Each chemical element produces a very distinct pattern of colors called a spectrum
- Spectra that have all colors = Continuous spectra
- Spectra that have only certain frequencies = Line Spectra of which there are 2 types:
 - Absorption spectrum: Some material in front of a continuous source absorbs some energy.
 - Emission (line spectrum): Some material giving off energy.

Absorption Spectrum:

When light from a hot object, such as a star, passes through a cooler gas, specific wavelengths are absorbed, creating dark lines in the spectrum. This absorption spectrum can indicate the presence of certain elements between the observer and the light source.

Example

- **Hydrogen Cloud:** Light passing through a hydrogen cloud will show dark lines at H-alpha, H-beta, and H-gamma wavelengths, indicating the presence of hydrogen.

Emission spectrum:

The line spectrum, also known as the emission spectrum, occurs when atoms or molecules emit light at specific, discrete wavelengths. This happens when atoms are heated or otherwise excited, and they release energy in the form of light as they return to their ground state. Each element has a unique emission spectrum, which acts as a fingerprint for identifying the element.

Explanation of Line Spectrum

- **Excitation of Atoms:** When an atom absorbs energy, its electrons move to higher energy levels. This energy can come from various sources such as heat or electrical energy.
- **Emission of Light:** As the electrons return to their lower energy levels, they emit light. This light consists of specific wavelengths unique to the element, creating the line spectrum.
- **Observation of Emission Spectrum:** For example, when different salts are exposed to a flame, the resulting colors are characteristic of the elements within those salts. Copper sulfate produces a green color, while sodium chloride (common salt) produces a red color.

Spectral Lines:

- **Hydrogen Spectrum:** The emission spectrum of hydrogen is particularly well-known. When hydrogen gas is excited, it emits light at specific wavelengths, creating distinct lines in the spectrum. The three primary lines in the visible spectrum are:
 - **H-alpha:** 656.5 nm (red)
 - **H-beta:** 486.3 nm (blue-green)
 - **H-gamma:** 434.2 nm (violet)

These lines correspond to specific transitions of electrons between energy levels in hydrogen atoms.

- **Helium Spectrum:** Helium emits light at different wavelengths compared to hydrogen. The spectrum of helium shows multiple lines, reflecting the various possible electron transitions in helium atoms. This is due to helium having more electrons and more complex electronic structure as shown in fig. 32.8.

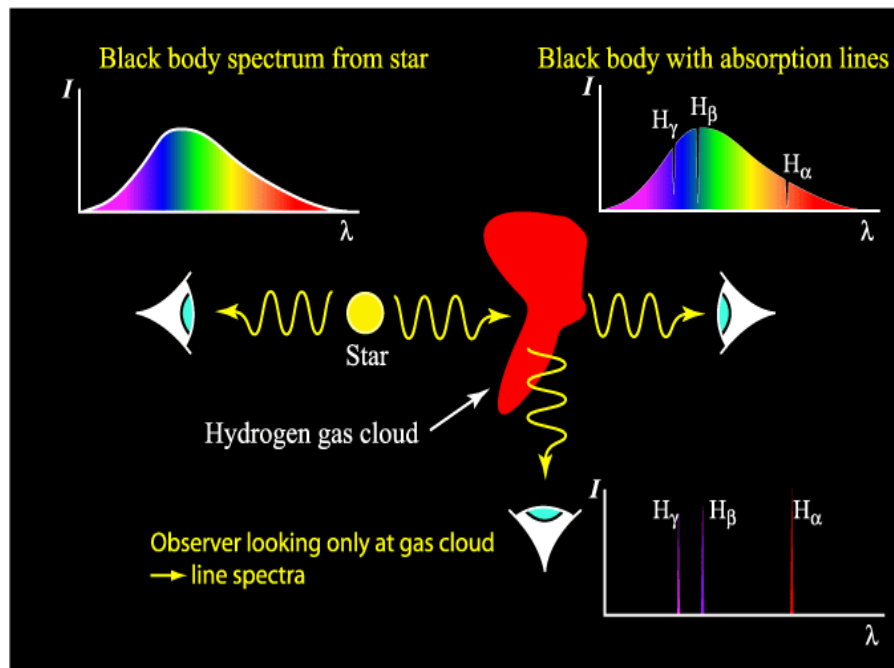


Fig. 32.8: Representation of spectral lines of Hydrogen and Helium. The Hydrogen spectrum shows distinct lines at 656.5 nm (H-alpha, red), 486.3 nm (H-beta, blue-green), and 434.2 nm (H-gamma, violet), corresponding to electron transitions. Helium's spectrum displays multiple lines due to its more complex electronic structure.

Why is the Sky Blue?

The blue color of the sky is due to the scattering of sunlight by small particles in the atmosphere. This scattering, known as **Rayleigh scattering**, is more effective at shorter wavelengths (blue light) than at longer wavelengths (red light).

Explanation

Air contains small particles that scatter blue light more effectively. The scattering of sunlight by these particles causes the sky to appear blue.

Conclusion

The emission and absorption spectra of elements, governed by their unique frequencies and wavelengths, help identify substances and understand celestial phenomena. Additionally, the scattering of light in the atmosphere explains the blue color of the sky.

Scattering:

Scattering is defined as:

“A change in the direction of motion of a particle because of a collision with another particle” as shown in fig. 32.9.

- **Atmospheric Molecules and Infrared Absorption:** Carbon dioxide (CO_2) and other atmospheric molecules like water vapor and nitrogen can absorb specific frequencies of light. When sunlight reaches Earth, it includes infrared radiation (heat). CO_2 absorbs this infrared radiation, preventing it from escaping back into space, thereby trapping heat and contributing to global warming.
- **Greenhouse Effect:** The absorption of infrared radiation by CO_2 traps heat near the Earth's surface making it harder for the heat to escape and contributing to the greenhouse effect, which leads to global warming.
- **Global Warming:** Increasing CO_2 levels from industrial emissions lead to higher global temperatures, causing polar ice to melt and other environmental changes.
- This effect is similar to **why the sky appears blue**. The sky is blue because there are fine particles in the air, such as dust. When sunlight hits these particles, the light scatters. The smaller particles scatter shorter wavelengths of light, like blue, more effectively than longer wavelengths like red. This scattering causes the sky to appear blue to us. If these small particles were removed, the sky would appear black, as there would be no scattering of light to produce the blue colour.

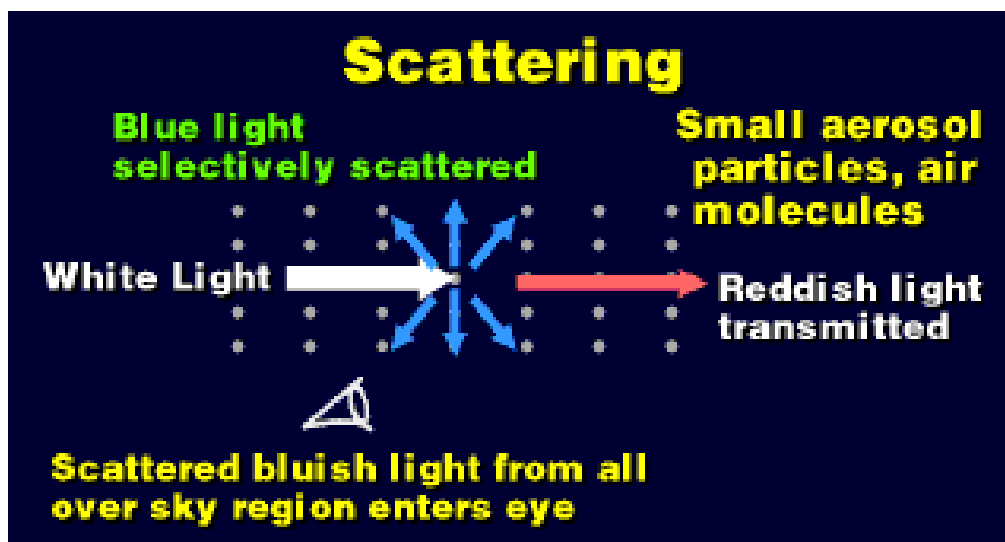


Fig. 32.9: When white light enters the atmosphere, it scatters upon colliding with air particles, with shorter blue wavelengths scattering more than longer red wavelengths, making the sky appear blue.

Light and Colour:

Objects appear coloured due to their interaction with light at specific frequencies.

- **Blue Sky:** Due to Rayleigh scattering, small particles in the atmosphere scatter blue light more than other colors.
- **Colorless Sky:** Without particles, the sky would appear black.
- **Red Rose:** Contains carotene, which absorbs all other frequencies except red.
- **Green Grass:** Contains chlorophyll, which absorbs blue and red light, reflecting green.

Resonance in Molecules:

Molecules resonate at specific frequencies, absorbing and emitting particular wavelengths of light.

- **Carbon Dioxide (CO₂):** Resonates in the infrared region, absorbing heat.
- **Oxygen Molecule (O₂):** Acts as a simple harmonic oscillator with a specific resonance frequency.

Transparency and Opacity:

The transparency or opacity of an object depends on how its molecules interact with light.

- **Transparent Materials:** Molecules are not significantly excited by light, allowing it to pass through.
- **Opaque Materials:** Molecules are excited and absorb light, preventing it from passing through.
- Metals reflect light due to free electrons that can move easily and re-emit the light.
- **Reflection:** Electrons in metals move in response to light, causing it to reflect.

Polarization:

- Polarization occurs when the electric field vector of a light wave oscillates in a particular direction. Reflected light is polarized, with applications in reducing glare (e.g., polarized sunglasses).
- **Unpolarized Light:** Contains electric field vectors in all directions.
- **Polarized Light:** Contains electric field vectors in a single direction as shown in fig. 32.10.

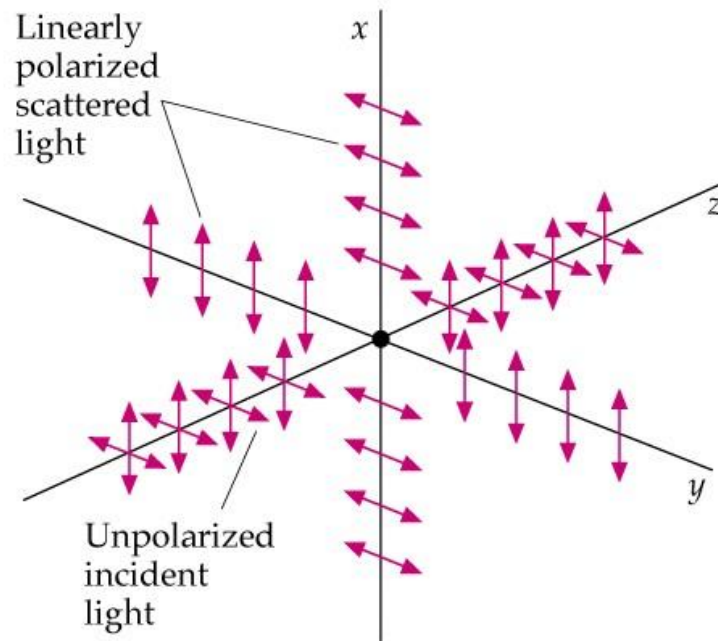


Fig. 32.10: Representation of Polarized and Unpolarized light. Polarized light has electric field vectors aligned in a single direction, while unpolarized light has electric field vectors in all directions.

- **Aligned Polarizers:** When two polarizers are aligned, they allow maximum light intensity to pass through as shown in fig. 32.11.
- **Perpendicular Polarizers:** When two polarizers are at a 90-degree angle to each other, they block most of the light, resulting in minimum intensity.

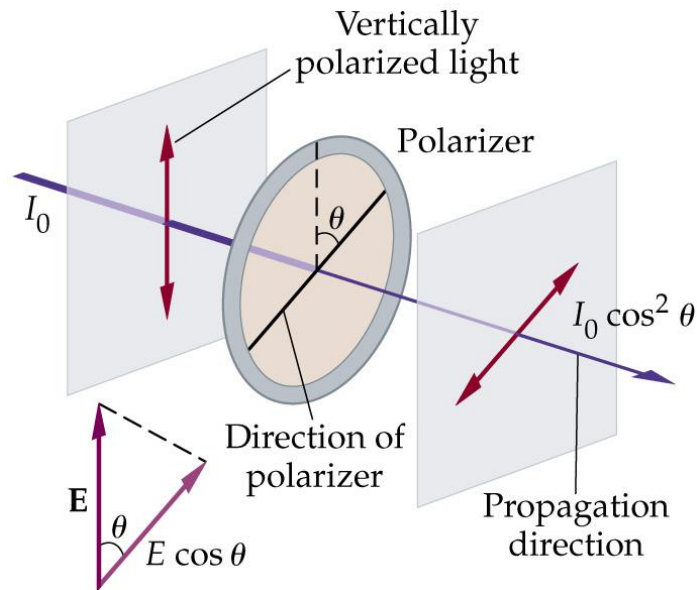


Fig. 32.11: Effect of polarizer alignment on light intensity. Aligned polarizers allow maximum light to pass, while perpendicular polarizers block most light, demonstrating the principle of attenuation for unpolarized light, with an average attenuation factor of $\langle \cos^2 \theta \rangle = 1/2$.

- Unpolarized light is the superposition of many waves, each with random polarization direction q , relative to fixed axis of polarizer.
- Each wave is attenuated (loss of energy) by factor $\cos^2 q$.
- Average attenuation is $\langle \cos^2 q \rangle = 1/2$
- **Applications:** Polarizing filters and sunglasses reduce glare by allowing only light polarized in a certain direction to pass through.
- **Example of Polarization:** A polarizer can be made using a metal plate with a slit. Light passing through the slit is polarized along the slit. If a second polarizer is placed in front of the first, rotating it will decrease the light intensity passing through.

Conclusion

Understanding the emission, absorption, and interaction of light with various substances provides insight into phenomena like the greenhouse effect, the colors of objects, and the polarization of light. These principles have practical applications in environmental science, optics, and everyday technologies like sunglasses.

Birefringence:

- Birefringent materials (crystals or stressed plastics) have the property that the speed of light is different in two transverse dimensions.
- Since the frequency of the wave is constant as the wave passes through the material, the wavelength is different in the two dimensions, producing a phase change.
- Birefringence can occur in any material that possesses some asymmetry in its structure, so that the material is springier in one direction than another.

Liquid Crystal Displays (LCDs):

Liquid crystal displays utilize the polarization of light and the properties of liquid crystals to create visual displays.

- **Liquid Crystals:** Contain long molecules whose alignment can be controlled by an electric field.
- **Polarization in LCDs:** The light passing through the liquid crystal is polarized along the direction of the molecules. By applying an electric field, we can change the orientation of the molecules, thus controlling the light passing through and creating images or numbers.
- **Refractive Index:** Materials can have different refractive indices based on the polarization of light.
- **Different Refractive Indices:** If light is polarized in one direction, it might have a different refractive index compared to light polarized perpendicularly. This property can be used in various optical devices to manipulate light.

Lecture#33

Interference and Diffraction of Light

Introduction to Light Interference:

It is often said that "iron cuts iron," but you have likely never heard that "light cuts light" or that light reduces the intensity of light. When you enter a room and switch on a bulb, the light from the first bulb combines with the second, resulting in a brighter room. You have never seen darkness created by adding more light sources. It seems contrary to our everyday experiences that light can cancel out light. However, this phenomenon does occur, and it is a fundamental aspect of the behaviour of waves, including light waves. In this lecture, we will explore the fascinating concept of light interference.

Interference of Light:

Interference occurs when two light waves overlap, resulting in an increase (constructive interference) or decrease (destructive interference) in intensity.

The natural consequence of light's wave nature reveals that, just like sound waves, light waves can interact in ways that either amplify or diminish their overall intensity. As we have learned in the study of sound, two sound waves can either increase each other's intensity or reduce it, depending on their phase relationship. Similarly, light waves can interfere constructively or destructively.

Constructive interference occurs when two light waves start simultaneously and their amplitudes combine, effectively doubling the wave's intensity. In contrast, **destructive interference** happens when two light waves are 180 degrees out of phase, meaning one wave is at its peak while the other is at its trough. When these waves intersect, they can cancel each other out, resulting in reduced or nullified intensity as shown in fig.1.

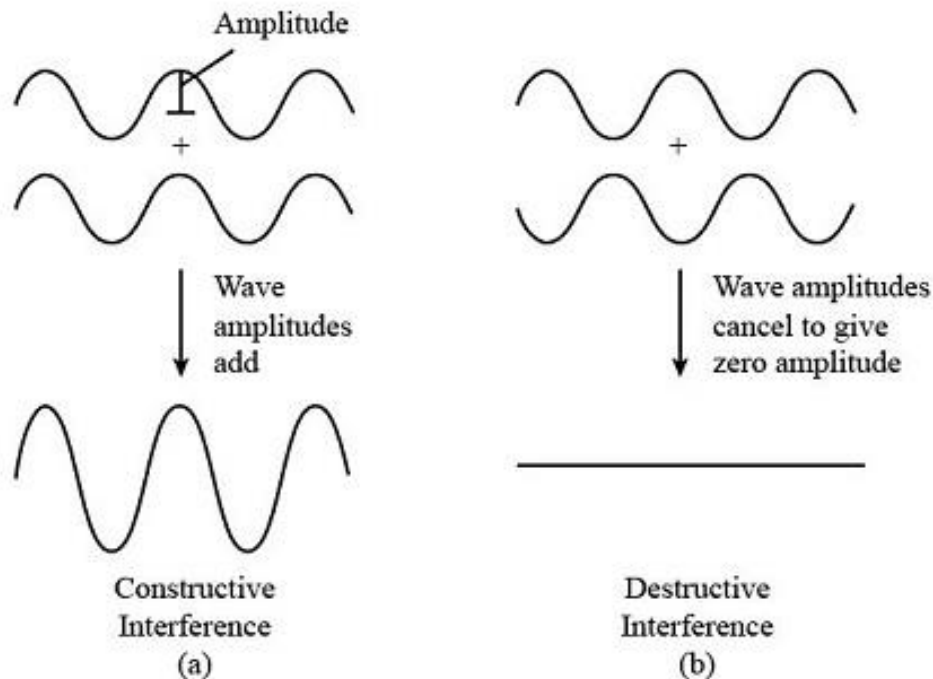


Fig. 33.1: Representation of the constructive and destructive interference of light waves. (a) Constructive interference results in increased intensity, while (b) destructive interference leads to reduced or nullified intensity.

Conditions for Interference:

- The two light waves must have the same frequency, If the frequencies differ, the waves will sometimes amplify and sometimes diminish each other, but not in a consistent manner.
- The phase difference between the waves must be constant, a condition known as coherence.
- It can't produce coherent light from separate sources ($f \approx 10^{14}$ Hz).
- Need two waves from single source taking two different paths having;
 - ❖ Two slits
 - ❖ Thin films for reflection as shown in fig.33.2.

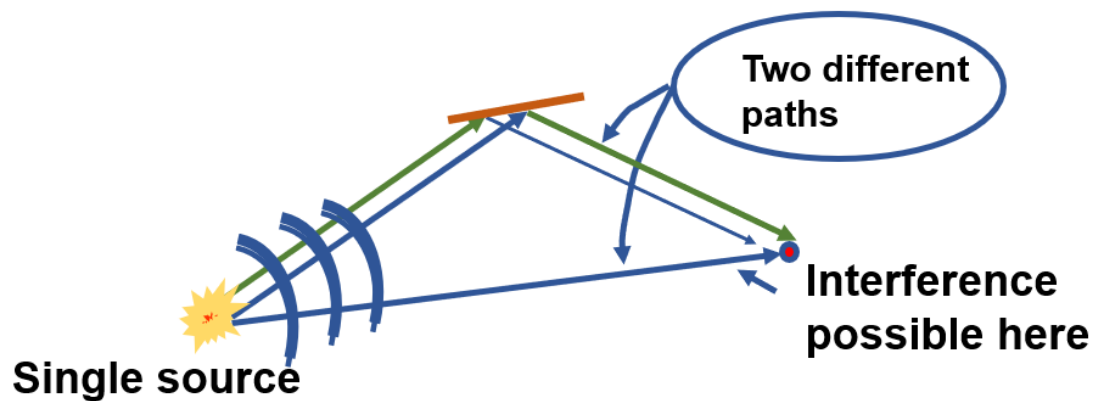


Fig. 33.2: Representation of the different paths to produce interference from a single source, including the use of two slits, and a thin film.

Coherent and Incoherent Light:

- Natural light sources like bulbs and the sun emit incoherent light, meaning the frequency and phase are not consistent.
- To observe interference, a single light source can be split into two coherent parts using slits or mirrors.

Double-Slit Experiment:

- Light passing through two slits produces an interference pattern of alternating bright and dark bands on a screen.
- The path difference between light from the two slits determines the interference pattern.
- An incoherent light source illuminates the first slit as shown in fig.33.3.
- This creates a uniform but coherent illumination of the second screen
- (from side-to-side on the screen, the light wave has the same oscillating phase).
- The two waves from the two slits S_1 and S_2 create a pattern of alternating light and dark fringes on the third screen.

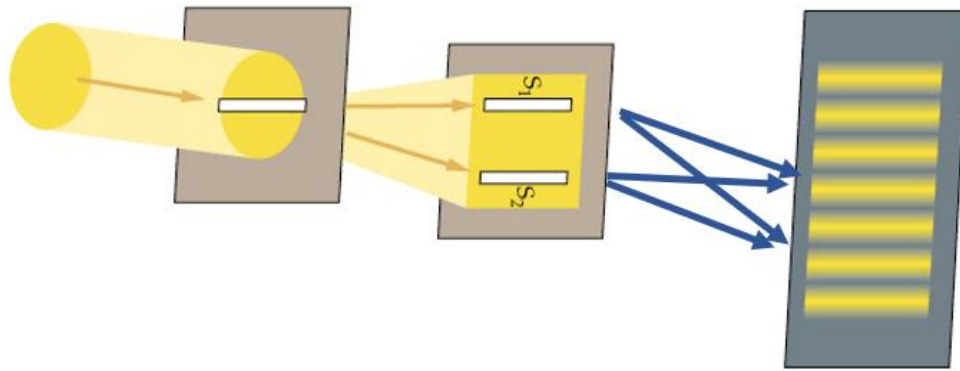


Fig. 33.3: Representation of the Interference pattern created by light passing through two slits. An incoherent light source illuminates the first slit, creating a uniform, coherent illumination on the second screen. The waves from slits S_1 and S_2 form alternating light and dark fringes on the third screen.

Path Difference and Phase Difference:

- The path difference ($d\sin\theta$) between the two slits affects whether constructive or destructive interference occurs.
- For constructive interference, the path difference must be an integer multiple of the wavelength.
- For destructive interference, the path difference must be an odd multiple of half the wavelength.

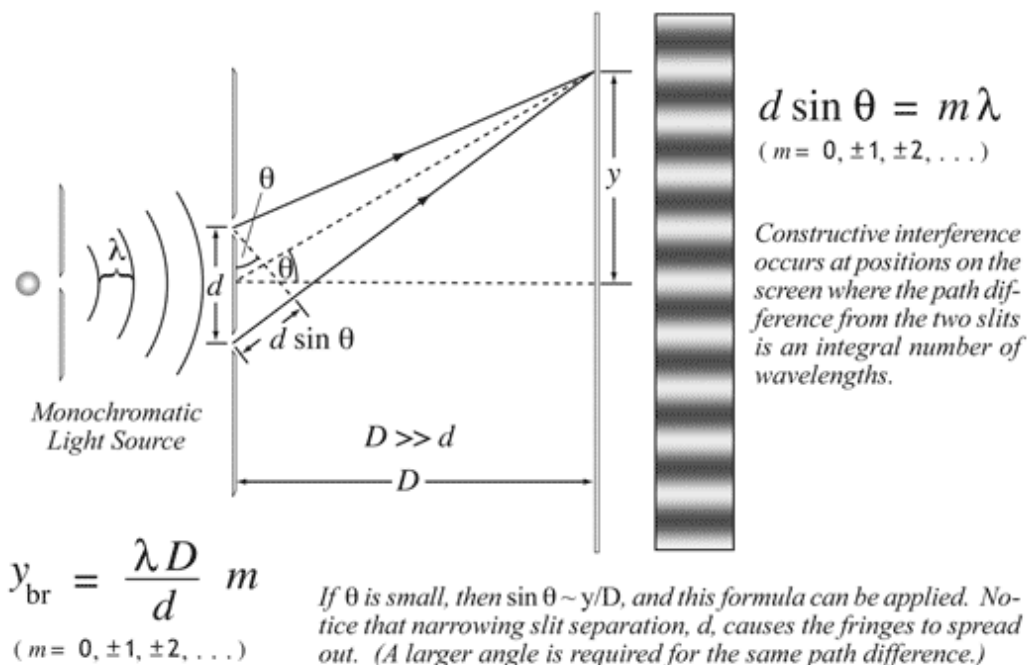


Fig. 33.4: Representation of the Path difference ($d\sin\theta$) determines interference: constructive interference occurs with integer multiples of the wavelength, and destructive interference occurs with odd multiples of half the wavelength.

Conditions for bright fringes (constructive interference):

$$d \sin \theta = m\lambda \quad (m = 0, \pm 1, \pm 2...)$$

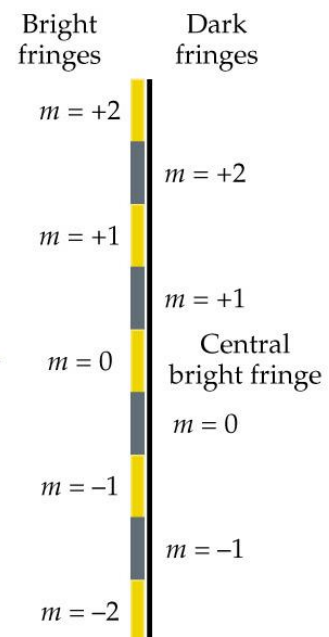
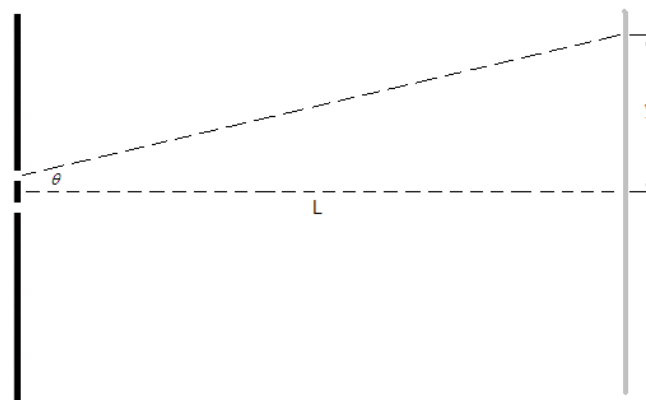
Conditions for dark fringes (destructive interference):

$$d \sin \theta = (m - \frac{1}{2})\lambda \quad (m = 0, \pm 1, \pm 2...)$$

m is called the order number.

Problem:

Two slits with a separation of $8.5 \times 10^{-5} \text{ m}$ create an interference pattern on a screen 2.3 m away. If the tenth bright fringe above the central is a linear distance of 12 cm from it, what is the wavelength of light used in the experiment?



Solution:

1. Calculate the angle to the tenth bright fringe:

$$y = L \tan \theta$$

$$\theta = \tan^{-1} \left(\frac{y}{L} \right) = \tan^{-1} \left(\frac{0.12 \text{ m}}{2.3 \text{ m}} \right) = 3.0^\circ$$

2. Use $\sin \theta = m\lambda/d$ to find the wavelength:

$$\lambda = \frac{d}{m} \sin \theta = \left(\frac{8.5 \times 10^{-5} \text{ m}}{10} \right) \sin(3.0^\circ)$$

$$\lambda = 4.4 \times 10^{-7} \text{ m} = 440 \text{ nm}$$

Phase of wave reflected by interface between two media:

- When light encounters a boundary between two media, it can be reflected or refracted.
- The phase change upon reflection depends on whether the light moves from a medium with a higher to a lower refractive index or vice versa.

Conditions for Phase Change

- **Reflection from a Denser Medium:**
 - When light reflects off a boundary with a medium that has a higher refractive index (denser medium), the reflected wave undergoes a phase change of π radians (180 degrees) as shown in fig.33.5.
 - Example: Light reflecting off a glass surface from air.
 - This is due to the wave experiencing a "hard" boundary, causing an inversion of the wavefront.
- **Reflection from a Less Dense Medium:**
 - When light reflects off a boundary with a medium that has a lower refractive index (less dense medium), there is no phase change as shown in fig.5.
 - Example: Light reflecting off an air surface from water or glass.
 - This is because the wave encounters a "soft" boundary, and the wavefront is not inverted.

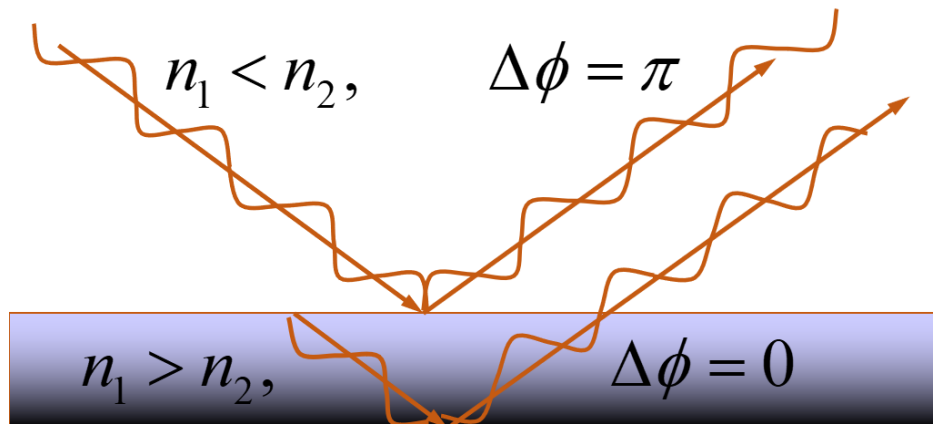
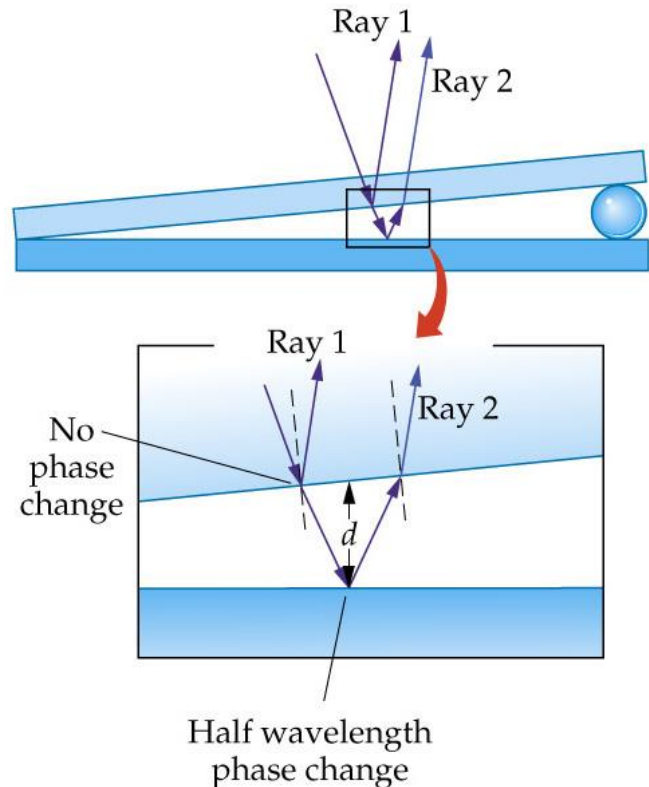


Fig. 33.5: Representation of the Phase change conditions upon reflection: a π radians (180 degrees) phase change occurs when light reflects from a denser medium, whereas no phase change occurs when reflecting from a less dense medium.

Interference in an Air Wedge:

Interference in an air wedge is a phenomenon observed when light waves reflect off the two surfaces of a thin film of air, often formed between two glass plates that are slightly angled to each other as shown in fig.33.6. This creates a wedge-shaped air film, leading to an interference pattern of bright and dark fringes.



Constructive interference:

$$\frac{1}{2} + \frac{2d}{\lambda} = m \quad (m = 1, 2, 3, \dots)$$

Destructive interference:

$$\frac{1}{2} + \frac{2d}{\lambda} = m + \frac{1}{2} \quad (m = 0, 1, 2, \dots)$$

Thin Film Interference:

- Thin films (like oil on water) create colourful interference patterns due to varying thicknesses.
- Light reflecting from different layers of the film interferes, resulting in constructive or destructive interference.

Fig. 33.6: Representation of the interference in an air wedge formed between two angled glass plates, producing interference fringes due to reflection from the upper and lower surfaces of the air film.

Phase Change upon Reflection:

- When light reflects off a surface, it can undergo a phase change depending on the refractive indices of the media involved.
- If light reflects from a medium with a higher refractive index to a medium with a lower refractive index, there is no phase change.
- If light reflects from a medium with a lower refractive index to a medium with a higher refractive index, there is a phase change of π (180 degrees) as shown in fig.33.7.

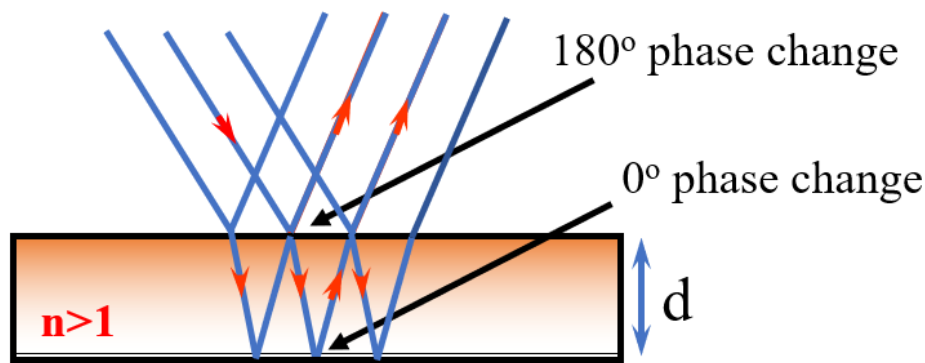


Fig. 33.7: Representation of the Reflection phase change: Light undergoes a π radians (180 degrees) phase change when reflecting from lower to higher refractive index media, and no change when reflecting from higher to lower index media.

Constructive interference:

$$2nd = m\lambda \quad (m = 0, 1, 2, \dots)$$

Destructive interference:

$$2nd = (m + \frac{1}{2})\lambda \quad (m = 0, 1, 2, \dots)$$

Where, n is the index of refraction for the medium.

Thickness of film selects wavelength:

In the study of light interference, one important aspect is how the thickness of a film can select specific wavelengths of light. When light waves encounter a thin film, such as a soap bubble or an oil slick on water, the interaction between the light waves reflected from the top and bottom surfaces of the film can lead to interference patterns. The thickness of the film plays a crucial role in determining which wavelengths of light will constructively or destructively interfere, thereby affecting the colors we observe.

When a light wave hits a thin film, part of the wave is reflected off the top surface, while the rest enters the film and is reflected off the bottom surface. The two reflected waves can then interfere with each other. For constructive interference (which results in a bright fringe), the path difference between the two waves must be an integer multiple of the wavelength. For

destructive interference (which results in a dark fringe), the path difference must be a half-integer multiple of the wavelength.

If the film thickness changes, the specific wavelengths also change, leading to different colors being observed. This is why thin films (soap bubble) often display a spectrum of colors, as different thicknesses within the film selectively amplify or diminish different wavelengths of light.

For Example:

- Thick portions of thin film appear blue, since the long-wavelength red light experiences destructive interference as shown in fig.8.
- Thinner portions appear red because the short-wavelength blue light interferes destructively.

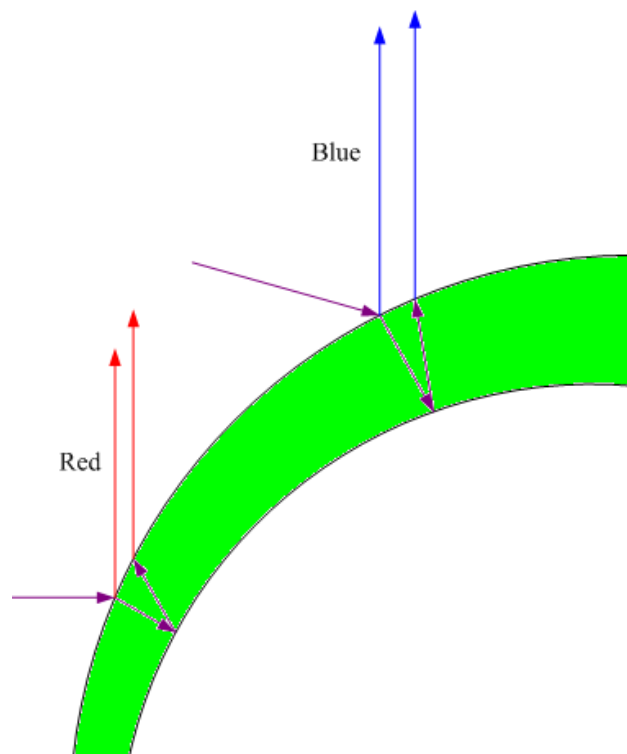


Fig. 33.8: Representation of the thin film interference causes thick portions to appear blue due to destructive interference of long-wavelength red light, while thinner portions appear red due to destructive interference of short-wavelength blue light.

Newton's Rings:

Newton's rings are a classic demonstration of the wave nature of light, particularly the phenomenon of interference. They are named after Sir Isaac Newton, who first studied and explained this interference pattern. The rings are formed by the interference of light waves reflected from two surfaces—a spherical surface and an adjacent flat surface.

Formation of Newton's Rings

Newton's rings are typically observed when a plano-convex lens is placed on top of a flat glass surface as shown in fig.33.9. The air film between the curved surface of the lens and the flat glass plate varies in thickness from zero at the point of contact to some maximum value at the edge of the lens. When monochromatic light (light of a single wavelength) is shone onto this arrangement, the light waves reflected from the top and bottom surfaces of the air film interfere with each other.

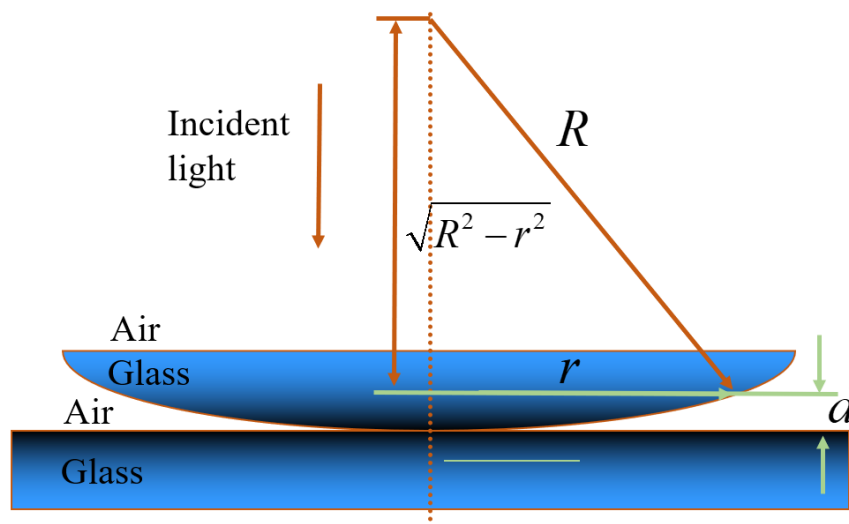


Fig. 33.9: Representation of the Newton's rings phenomenon observed with a plano-convex lens placed on a flat glass surface. Interference occurs between light waves reflected from the air film's top and bottom surfaces, producing concentric bright and dark rings due to varying air film thickness.

Constructive and Destructive Interference

The interference of the reflected light waves produces a pattern of concentric bright and dark rings as shown in fig.33.10. These rings are formed due to the constructive and destructive interference of the light waves.

- An example of interference patterns formed by placing a convex lens on a flat glass surface and observing concentric rings.
- The radius of the rings depends on the wavelength of light and the curvature of the lens.

Condition for constructive interference,

$$2d = \left(m + \frac{1}{2}\right)\lambda, \quad m = 0, 1, 2, \dots \text{(maxima)}$$

$$d = R - \sqrt{R^2 - r^2}$$

$$d = R - R \left[1 - \left(\frac{r}{R}\right)^2 \right]^{1/2}$$

Using Binomial theorem,

$$\therefore (1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots$$

$$d = R - R \left[1 - \frac{1}{2} \left(\frac{r}{R}\right)^2 + \dots \right]$$

$\therefore \frac{r}{R} \approx$ small quantity, hence ignoring higher order terms,

$$d = R - R + R \cdot \frac{r^2}{2R^2}$$

$$d \approx \frac{r^2}{2R}$$

As,

$$2d = \left(m + \frac{1}{2}\right)\lambda$$

Putting the value of 'd', we get

$$2 \cdot \frac{r^2}{2R} = \left(m + \frac{1}{2}\right)\lambda$$

$$r = \sqrt{\left(m + \frac{1}{2}\right)\lambda R}, \quad m = 0, 1, 2, \dots \text{(maxima)}$$

where:

- n is the order of the ring (1, 2, 3, ...).
- λ is the wavelength of the light used.
- R is the radius of curvature of the lens.



Newton's Rings due to
Reflected Light



Newton's Rings due to
Transmitted Light

Fig. 33.10: Representation of the Newton's rings due to the reflected light and transmitted light. The central ring is dark, illustrating the interference pattern where light waves reflected from the air film between a plano-convex lens and a flat glass surface are out of phase. The central ring is bright, showing an interference pattern opposite to that of reflected light. If there were no phase change, the rays reflected from two surfaces in contact would be in the same phase at the center, resulting in a bright central spot.

Diffraction:

- Diffraction is a type of interference occurring when light bends around obstacles or passes through small apertures as shown in fig.33.11.
- Single-slit diffraction patterns consist of central bright bands flanked by alternating dark and bright bands.
- Diffraction occurs when light passes through very small apertures or near sharp edges.
- We can get an interference pattern with a single slit provided its size is approximately λ (neither too small nor too large).
- **For example**, there is a knife which has a very sharp edge. If you look from above it, then the light gets diffracted and appears in your eyes. It is not always easy to see, but if you take two knives and place them side by side, then you will see diffraction from their edges.

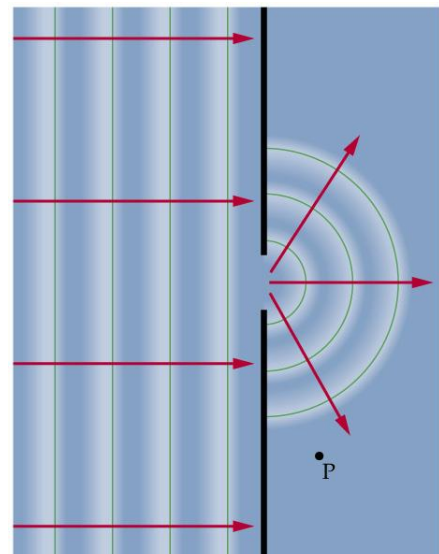


Fig. 33.11: Representation of the diffraction, when light bends around obstacles, producing single-slit patterns with central bright bands and alternating dark and bright bands.

Single Slit Diffraction:

A thin strip or slit as shown in fig.33.12, which has been cut in a metal plate or in paper. Light comes from above it and as a result, light comes out from the edges of the slit, waves emerge and they interfere with each other, as a result of which an interference pattern is formed. This pattern is visible to us near a sharp edge. There also we get dark and bright bands at other places. The point is that in diffraction we need only one source, but different parts of that source can interfere with each other.

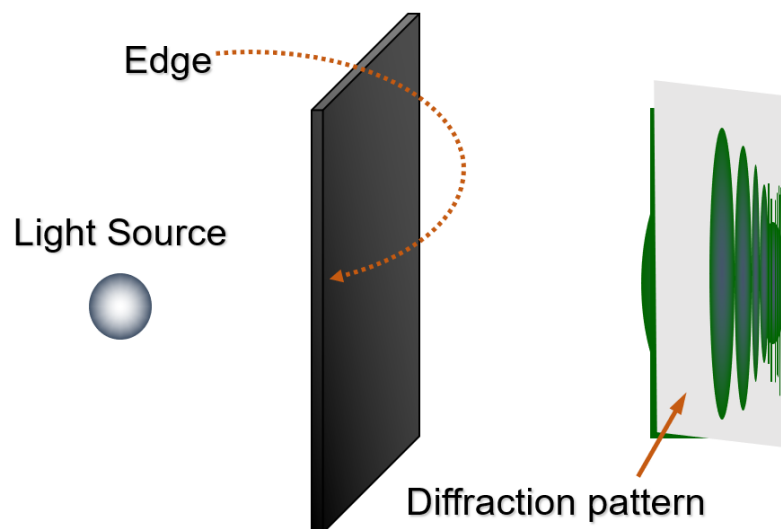


Fig. 33.12: Representation of the diffraction pattern from a thin slit where light waves emerging from the edges interfere, producing visible dark and bright bands near the sharp edge, showcasing interference from different parts of a single source.

Calculating Path Difference

In a single slit diffraction, a slit having width d . The phase difference between these is $w \sin \theta$, if this distance is equal to some integer wavelength, then interference will occur, otherwise not. In such a case dark fringe will be generated and then we can determine the interference pattern from this.

Conditions for dark fringes in single-slit diffraction,

$$W \sin \theta = m\lambda$$

$$m = \pm 1, \pm 2, \pm 3 \dots$$

Problem:

Light with wavelength of 511 nm forms a diffraction pattern after passing through a single slit of width $2.2 \times 10^{-6} \text{ m}$. Find the angle associated with (a) the first and (b) the second dark fringe above the central bright fringe.

Solution:

1. Solving for using $m = 1$:

$$W \sin \theta = m\lambda$$

$$\theta = \sin^{-1} \left(\frac{m\lambda}{W} \right) = \sin^{-1} \left(\frac{(1)(511 \times 10^{-9} \text{ m})}{2.20 \times 10^{-6} \text{ m}} \right) = 13.4^\circ$$

2. Solving for using $m = 2$:

$$\theta = \sin^{-1} \left(\frac{m\lambda}{W} \right) = \sin^{-1} \left(\frac{(2)(511 \times 10^{-9} \text{ m})}{2.20 \times 10^{-6} \text{ m}} \right) = 27.7^\circ$$

Resolution:

Resolution refers to the ability of an optical instrument, such as a telescope or microscope, to distinguish between closely spaced objects. The phenomenon of diffraction significantly impacts resolution.

When light passes through an aperture or around the edges of an object, it bends and spreads out, causing interference patterns. The central bright spot and surrounding rings of these patterns can overlap, blurring the details of the objects being observed.

The formula for the angle θ at which the first dark fringe occurs in a diffraction pattern is given by:

$$\sin \theta = 1.22 \left(\frac{\lambda}{D} \right)$$

Rayleigh's criterion:

$$\theta_{\min} = 1.22 \frac{\lambda}{D}$$

where, λ is the wavelength of light, D is the diameter of the aperture.

- From this equation, it is evident that increasing the diameter of the aperture D results in a smaller angle θ , leading to less diffraction and better resolution.
- Conversely, using light with a shorter wavelength also enhances resolution, as shorter wavelengths diffract less.

- This also proves that if λ is less which means that if the frequency is higher then the resolution will be better accordingly, because a good telescope has a larger diameter and with this, we can see distant things and we can create difference in the close things.

Diffraction Gratings:

A diffraction grating is an optical component with a pattern of regular, closely spaced lines or slits that diffract light into several beams. These beams interfere with each other, producing a pattern of light and dark bands or lines, known as an interference pattern, as shown in fig.13.

- A diffraction grating consists of multiple slits that disperse light into its component wavelengths.
- The angle of diffraction depends on the wavelength and the spacing of the slits.

The principle behind diffraction gratings relies on the interference of light waves. When light encounters the grating, it is split into multiple paths. The condition for constructive interference, where the light waves reinforce each other to create bright bands, is given by:

$$d \sin \theta = m \lambda$$

where, d is the distance between adjacent slits (grating spacing), θ is the angle of diffraction, m is the order of the maximum (an integer) and λ is the wavelength of the light.

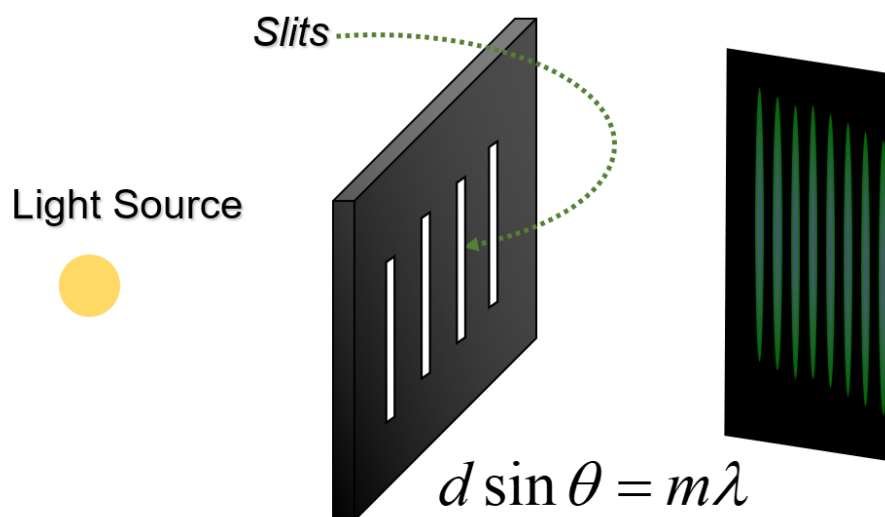


Fig. 33.13: Representation of the diffraction grating disperses light into beams that interfere, forming an interference pattern of light and dark bands on the screen.

Diffraction gratings can be used for various purposes, such as:

- **Analyzing Light Spectra:** By separating light into its component wavelengths, diffraction gratings are essential tools in spectrometry for analysing the spectral composition of light from various sources.
- **Optical Instruments:** They are utilized in devices like monochromators and spectrographs to isolate specific wavelengths of light.
- **Compact Discs and DVDs:** The fine grooves on CDs and DVDs act as diffraction gratings, reflecting different colors of light and creating the characteristic rainbow pattern seen when light reflects off their surfaces.

CDs and DVDs: Practical Application of Diffraction Gratings

- CDs (Compact Discs) and DVDs (Digital Versatile Discs) are practical examples of diffraction gratings in everyday use. The surface of these discs contains tiny pits and lands arranged in a spiral track as shown in fig.14.
- When light, particularly from a laser, shines on the disc, it gets diffracted by these microscopic features. The distance between the grooves on a CD or DVD is on the order of hundreds of nanometers, close to the wavelength of visible light. As a result, the discs create interference patterns that can be read by a laser in a CD or DVD player. This diffraction and interference mechanism is what allows the digital information encoded on the disc to be read and interpreted by the device.
- A laser puts information on the CD, which is then read by a CD reader or drive inside the computer. The CD has a layer of metal over the plastic with varying thickness. A laser beam hits it, bounces back, and creates interference. Where $d \sin \theta = n\lambda$, interference is created, forming a pattern of zeros and ones. This digital information is then read by the laser.
- The interference conditions, given by $d \sin \theta = m\lambda$, explain the varying colors observed when viewing a CD or DVD under white light. Different wavelengths (colors) of light meet the interference condition at different angles, producing a spectrum of colors. On one side, you can see a bright part where information has been recorded. When you shake it, you see colors changing due to light diffraction. If you look at it under a microscope, you will see rings where a laser has put information, with a distance of about five or 6000 angstroms between them.
- As the angle of a CD or DVD changes, the conditions of interference change, resulting in different colors. The condition of interference is $d \sin \theta = n\lambda$, so the value of θ will be

different for each λ . Changing the angle results in different constructive interference patterns.

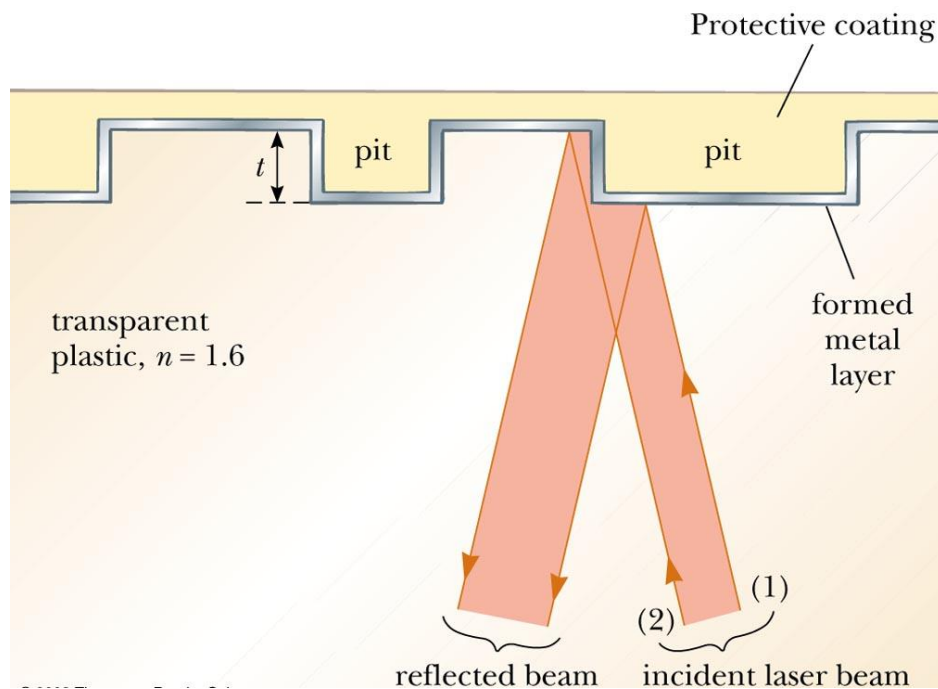


Fig. 33.14: Representation of the CDs and DVDs use microscopic pits and lands on their surfaces to diffract light, creating interference patterns read by lasers to encode and decode digital information.

X-ray Diffraction:

X-ray diffraction (XRD) is a powerful technique used to study the structure of crystalline materials. It involves directing X-rays at a sample and analysing the patterns produced by the diffracted rays. This technique allows scientists to determine the arrangement of atoms within a crystal and gain insights into the material's properties.

1. Principle of X-ray Diffraction

The fundamental principle of X-ray diffraction is based on the interference of X-rays scattered by the atoms in a crystal lattice. The condition for constructive interference, which results in the diffraction peaks, is given by Bragg's law:

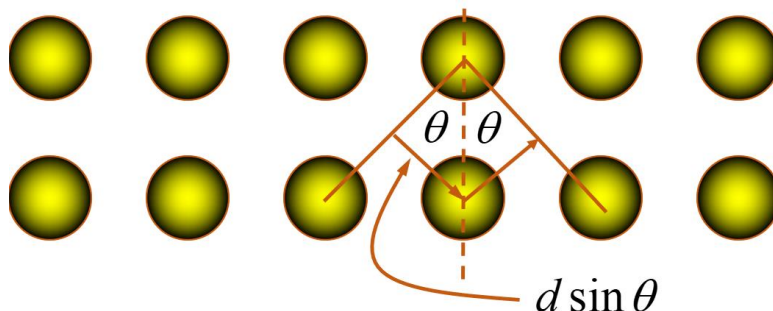
$$2d\sin(\theta) = m\lambda$$

where:

- d is the distance between crystal planes (interplanar spacing) as shown in fig.15.
- θ is the angle of incidence.
- m is an integer (the order of the maximum).

- λ is the wavelength of the X-rays.

When X-rays strike a crystal, they are scattered by the atoms in the crystal lattice. The scattered waves interfere with each other, and constructive interference occurs when the path difference between the waves is an integer multiple of the wavelength. This results in a diffraction pattern that can be analysed to determine the crystal structure.



$$2d \sin \theta = m\lambda$$

Fig. 33.15: Representation of the X-ray diffraction relies on Bragg's law, $2d\sin(\theta) = m\lambda$, where d is the interplanar spacing in a crystal lattice. X-rays scattered by atoms in the lattice interfere constructively at specific angles, producing diffraction peaks that reveal the crystal's structure.

2. X-ray Crystallography

Using X-ray crystallography, we learn the properties of different salts like sodium chloride and copper sulphate. This method reveals the arrangement of atoms. However, to look inside atoms, we need Gamma rays with higher energy and smaller wavelengths.

While X-ray diffraction is useful for studying the structure of crystals and molecules, gamma rays, which have even shorter wavelengths than X-rays, can be used to explore even smaller structures, such as atomic nuclei.

To see the nucleus, we use Gamma rays, as the nucleus size is approximately 10 femtometres. Gamma rays with appropriate wavelength help us see inside the nucleus.

Conclusion

Whatever you want to see, the light used depends on the size of that thing. For smaller objects, we need smaller wavelengths and higher frequencies. This principle is essential for understanding atomic and subatomic structures. The lecture concludes with an emphasis on the importance of diffraction and interference in various fields.

Lecture #34

THE PARTICLE NATURE OF LIGHT

A century ago, physics faced a profound question regarding light: whether light behaved as a wave or as particles, serving as a source of energy. Around 350 years ago, Newton's study of light suggested it comprised particles, based on observations of sunlight's colors. This view gained widespread acceptance due to Newton's theory. However, subsequent experiments, notably the double-slit experiment (fig 34.1), challenged this view.

Here, light passing through two narrow slits onto a screen, created an interference pattern, indicating wave-like behavior. This phenomenon arises due to variations in the time taken for light to reach the screen through different slits, resulting in wave interference and modulation of amplitude. This finding posed a significant challenge to the particle theory of light.

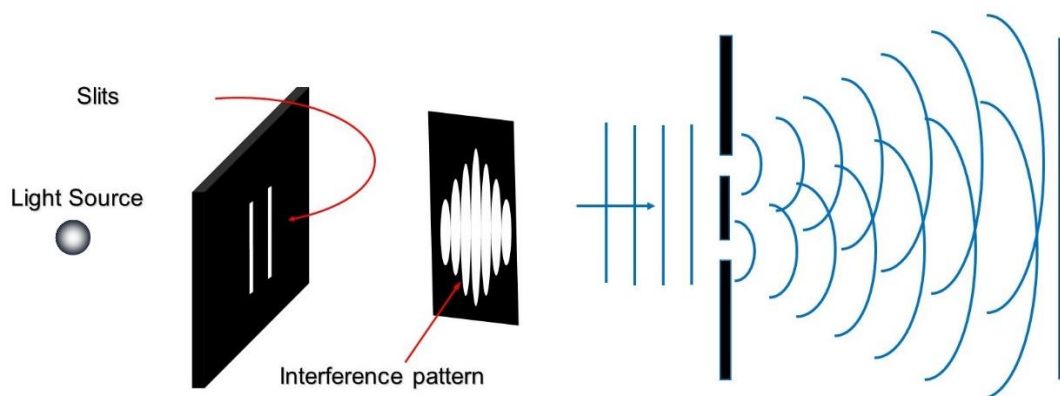


Figure 34. 1. The double-slit experiment showing light's wave-like behavior through interference patterns.

In the 19th century, Michael Faraday and James Clark Maxwell confirmed that light behaves as electromagnetic waves, as discussed in previous lectures. They demonstrated that changes in electric fields produce magnetic fields, and vice versa, forming electromagnetic waves.

Heinrich Hertz, a German scientist, further validated this theory in 1880 through experiments with transmitters and receivers, confirming the existence of these waves. Electromagnetic waves feature perpendicular electric and magnetic fields, with the electric field along the x-axis, the magnetic field along the y-axis, and the wave propagation along the z-axis. As the electric field increases, the magnetic field decreases, allowing these fields to propagate through space.

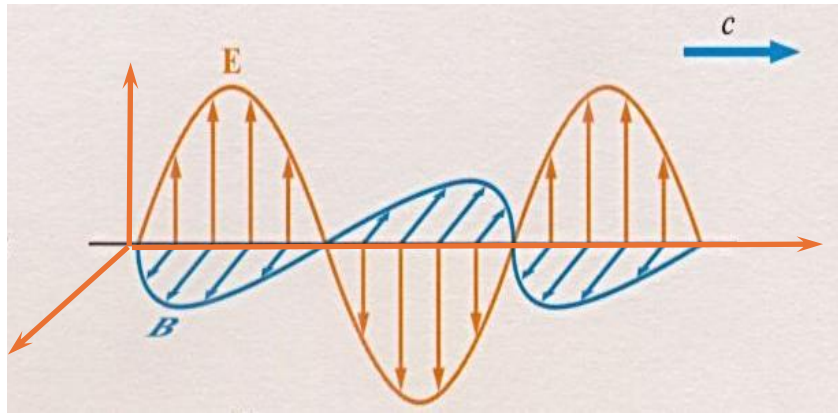


Figure 34. 2. Electromagnetic waves featuring perpendicular electric and magnetic fields.

Energy of Electromagnetic Wave

Electromagnetic fields carry energy, which can be calculated based on the product of the electric and magnetic fields as $E \times B$. The power in an electromagnetic wave is proportional to the square of the maximum intensity of either the electric field (E_0) or the magnetic field (B_0). Increasing the wave's amplitude, represented by E_0 or B_0 , results in a squared increase in energy.

$$\text{Average power per unit area} = \frac{E_{\max} B_{\max}}{2\mu_0} = \frac{E_{\max}^2}{2\mu_0 c}$$

$$\therefore \frac{E}{B} = c$$

$$\text{Average power per unit area} = \frac{c B_{\max}^2}{2\mu_0}$$

Additionally, electromagnetic waves possess momentum, despite us typically only sensing their warmth when exposed to sunlight. The linear momentum (p) is given by:

$$p = \frac{2U}{c}$$

Measurement of momentum

To detect momentum, a highly sensitive tool is required. In this setup depicted in fig 34.3, a mirror and a black disc are positioned such that when light strikes the mirror, it reflects, causing the apparatus to rotate. The rotation angle increases with higher light intensity, reflecting the energy density and momentum.

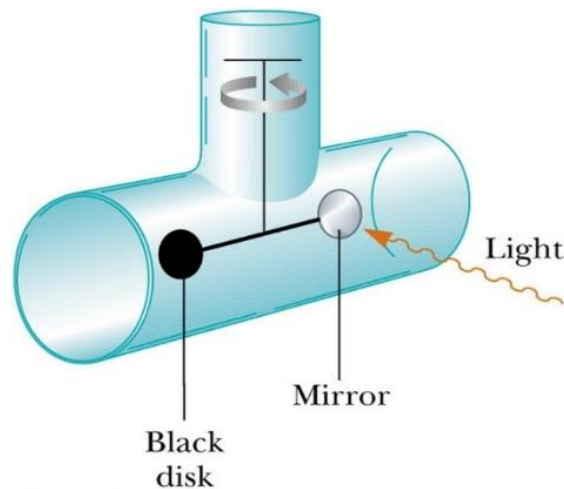


Figure 34. 3. Rotating mirror and black disc setup for momentum measurement.

Photoelectric Effect

In 1887, Heinrich Hertz conducted a groundbreaking experiment. Light illuminated a metal surface, causing electrons to be emitted and collected elsewhere, known as the photoelectric effect and the electrons that come out of the light are called photoelectrons. This phenomenon revolutionized physics, leading to further exploration.

Experimental Setup

Light of specific frequency strikes a metal plate within a vacuum chamber to ensure no air molecules interfere. Electrons are ejected from the metal upon light collision and move toward the positive terminal, connected to a battery. The resulting electron flow creates a measurable current, detectable by an ammeter.

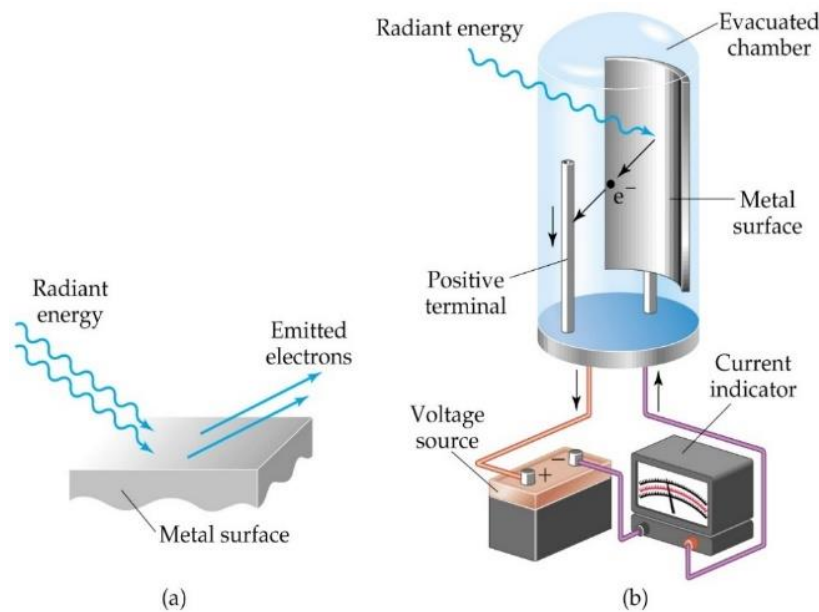


Figure 34. 4. Experimental setup for demonstrating photoelectric effect.

Conclusions

This experiment yielded significant results.

1. Firstly, the kinetic energy of electrons reaching the anode is independent of light intensity, contradicting classical theories from Newton to Maxwell. Increasing light intensity doesn't result in higher electron energy, puzzling classical theory.
2. Secondly, electrons are not emitted unless light frequency exceeds a certain threshold, regardless of intensity. Classical theory fails to explain why increasing electromagnetic field intensity doesn't cause electron emission until a specific frequency is reached.
3. The third crucial observation is that when light strikes a surface, classical theory predicts a time delay before atoms gain sufficient energy to eject electrons. However, electrons are emitted instantaneously upon light exposure, contradicting classical predictions. These discrepancies caused a major upheaval in physics, challenging classical mechanics and electromagnetic theory.

Einstein's solution to photoelectric puzzle

In 1905, Einstein proposed a groundbreaking theory, which resolved these issues but also triggered a conceptual crisis. Einstein suggested that light consists of discrete particles called quanta, each carrying a specific amount of energy. These energy quanta, or photons, are characterized by the equation:

$$E = h\nu$$

$$(\text{ or } E = \hbar\omega \text{ where } \hbar = \frac{h}{2\pi} \text{ and } \omega = 2\pi\nu)$$

$$h \text{ (Planck's constant)} = 6.626 \times 10^{-34} \text{ Joule-seconds.}$$

This equation ensures that as the frequency increases, so does the energy of the photons. Thus, Einstein's theory revolutionized our understanding of light, introducing the concept of quantized energy and laying the foundation for modern quantum mechanics.

Radio waves have lower frequency, which means that each quantum of a radio frequency wave carries less energy compared to a quantum of a gamma ray.

To illustrate this concept, let's use the analogy of climbing stairs. Imagine each step on the staircase represents a quantum of energy. When you climb the stairs, you can only move up by one full step at a time. You can't take half or partial steps. Similarly, energy is quantized, meaning it comes in discrete units or "quanta." Electrons, for instance, are only emitted from a material when the frequency of the incoming light exceeds a certain threshold frequency. Below this threshold, no electrons are emitted. However, once the threshold frequency is surpassed, the number of emitted electrons depends on the intensity of the light.

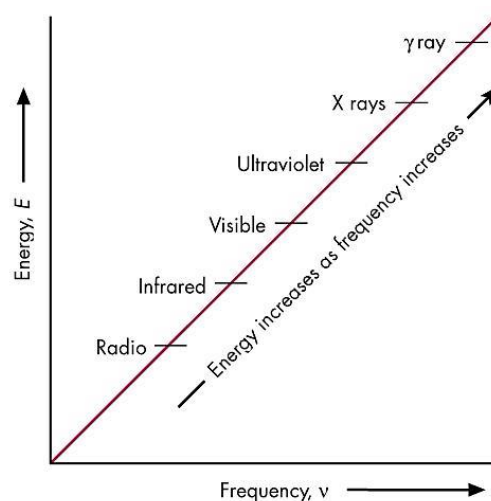


Figure 34. 5. Graph showing the direct proportionality between the energy and frequency of electromagnetic waves, illustrating that higher frequency waves have higher energy.

Black Body Radiation

Classical theory failed to explain the photoelectric effect and faced another challenge regarding

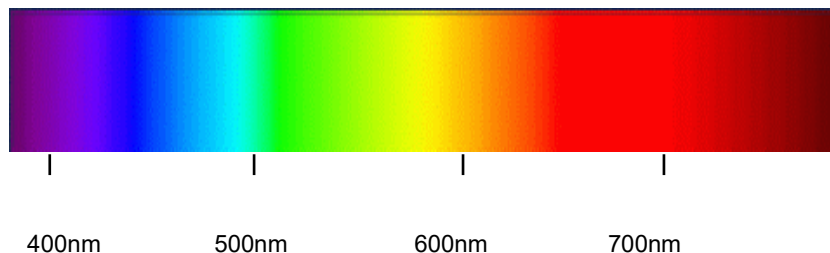


Figure 34. 6. visible light spectrum

black body radiation. When observing the frequencies emitted by a heated black body, a continuous spectrum spanning from deep blue to deep red is observed. However, heating individual atoms results in only specific colours of light being emitted. This phenomenon puzzled scientists as classical mechanics and electromagnetic theory couldn't account for it. Spectroscopic analysis of elements like hydrogen, helium, neon, and carbon revealed distinct spectral lines, such as H_α , H_β , and H_γ for hydrogen. The origin of these lines remained a mystery, leaving scientists perplexed about atomic behaviour.

Rutherford's Experiment

Rutherford's experiment challenged the Plum Pudding Model of the Atom, which proposed that electrons and protons were evenly distributed throughout. Rutherford's experiment involved bombarding atoms with alpha particles. Surprisingly, while some alpha particles passed straight through, others were deflected, and some even rebounded. This unexpected result led Rutherford to propose that atoms contained a dense central core, which he termed the nucleus. This nucleus, he reasoned, must be positively charged to repel the positively charged alpha particles. Thus, electrons orbit the nucleus much like planets orbit the sun. This revolutionary insight, confirmed around a century ago.

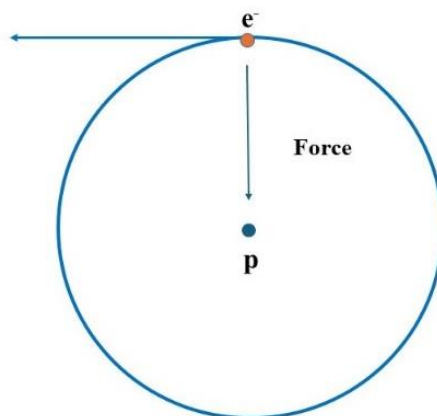


Figure 34. 7. Simplified model of a hydrogen atom with a single electron orbiting a proton, highlighting the balance between attractive and centrifugal forces.

In the case of the simplest atom, hydrogen, a single proton forms the nucleus around which an electron orbit (see fig 34.7). This orbital motion is balanced by the attractive force of the proton

and the centrifugal force exerted by the electron's motion. However, this model posed new questions, suggesting a problem that demands further exploration.

As the electron orbits the nucleus, its velocity continuously changes due to rotation, resulting in acceleration. According to electromagnetic theory, accelerating charges emit light waves, releasing energy. This energy cannot come from the potential energy between the proton and electron, as separating opposite charges requires work. Therefore, the electron's energy loss implies it moves closer to the nucleus over time. Calculations suggest that a hydrogen atom would collapse within 10^{-8} s due to this radiation. This phenomenon applies not only to hydrogen but to all atoms, leading to electron-nucleus collisions and destruction of atom.

Bohr's Atomic Model

To prevent this destruction, Niels Bohr, a Danish scientist, proposed a hypothesis suggesting that electrons could remain stable by orbiting the nucleus in specific circular paths, where their angular momentum is quantized as n times $\hbar$ ($\hbar = h/2\pi$), with n being integers (1, 2, 3, etc.). If an electron's angular momentum follows this quantization, it will not emit radiation or lose energy, allowing the atom to maintain stability. This hypothesis was further investigated to determine its validity and implications for understanding the atomic structure.

The Bohr Atom

This atom exhibits equilibrium when the electrical force (between the nucleus (+e) and electron (-e)), balances the centrifugal force.

$$F = ma = \frac{(e)(e)}{r_n^2} = m \frac{v^2}{r_n}$$

$$\frac{e^2}{r_n^2} = \frac{mv^2}{r_n} \dots\dots\dots(1)$$

According to Niels Bohr's proposal, angular momentum is quantized as

$$L = r_n \times p = r_n p \sin 90^\circ = r_n p = mvr_n = n \frac{h}{2\pi} \equiv n\hbar$$

$$v = \frac{n\hbar}{mr_n}$$

By putting the value of v in eq (1), we get the value of r_n

$$r_n = \frac{e^2}{mv^2} = \frac{e^2 m r_n^2}{n^2 \hbar^2}$$

$$r_n = \frac{n^2 \hbar^2}{me^2}$$

From this, we deduce that the electron can orbit the nucleus without emitting radiation, provided its distance equals $\frac{n^2 \hbar^2}{me^2}$. This allows for specific orbit radii. $\frac{n^2 \hbar^2}{me^2} = 0.5 \text{ \AA}$, implying one possible value of r_n is $0.5 \text{ \AA}$, followed by subsequent values increasing by squares (4, 9, 16, etc.). Thus, the electron can occupy specific orbits but not intermediate ones.

If we add both potential and kinetic energy,

$$PE = -eV = -\frac{e^2}{r}$$

$$E_n = \frac{1}{2}mv^2 - \frac{e^2}{r_n} = \frac{1}{2}m\left(\frac{n\hbar}{mr_n}\right)^2 - \frac{e^2}{r_n} = \frac{1}{2}\frac{n^2\hbar^2}{mr_n^2} - \frac{e^2}{r_n}$$

putting the value of r_n

$$\frac{1}{2} \frac{me^4 n^2 \hbar^2}{n^4 \hbar^4} - \frac{me^4}{n^2 \hbar^2} = \frac{1}{2} \frac{me^4}{n^2 \hbar^2} - \frac{me^4}{n^2 \hbar^2}$$

$$E_n = -\frac{e^4 m}{2n^2 \hbar^2}$$

$$E_1 = -13.6 \text{ eV}$$

What does "minus" signify? It represents the energy required to remove an electron from the atom, known as ionization potential, which is 13.6 eV. Additionally, various energy levels exist, with values decreasing in multiples of 4 (13.6, 13.6/4, 13.6/9, 13.6/16, etc.).

Niels Bohr proposed that when electrons transition between these levels, light is emitted. The colour or frequency of this light is determined by the energy difference between the levels.

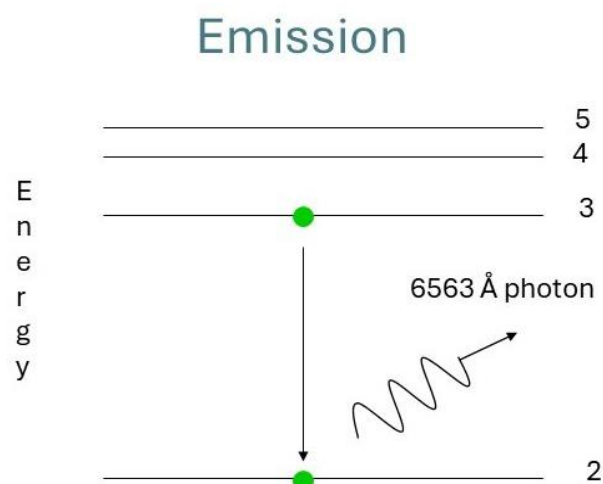


Figure 34. 8. Bohr's model: Electron transition between energy levels emitting light, with frequency determined by the energy difference.

Consider an electron transitioning from level 3 to level 2, with the extracted energy given by $\hbar\nu = 13.6 [(1/2)^2 - (1/3)^2]$. Just as the atom emits light, it becomes excited in the same manner. For instance, to transition from $n = 2$ to $n = 3$, the photon's energy must satisfy $\hbar\nu = E_3 - E_2$. Deviations from this energy level won't induce light emission.

In the hydrogen atom spectrum, depicted here, transitions from the fifth to the second level emit blue light with a wavelength of 434 nm. Conversely, the transition from level 3 to level 2 results in red light, with the lowest energy photon having a wavelength of 656 nm. The strip below showcases four different colours, each representing distinct frequencies generated by the atom. These differences arise from variations in spacing between the energy levels.

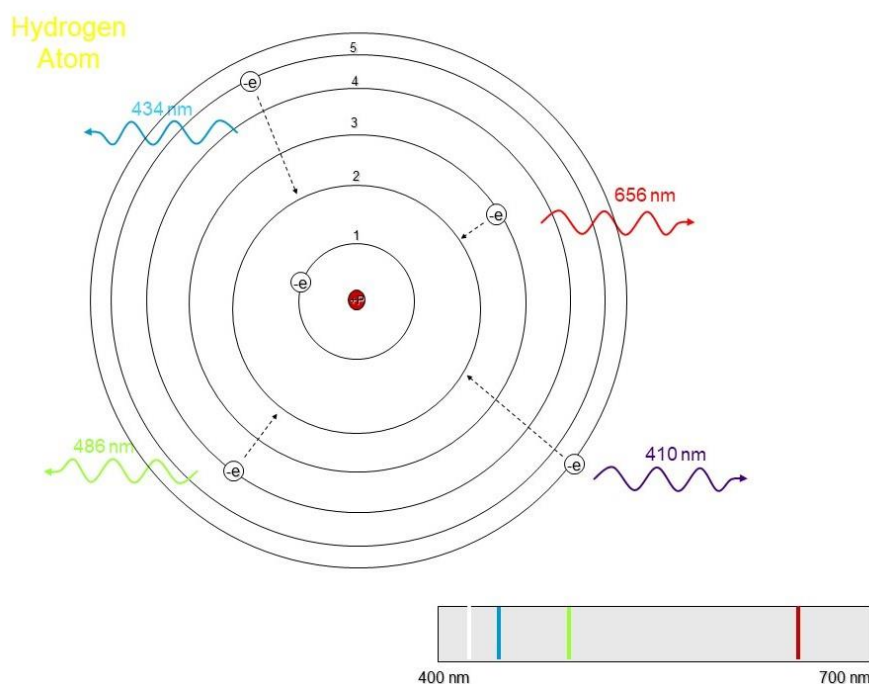
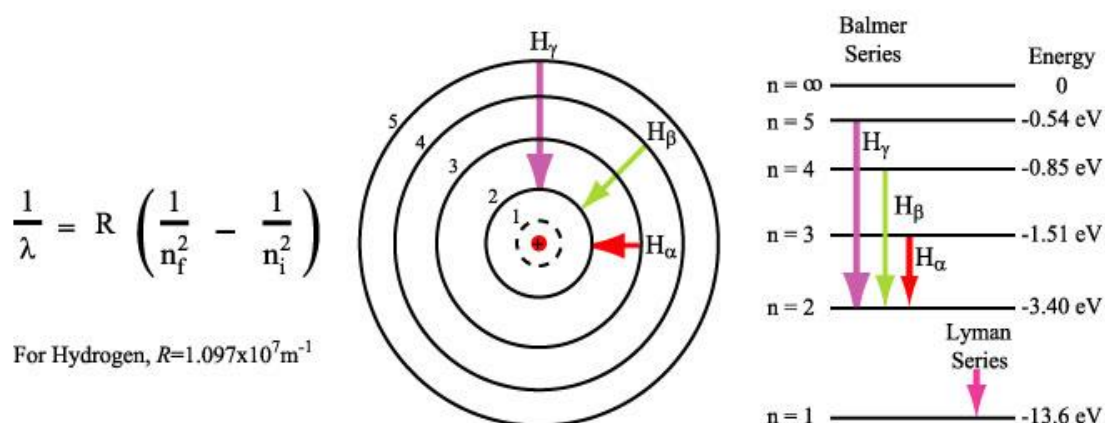


Figure 34. 9. Hydrogen spectrum: transitions from higher to lower energy levels emit light at specific wavelengths, creating distinct colors.



In the hydrogen atom spectrum, transitions like H_α , H_β , and H_γ occur when the atom is excited. These transitions emit specific wavelengths of light.

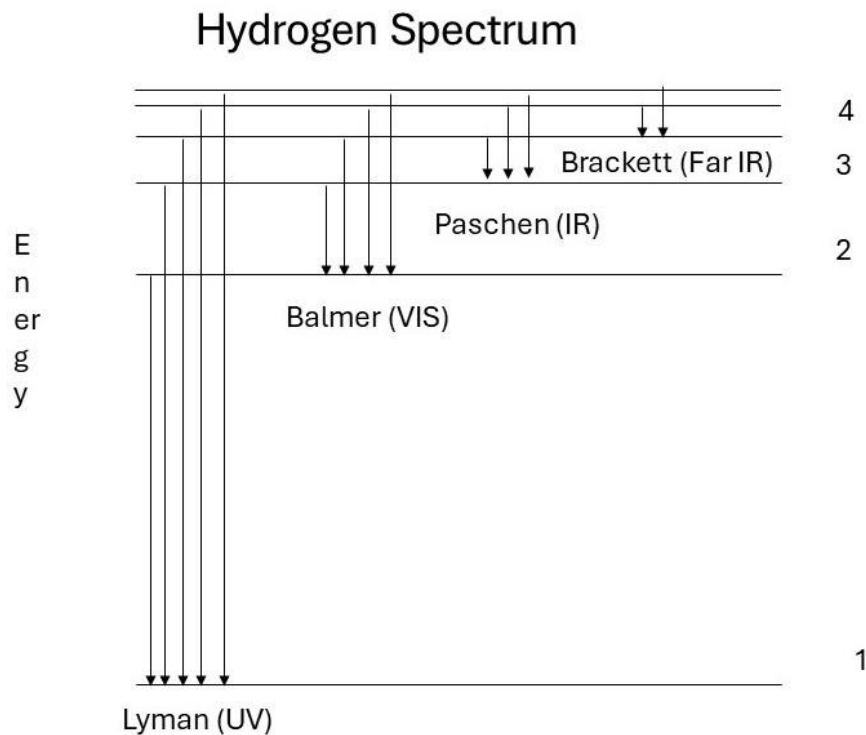


Figure 34. 10. The hydrogen spectrum is well-documented, with series such as Lyman, Balmer, Paschen, Brackett, etc., each depicting different transitions.

Fluorescence

When light strikes an object and is reflected, the frequency remains unchanged. However, this isn't always the case. In fluorescence, incident light excites atoms, causing them to temporarily transition to higher energy states before returning to lower states. Here, an ultraviolet photon collides with an electron, elevating it to a higher state. After a brief pause, the electron descends to a lower state, emitting a photon.

The emitted photon usually has a longer wavelength, hence a different color, compared to the incident ultraviolet light as shown in fig. 34.11. This color difference, known as the **Stokes shift**, is determined by the energy difference between the excited and ground states, which is influenced by the atomic structure of the material. It highlights that incoming and outgoing light can exhibit varying colors."

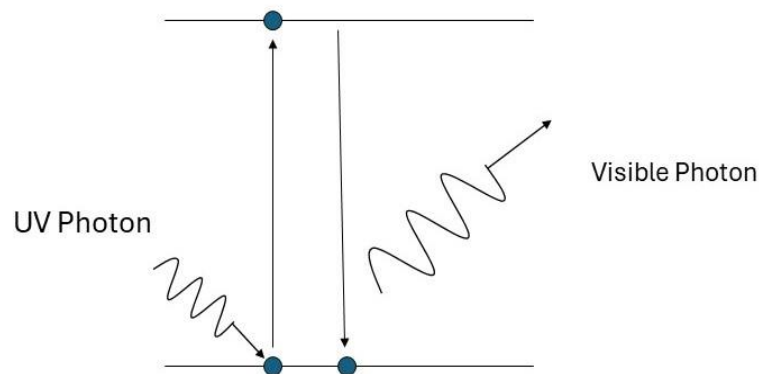


Figure 34. 11. Fluorescence: Incident light excites atoms, resulting in emission of longer wavelength photons, known as the Stokes shift.

Phosphorescence

Phosphorescence is like fluorescence but with a key difference in how long the light lasts.

1. An ultraviolet photon hits an atom and excites an electron, making it move to a higher energy level.
2. The electron then falls to a slightly lower energy level and stays there for a longer period.
3. Finally, the electron drops back to its original level, emitting light in the process.

In fluorescence, this process happens almost instantly, so the emitted light fades quickly. In phosphorescence, the electron stays in the intermediate level longer before returning to its original state. This delay means the material can glow in the dark, as it continues to emit light even after the original light source is removed. This glow can last for a much longer time, making phosphorescent materials useful for things like glow-in-the-dark items.

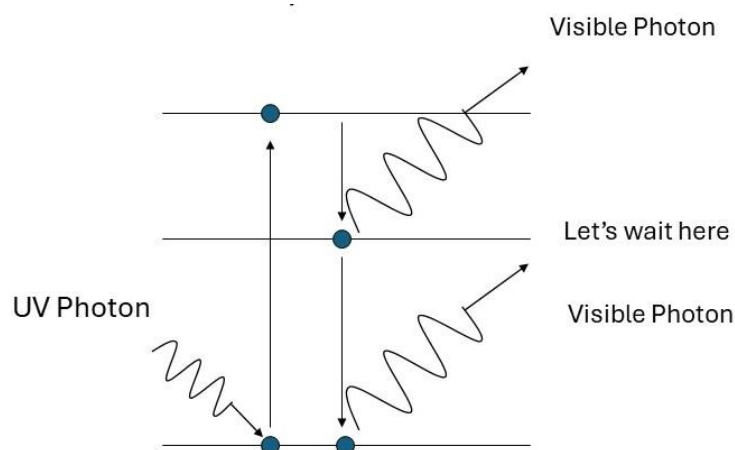


Figure 34. 12. Phosphorescence: After exposure to light, atoms release energy gradually, emitting light over an extended period.

LASER

Lasers, which are now common in many applications, work on a principle called "light amplification by stimulated emission of radiation" (LASER).

1. **Light Amplification:** This means increasing the number of photons or increasing the amplitude of EM wave. In other words, making the light brighter or stronger.

2. **Stimulated Emission:** Normally, when an atom's electron drops from a higher energy level to a lower one, it emits a photon. In stimulated emission, an incoming photon triggers the excited electron to drop to a lower energy level and emit a photon. The emitted photon has the same energy, phase, and direction as the incoming photon, creating a coherent light wave.

Working Mechanism:

- **Excitation:** Atoms in the laser material are excited to a higher energy level by an external energy source, like an electrical current or another light source.
- **Population Inversion:** More atoms are in the excited state than in the lower energy state.
- **Stimulated Emission:** When one of these excited atoms encounters a photon of the right energy, it is stimulated to emit a photon that matches the incoming one. This triggers a chain reaction where more and more photons are emitted in the same phase and direction.
- **Amplification:** These photons bounce back and forth between mirrors at each end of the laser medium, stimulating more emissions. One of the mirrors is partially transparent, allowing some of the light to escape as the laser beam.

The success of the laser has confirmed that photons are fundamental to light. Without the concept of photons, the development of lasers and many other technologies would not have been possible.

Two key points illustrate the importance of photons:

1. The photoelectric effect cannot be explained by classical theory.
2. The spectra of different atoms cannot be understood without considering photons.

There is also a third compelling argument in support of photons, which is equally significant i.e. Compton effect.

Compton Scattering

In Compton scattering, when light hits an electron, it scatters, and the outgoing light does not necessarily have the same frequency as the incoming light. This might seem like fluorescence and phosphorescence, where light excites an atom and then the emitted light has a different frequency. However, there's a key difference: in fluorescence and phosphorescence, the light transition occurs between specific atomic energy levels.

In contrast, Compton scattering involves free electrons, which can move freely and have any momentum. A bound electron in an atom can only occupy certain discrete energy levels, whereas a free electron is not restricted in this way.

Classically, when a light wave interacts with an electron, the electric field of the wave should cause the electron to oscillate and emit radiation at the same frequency as the incoming light. However, in Compton scattering, the scattered light has a different frequency, indicating that classical theory does not fully explain the phenomenon.

Quantum Mechanical Approach: In Compton scattering, a photon with energy $h\nu$ (where ν is the frequency of blue light) collides with an electron. This collision causes the electron to gain momentum and move with increased kinetic energy. The scattered photon, now with less energy, emerges with a lower frequency than the incoming photon.

Two quantities remain conserved in this process: energy and momentum. Initially, the energy consists of the photon's energy plus the electron's rest energy, which has no kinetic component as the electron is initially at rest. Thus, the initial energy is the photon's energy.

After scattering, the total energy includes the energy of the scattered photon and the kinetic energy gained by the electron. For energy conservation to hold, the energy of the scattered photon must be less than the energy of the incoming photon, as some energy is transferred to the electron. Therefore, if a blue photon collides with an electron and a red photon is emitted, it indicates a loss of energy from the photon to the electron. This phenomenon supports the quantum theory of light and cannot be explained by classical theory alone.

Do we see photons? Where are they? How many are there, and what is their energy?

- Sunny day (outdoors): 10^{15} photons per second enter eye (2 mm pupil)
- Moonlit night (outdoors): 5×10^{10} photons/sec (6 mm pupil)
- Moonless night (clear, starry sky): 10^8 photons/sec (6 mm pupil)

- Light from dimmest naked eye star (mag 6.5): 1000 photons/sec entering eye

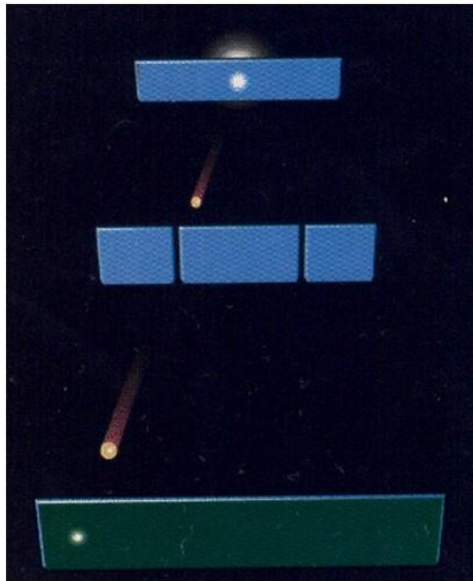
So far, we have presented three arguments: the photoelectric effect, atomic spectra, and the Compton effect. These arguments indicate that light consists of photons, which are quanta of energy. A photon's energy is given by $\hbar\omega$ or $h\nu$, meaning it comes in discrete packets. There is a fourth argument related to black body radiation, which will be discussed later when we study heat.

Wave-Particle Duality

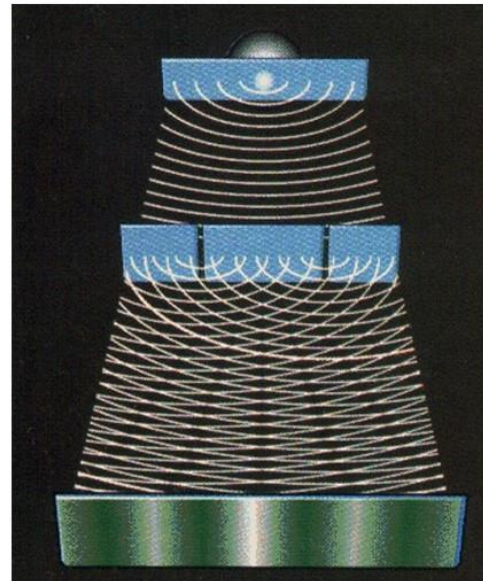
Presently, we have two different theories: the particle theory, which involves photons and is associated with Einstein, and the wave theory of electromagnetic waves, related to Maxwell and Faraday. In 1924, Einstein remarked that *there are two theories of light, both indispensable and without any logical connection*. One suggests that light is composed of particles, while the other suggests it is composed of waves. This raises a dilemma: if one theory is correct, does it invalidate the other? This contradiction has led to considerable complexity in understanding the nature of light.

To explore this problem further, let's revisit the double-slit interference experiment. In this experiment, a light source is split into two paths that interfere with each other. We will examine this from two perspectives: light as particles and light as waves.

If we consider light as photons, individual atoms emit these photons. A photon can pass through either one slit or the other and then hit a screen, leaving a mark. If we imagine using bullets from a rifle instead of photons, the bullets will scatter randomly with no distinct pattern, unlike waves. Waves can reinforce or cancel each other out, creating an interference pattern. This suggests that if light exhibits interference, it must be waves, contradicting the particle-based explanations of the photoelectric effect, atomic spectra, and the Compton effect.



particle?



wave?

However, this is not the whole story. While particles like photons typically don't interfere in the way waves do, photons can indeed interfere under certain conditions. The behavior of photons depends on the type of experiment performed. In some experiments, photons exhibit particle properties, such as in the photoelectric effect. In others, like the double-slit experiment, they exhibit wave properties. This dual nature is known as **wave-particle duality**.

Initially, Einstein found this concept challenging, but it is now a cornerstone of modern physics and the foundation of quantum mechanics. We will explore this further in another lecture.

Lecture 35 – GEOMETRICAL OPTICS

This lecture focuses on geometrical optics, the study of light as rays traveling in straight lines. While light is fundamentally a wave, exhibiting interference and diffraction, these wave properties are not considered in geometrical optics. Instead, we observe that light, such as from a candle, casts shadows when obstructed. Shadows form in the direction opposite to the light source, a phenomenon observable both on Earth and in the celestial bodies we see in the sky. For instance, the moon also casts its shadow on the Earth, leading to solar eclipses when the Sun's rays are blocked. This shadow has two distinct parts: a completely dark region, and the region where the darkness is partial. As the Moon orbits, these shadow regions move along with it. Similarly, lunar eclipses occur when the Sun's rays are blocked by the Earth and cast its shadow on the Moon.

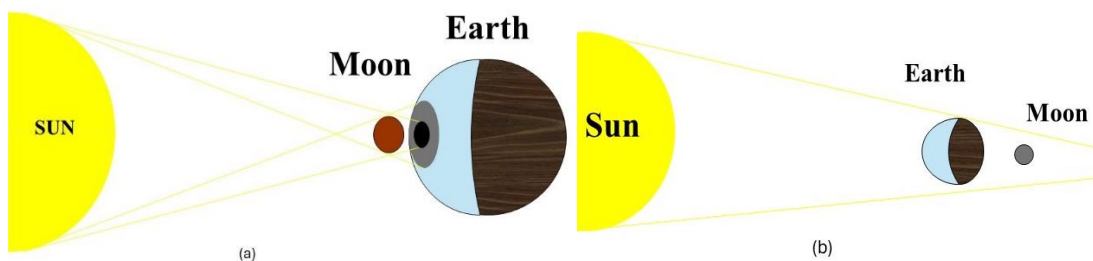


Figure 35. 1. (a) Solar eclipse (b) Lunar eclipse

Reflection and Mirrors

1. Plane Mirrors

When you look into a mirror, the image you see is known as a virtual image. It's formed by the way light rays reflect off the mirror. Imagine a candle in front of a mirror emitting light rays as shown in fig 35.2. The rays that hit the mirror reflect at the same angle. If we extend these reflected rays backward using dashed lines, they appear to converge at a point behind the mirror. This point, where the extended rays meet, is where the virtual image of the candle appears.

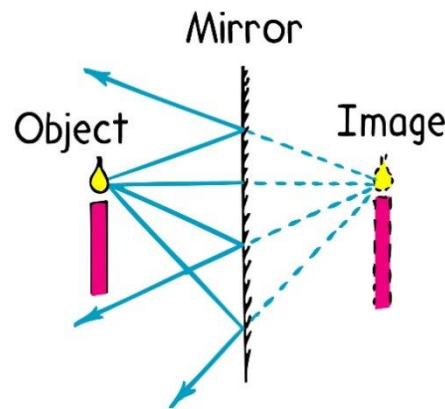


Figure 35. 2. A virtual image of a candle forms behind a mirror where reflected light rays appear to converge.

The **characteristics** of a virtual image in a plane mirror are:

- The rays appear to converge behind the mirror, although they do not physically do so.
- The image appears to be the same distance behind the mirror as the object is in front of it.
- The image is upright and the same size as the object.
- The image is laterally inverted, meaning left and right are switched.

This behavior of light is fundamental to the definition of a plane mirror, which reflects light in such a way that the virtual image seems to exist behind the mirror's surface.

The image created by a mirror is known as a virtual image because it forms where the extended rays appear to converge, but no actual object exists there. Conversely, a real image is formed when light rays converge at a point.

In a plane mirror, the angle of incidence equals the angle of reflection, meaning the light rays reflect symmetrically. This holds true for any smooth section of the mirror. However, if the mirror's surface is rough, the light rays scatter in various directions, preventing the formation of a clear image (fig 35.3).

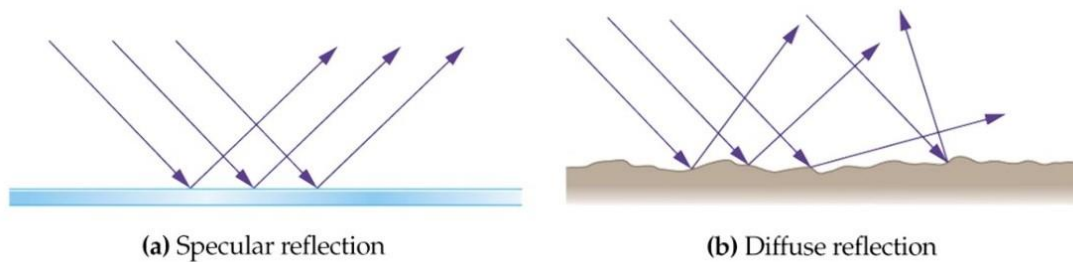


Figure 35. 3. (a) Smooth surfaces cause specular reflection with clear images, while (b) rough surfaces cause diffuse reflection, scattering light.

For a mirror to reflect light properly, its surface must be extremely smooth. Given that the wavelength of visible light ranges from 4000 to 7000 Å (or atomic lengths), the surface roughness must be significantly smaller, ideally at least eight times smaller than these wavelengths. This ensures that the light reflects accurately, maintaining the equality of the angle of incidence and the angle of reflection.

On the other hand, consider satellite dishes, which use microwaves with much larger wavelengths compared to visible light. Despite the use of a wire mesh in these dishes, they still function effectively because the large wavelengths of microwaves can reflect off a less smooth surface.

Another example of forming an image involves a source of light placed in front of a bi-convex lens which bends the light as shown in fig 35.4 (a). The light rays from the bulb pass through the lens and reach our eyes. If we trace these rays backward, they converge at a single point, forming an image.

A partially silver mirror allows light to both reflect off it and pass through it. When light from a source is directed at such a mirror, it can create images in the same location, regardless of the source's position (see fig 35.4 (c)).

If we position the same light source in different locations, its image will still be formed at the same point. For instance, fig 35.4 (a) shows light originates from the source, and when traced back, converges at a point, creating a virtual image. fig 35.4 (b) depicts identical object and image positions, while the third employs a mirror, resulting in identical mirror images. Thus, discerning between the three is challenging.

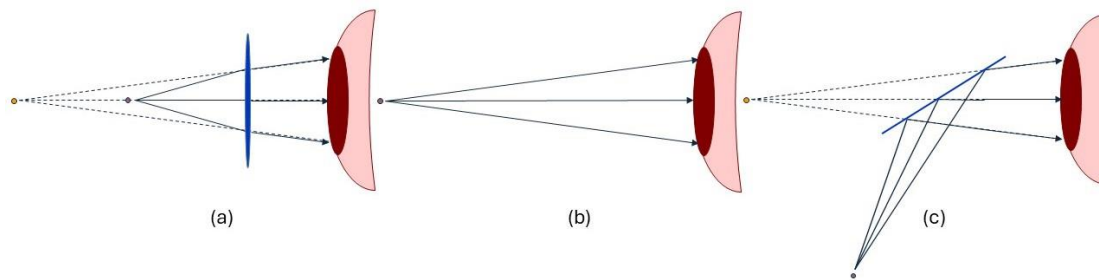


Figure 35. 4. (a) Light bending through a bi-convex lens to form an image as rays converge at a single point. (b) Identical object and image positions showing consistent image formation. (c) A partially silvered mirror creating images in the same location regardless of the source's position.

To determine if an image is real or virtual, trace the light rays backward to their apparent origin. If they seem to emanate from a point behind the mirror, it's a virtual image. It's important to note that the distance from the object to the mirror is equal to the distance from the mirror to the image.

2. Spherical Mirrors

Not all reflective surfaces are flat like glass. For example, by cutting a sphere, we can create curved mirrors that form images with different properties.

Imagine cutting a portion of a spherical object and coating its surface with reflective material illustrated in fig 35.5. Depending on whether the coating is applied to the inner or outer surface, two types of mirrors are created: **concave and convex mirrors**. Each mirror has unique reflective properties.

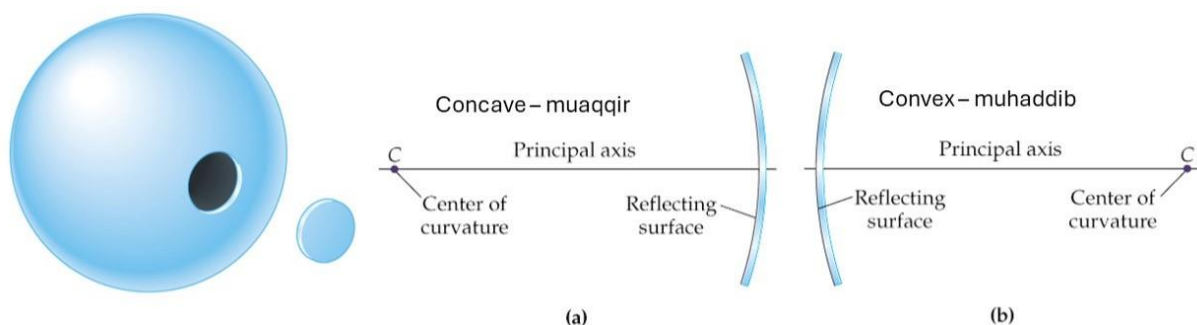


Figure 35. 5. (a) concave and (b) convex mirrors created by coating the inner and outer surfaces of a cut spherical object, respectively.

Let's focus on the concave mirror, which is relatively inexpensive. When positioned at various distances, it produces different images of surrounding lights on a screen. As the mirror is brought closer, the image clarity improves until reaching a specific position, beyond which the

image becomes blurry again. This phenomenon can be explained through a diagram, where parallel lines originating from a distant source interact with the mirror.

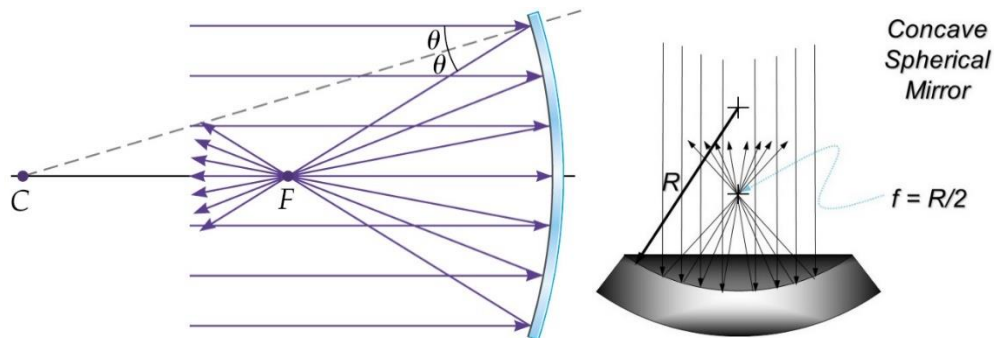


Figure 35. 6. (a) Parallel light rays converge at the focal point (F) after reflection from a concave mirror, (b) the focal point is located at $R/2$ from the center of the mirror, known as the focal length.

Parallel light rays that strike a mirror converge at a single point after reflection, known as the focal point (F) as shown in fig 35.6. For a concave mirror, the focal point is located at a distance of $R/2$ from the centre of mirror, known as the focal length.

When parallel rays strike a convex mirror, they appear to originate from a specific point behind the mirror known as the **virtual focus**. Despite seeming to originate from this point, the rays reflect off different portions of the mirror (fig 35.7). In optics, we use a convention where the focal length is negative if the focus is behind the mirror. Thus, for a convex mirror, the focal length is negative, while for a concave mirror, it's positive, at $R/2$.

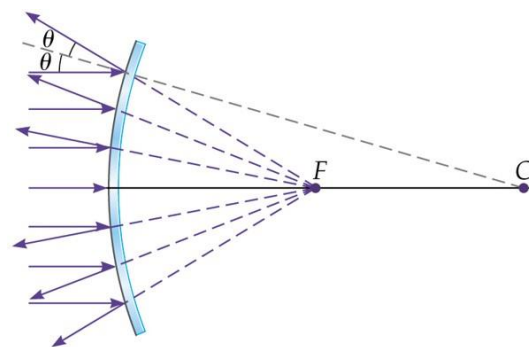


Figure 35. 7. Parallel rays striking a convex mirror appear to converge at a virtual focus point behind the mirror, reflecting off different portions of the surface.

Image formation in concave mirror

A concave mirror reflects light in a way that forms images with varying sizes. Three principal rays are considered:

- P-ray: Travels parallel to the axis, reflects through the focal point.
- F-ray: Strikes the focal point, reflects parallel to the axis.
- C-ray: Passes through the center, reflects on itself.

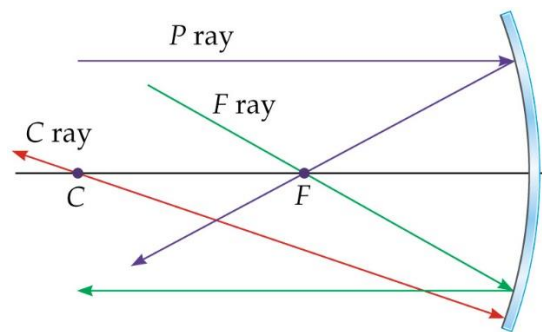


Figure 35. 8. Principal rays for a concave mirror: P-ray reflects through the focal point, F-ray reflects parallel after striking the focal point, and C-ray reflects through the center, illustrating image formation.

When an object, like a pencil with a lit tip, is placed before a concave mirror, the light rays it emits reflect off the mirror. If the object is placed beyond the center (C), the reflected rays, P and F, intersect, forming a real axis where the image appears smaller than the object shown in fig 35.9 (a). If the object is between the center (C) and the focal point (F), the reflected rays, P and F, intersect, forming a real axis where the image appears larger than the object shown in fig 35.9 (b). This magnification property is why concave mirrors are used in applications like shaving mirrors, where an enlarged view is beneficial.

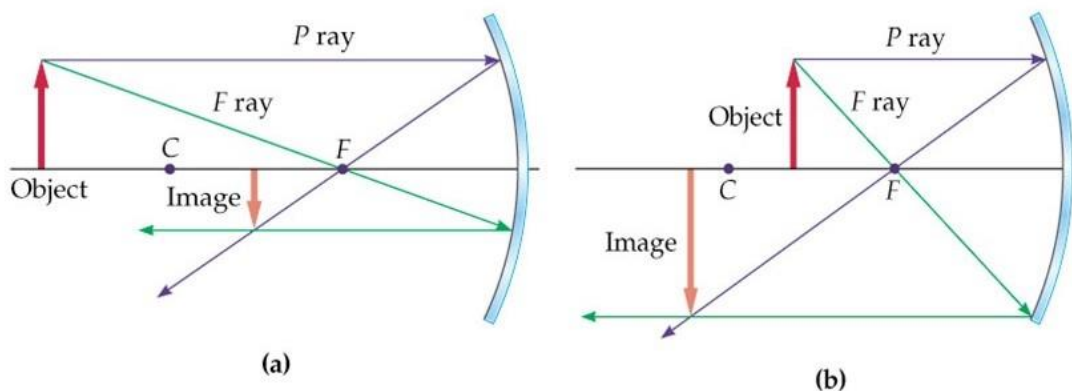


Figure 35. 9. Image formation by a concave mirror: (a) a smaller image formed when the object is placed beyond the center (C), while fig (b) illustrates a larger image formed by the object placed between the center (C) and the focal point (F), highlighting the mirror's magnification property.

In a concave mirror, an object like a pencil creates a real image. The ratio of the image size (h') to the object size (h) is known as the magnification factor. Mathematically, it's expressed as h'/h or $-q/p$, where q is the distance from the mirror to the image, and p is the distance between the object and the mirror.

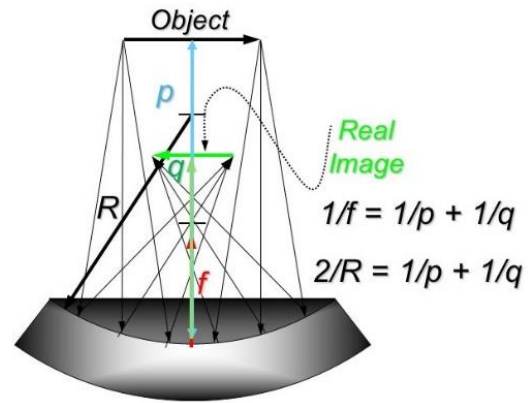


Image formation in convex mirror

When an object, such as a pencil, is placed in front of a convex mirror, the reflected rays—P-rays, F-rays, and C-rays—diverge. P-rays reflect upward, C-rays reflect along the same path, and F-rays reflect parallel to the axis. Extending these rays backward, they seem to converge at a point behind the mirror, forming a diminished, upright image of the object. Unlike concave mirrors, which produce inverted images (hence the negative magnification factor), convex mirrors always create upright images. The focal length for a convex mirror is represented as $-R/2$, indicating the virtual nature of the focus.

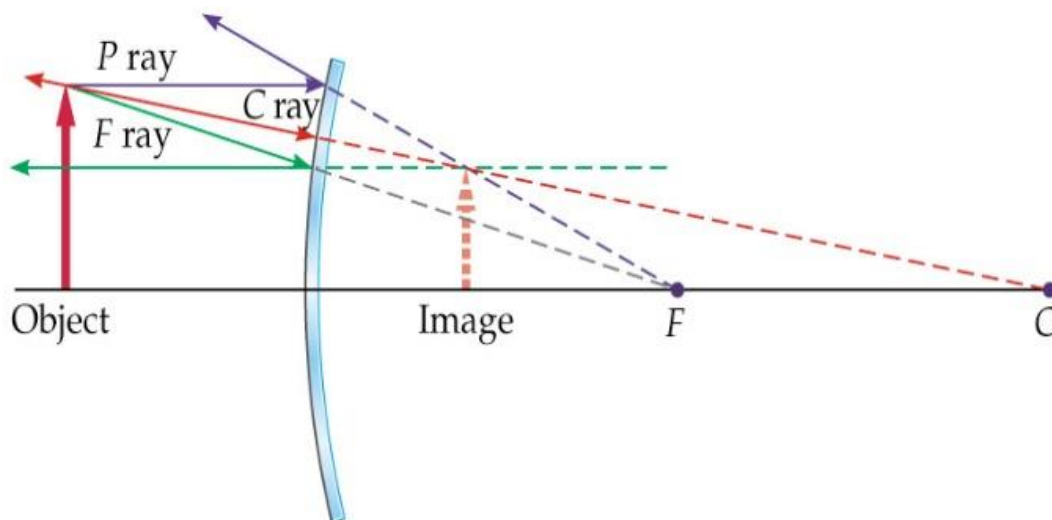
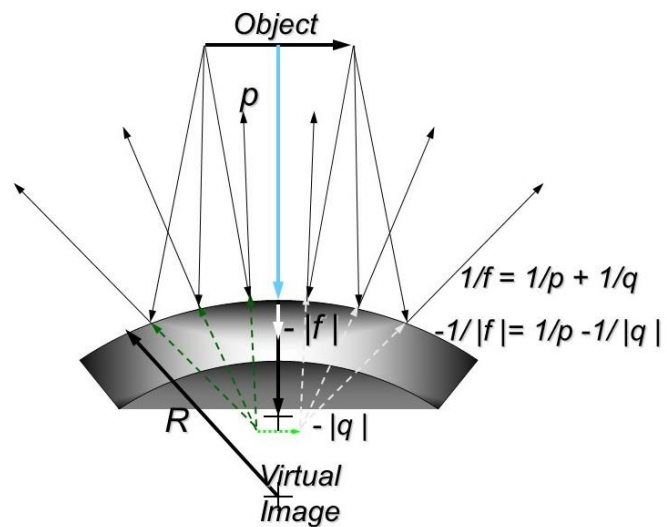


Figure 35. 10. A virtual, upright and diminished image formed by a convex mirror with diverging reflected rays appearing to converge behind the mirror.

In convex mirrors, reflected light rays diverge, creating a scattered effect. Yet, when these rays are extended backward, they seem to converge at a point, forming a virtual image. The magnification factor, (M), is the ratio of the image height to the object height, expressed as ($M = h'/h$), which is also (q/p). Unlike images formed by concave mirrors, images in convex mirrors remain upright, hence the absence of a negative sign in the magnification formula.



Refraction and Lenses

Lenses focus or disperse light due to the change in speed caused by the difference in refractive index between the lens material and the surrounding medium, like air or vacuum. The refractive index (n) is defined as the ratio of the speed of light in a vacuum to its speed in the medium. When light enters a lens, it slows down and bends towards normal due to refraction. In the case of a convex lens, the light converges towards the center, while in a concave lens, it diverges.

In a convex lens, parallel light rays converge to a point, focusing the light. When a light source is positioned on the focal point of a lens, the emitted rays are parallel.

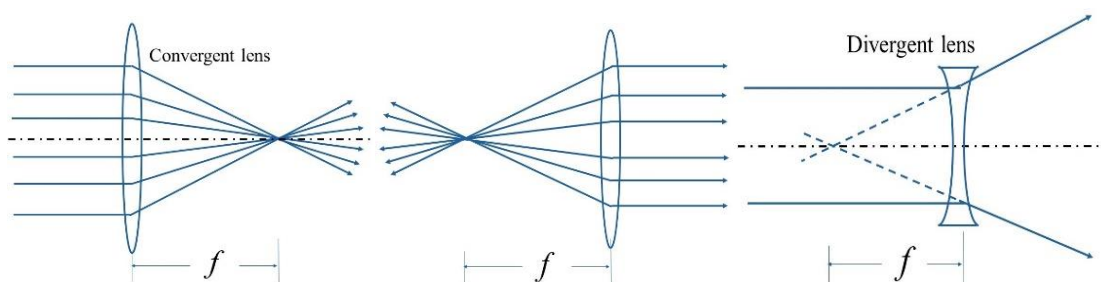
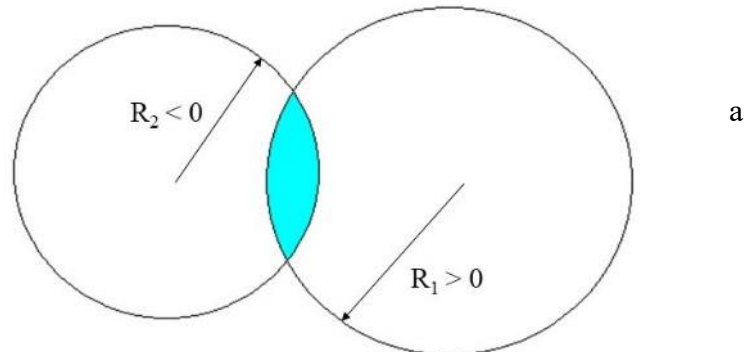


Figure 35. 11. Diagrams can help visualize how convex lenses (a) converge light to a focal point, (b) Light emitted from the focal point of a lens emerges as parallel rays, while (c) concave lenses cause light to diverge, affecting the size and focus of the resulting image.

If parallel light rays encounter a diverging (concave) lens, they scatter. Extending these diverging rays backward, they seem to meet at a virtual focal point, denoted as negative for concave lenses, while it remains positive for convex lenses.

Lens maker formula for Spherical Single Lens

The focal length of a spherical lens is determined using the **lens maker formula**. It accounts for the lens's curvature on both sides, which may differ, leading to different radii of curvature (R_1 and R_2). A larger radius corresponds to a flatter lens, whereas a smaller radius indicates more curved and thus more powerful lens.



The formula is:

$$\frac{1}{f} = (n-1) \left(\frac{1}{R_1} - \frac{1}{R_2} \right)$$

Here, n is the refractive index of the lens material, and 1 is the refractive index of air. R_1 and R_2 are the radii of curvature for the lens's surfaces. R_2 is considered negative when the lens's curvature is inward from the light source's perspective. The power of a lens increases as the radii of curvature decreases. Since R_2 is negative, its magnitude contributes positively to the lens's power.

Different **lens shapes** include biconvex (curved outward on both sides), biconcave (curved inward on both sides), or a combination of convex and concave, including plano-convex (flat on one side and curved outward on the other), and plano-concave (flat on one side and curved inward on the other) configurations. Each type has unique properties affecting how light is refracted through them (fig 35.12).

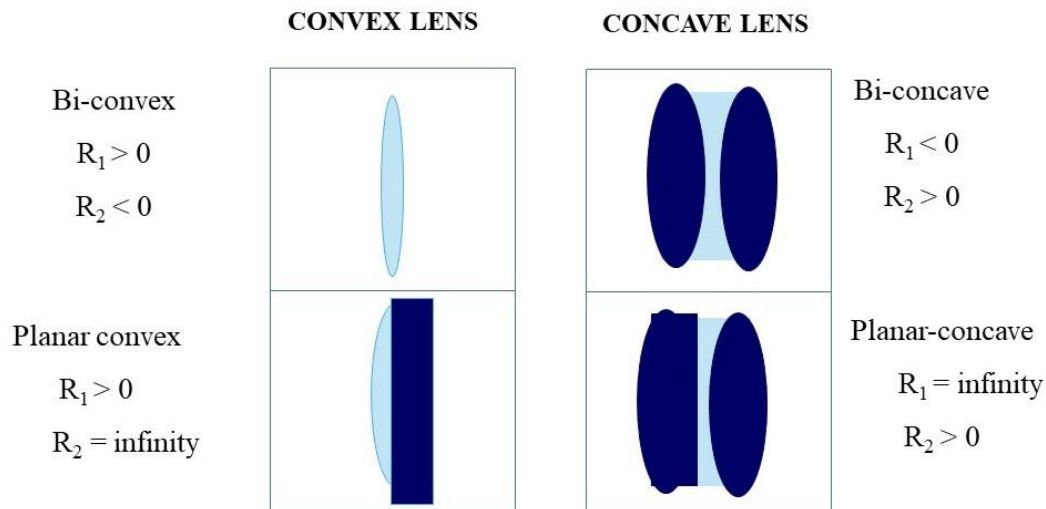


Figure 35. 12. Various lens shapes have distinct properties influencing light refraction.

Diopter

The **dioptr** is the reciprocal of a lens's focal length (f) in meters, expressed as

$$\text{Diopter} = 1/f$$

The lens's power, measured in diopters, depends on its refractive index (n) and the radii of curvature (R_1 and R_2) on each side. A lens with greater power can focus or diverge light more effectively.

Image formation by convex Lens

To understand how a lens shapes light, we trace the path of light rays through the lens. For a convex or converging lens, the image formed is always **inverted**. If an object, like a pencil, is placed close to the lens, the image appears larger (magnified) and virtual because the light rays diverge and do not meet after passing through the lens indicating magnification greater than 1. As the object moves further from the lens as shown in fig 35.13, the image becomes smaller and real, as the light rays converge to a point. The magnification changes with the object's distance from the lens, affecting the size of the image produced. As the distance between the object and the lens changes, the nature of the image also changes.

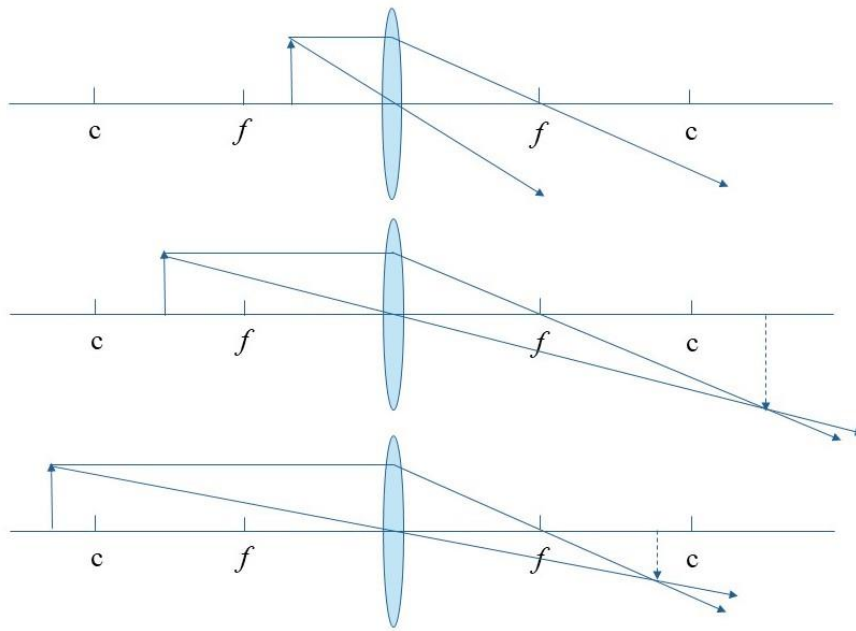
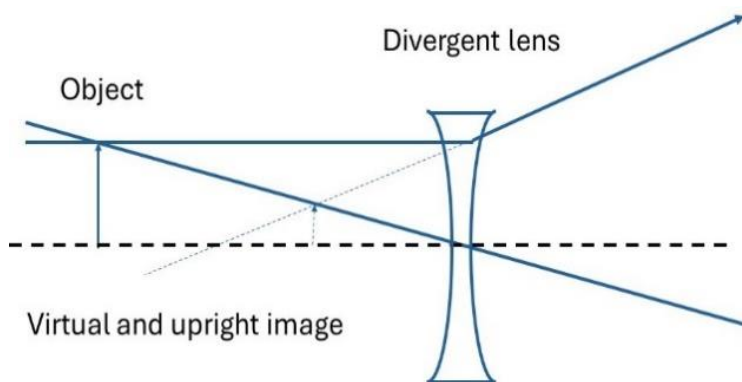


Figure 35. 13. "Figures illustrate the behavior of convex lens: close objects create larger, virtual, and magnified images, while distant objects form smaller, real images due to light ray convergence, demonstrating varying magnification and image properties."

Image formation by concave Lens

On the other hand, concave lenses produce a smaller, virtual image as they diverge from light rays. While concave lenses reduce the size of images, they are beneficial for correcting vision and other applications.



Lens Defects:

- i. A concave lens's primary function is to diverge rays coming from a distant source, such as parallel rays from the Sun. However, lens makers face challenges like **chromatic aberration** (*Aberration refers to imperfection*), where different colors focus at different points due to varying refractive indices. This means that red and blue light, for instance,

will converge at slightly different focal points, leading to a blurry image. This issue is tackled by employing various techniques, including applying specialized coatings to the lenses to minimize the difference in focal points for various colors, ensuring clearer images.

- ii. **Spherical aberration** occurs when rays striking different parts of a wide lens do not converge at a single point, leading to a distorted image. This defect is particularly pronounced in wide lenses where parallel rays don't meet at one focal point. Reducing the lens's aperture can minimize aberration, allowing parallel rays to converge more effectively and produce a clearer image.

Let's explore why do objects appear larger or smaller at different distances, and how can we calculate this?

Optical Instruments

The simple Magnifier

Consider a convex lens, when it is held close to the thumb, the thumb appears its normal size. As the lens moves away from the thumb, the size of image changes, becoming larger, until it eventually defocuses.

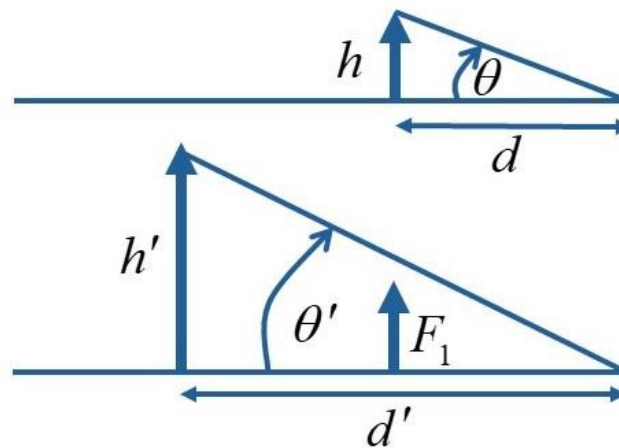


Figure 35. 14. A simple magnifier

To calculate this, we use magnification, defined as the ratio of θ' to θ .

$$m_{\theta} = \frac{\theta'}{\theta} = \frac{h / f}{h / 25\text{cm}}$$

$$m_{\theta} = \frac{25\text{cm}}{f}$$

25 cm is the typical closest distance at which the human eye can focus comfortably, about 8 inches. Closer than this, objects defocus. A shorter focal length results in greater magnification, making small objects appear larger. However, there's a limit to magnification with a single lens, as too short a focal length can cause aberrations. Thus, there are practical limitations to how much a single lens can magnify.

Compound Microscope

In a compound microscope shown in fig 35.15, magnification is achieved through two lenses: the objective and the eyepiece. The object's height is denoted as h . Light rays originating from the object pass through the objective lens, creating an intermediate image of height h' at F_1' . This image is then magnified by the eyepiece lens, resulting in a larger final image.

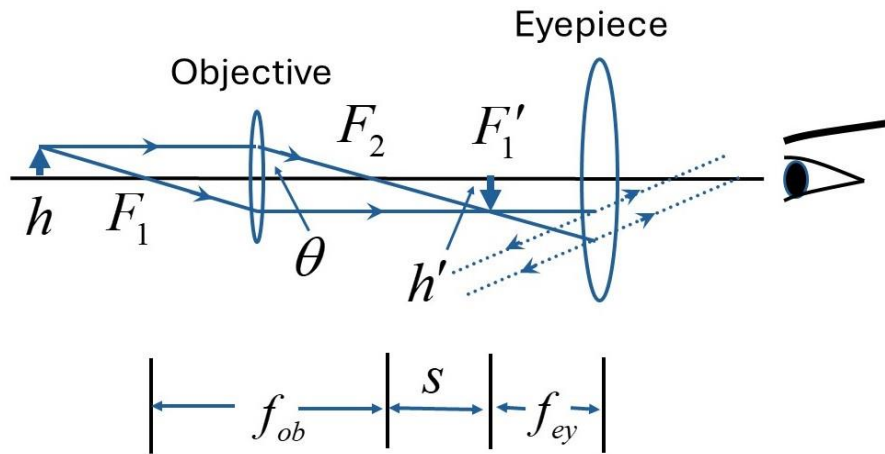


Figure 35. 15. A compound microscope, featuring multiple lenses, magnifies small objects by combining the magnification powers of its objective and eyepiece lenses.

The total magnification M of the microscope is the product of the magnifications by the objective (m) and the eyepiece (m_θ) which is calculated as:

$$M = mm_\theta = -\frac{s}{f_{ob}} \frac{25cm}{f_{ey}}$$

Here, s is the tube length, f_{ob} is the focal length of the objective lens, and f_{ey} is the focal length of the eyepiece. The negative sign indicates that the intermediate image is inverted. The dual-lens system allows for greater magnification than a single lens.

Refracting Telescope

A binocular or refracting telescope can also be constructed using two lenses to enhance the apparent size of distant objects (fig 35.16). The parallel rays are focused to form a small angle since the object is far away. This angle is the size of the object divided by the distance, a very

small quantity. We aim to increase this angle. The objective lens captures the object at a distance, creating an image at the focal point of size h' , which is then enlarged by the eyepiece for clearer viewing. The overall design allows for an increased angle of view, making distant objects appear closer and larger to the observer.

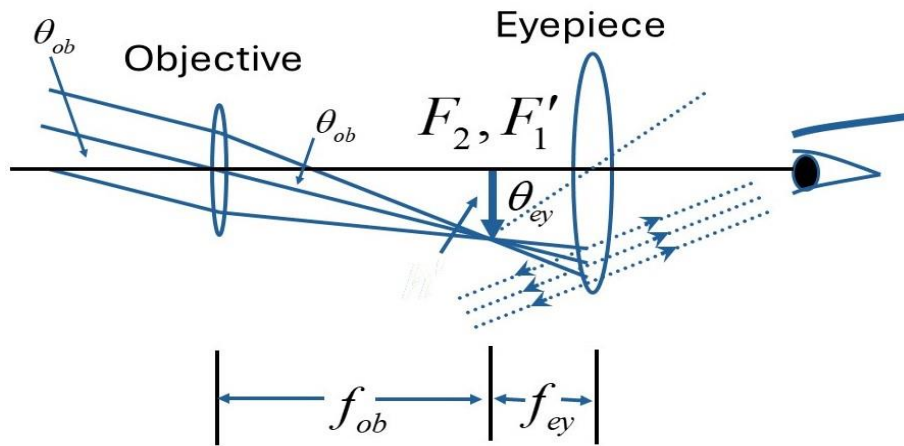


Figure 35. 16. A refracting telescope comprised of two lenses to gather and focus light from distant objects, providing magnified views of celestial bodies.

Human Eye

In a normal eye, light focuses precisely on the retina, creating a clear image. This sharp focus allows the retina to transmit electrical signals to the brain, enabling us to perceive the image. People with poor vision particularly benefit from corrective lenses.

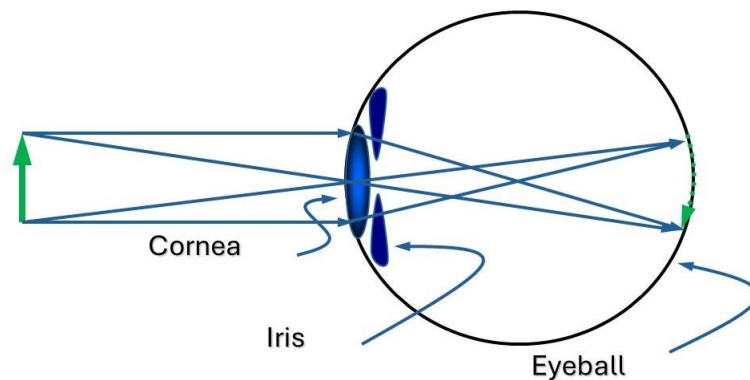


Figure 35. 17. Image formation in normal human eye.

Eye Defects and their Correction

There are two common types of vision problems:

- **Farsightedness (Hypermetropia):** Difficulty seeing near objects because the image forms behind the retina. To **correct** hypermetropia, a convex lens is used, which adjusts the focal point, so the image falls on the retina.

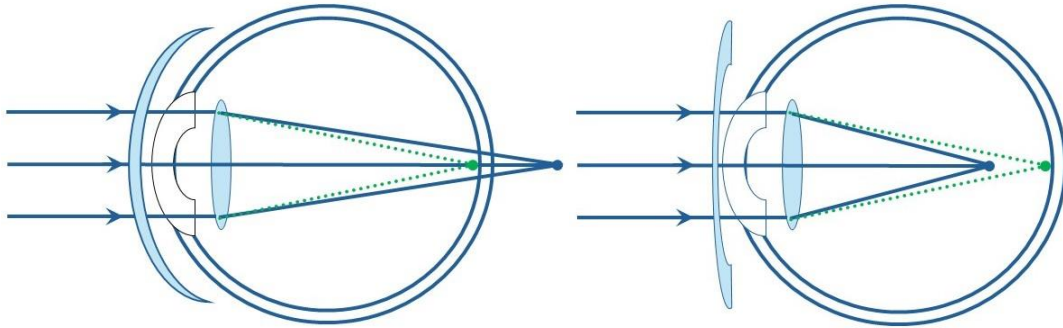


Figure 35. 18. Eye defects and their correction using spherical lenses: Hypermetropia (left), Myopia (right)

- **Nearsightedness (Myopia):** Difficulty seeing distant objects because the image forms in front of the retina. For myopia, a concave lens is employed to diverge the light rays, moving the focal point forward onto the retina. Thus, lenses help restore clear vision by compensating for the eye's focusing deficiencies.

Prism and total Internal Reflection

Prisms are fascinating optical devices that not only disperse light into its constituent colors but also enable total internal reflection. This phenomenon occurs when the angle of incidence exceeds a critical value, causing light to reflect entirely within the prism rather than passing through it. When light enters a prism and reflects internally, it can create an image that is the same size as the object if the rays are parallel. If the prism is rotated, the image can be inverted (see fig 35.19). This principle is utilized in binoculars to correct the orientation of the observed image, ensuring that it appears right-side-up to the viewer.

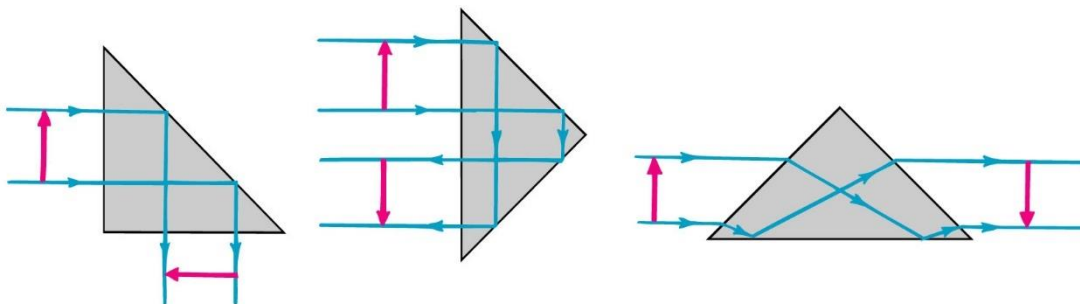


Figure 35. 19. Total internal reflection within a prism creates an image the same size as the object with parallel rays, capable of inversion when the prism is rotated.

In binoculars, prisms play a crucial role. They are used to extend the optical path without increasing the physical length of the binoculars, which keeps them compact. The prisms inside binoculars cause light to undergo total internal reflection, effectively lengthening the path the light travels before reaching the eyes. This design allows for greater magnification power while maintaining a portable size.

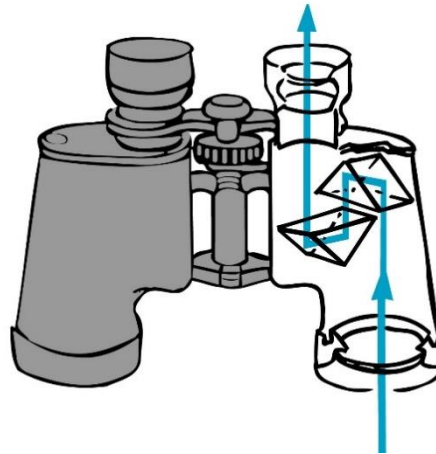


Figure 35. 20. The diagram illustrates light reflecting through multiple prisms, effectively extending the path length within a shorter physical space.

Prisms can be used as mirrors through the phenomenon of total internal reflection. When parallel rays strike the prism, it can be reflected off the internal surfaces if the angle of incidence is greater than the critical angle for the material. This reflection can effectively turn the prism into a mirror without the need for a reflective coating.

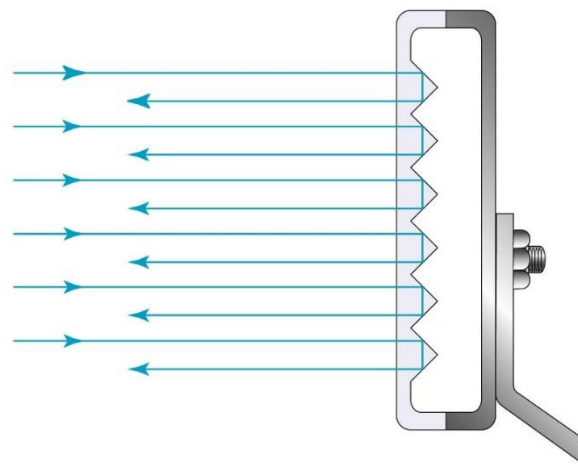


Figure 35. 21. Prisms utilizing total internal reflection to act as mirrors for parallel rays, reflecting internally without external coatings.

Advanced Optics: Gravitational Lensing

Light typically bends when it passes through materials with different refractive indices, such as air or glass. However, in space, light can also bend due to gravity, a phenomenon predicted by Einstein's general theory of relativity. This bending occurs near massive objects like stars or black holes having strong gravitational fields, creating what is known as a gravitational lens. Gravitational lensing allows us to observe distant objects like quasars, as their light is focused by these massive celestial bodies. This effect has been observed and helps astronomers study the distribution of matter in the universe.

Lecture 36 – THERMAL PHYSICS I

Introduction to Heat and Temperature:

Energy is a fundamental concept, originating from small particles like electrons and protons, as well as stars and galaxies. It can be of various forms, including chemical and atomic energy etc. **Heat** differs slightly from other forms of energy and flows from a system at high temperature to a system at low temperature. Heat has played a crucial role in human history. Heat has been pivotal in human progress. The invention of steam engines revolutionized transportation and communication, leading to the development of diesel and petrol engines, enabling long-distance travel and changing societal structures. We instinctively recognize the difference between hot and cold objects, associating heat with a higher temperature. Scientifically, **temperature** is the common property that equalizes when bodies are in contact, leading to thermal equilibrium. **Thermal equilibrium** is a state where heat transfer between two or more bodies ceases due to equal temperatures as shown in fig 36.1.

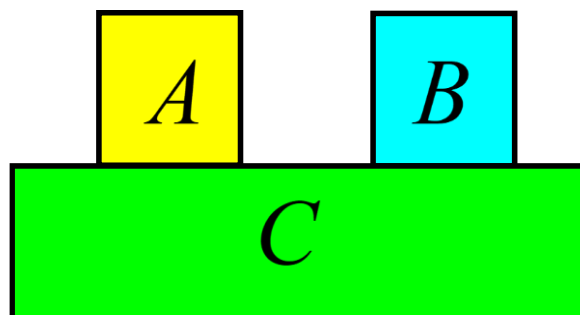


Figure 36. 1. Three bodies A, B and C are in thermal contact with each other.

Measuring Temperature

Temperature is measured using various methods. One way is by observing color changes in heated objects, such as iron turning from yellow to white as it heats. Even from a distance, like with the sun, we can infer temperature by analysing light and color.

Thermometers

Thermometers measure temperature based on the principle that *substances expand when heated*. For instance, alcohol in a thermometer expands and rises in a tube, as it gets hotter. Demonstrating with a candle, we observe the alcohol rise with heat and fall when placed in cold water, showcasing the thermometer's response to temperature changes.

Gas Thermometers: Thermometers can utilize various liquids (like mercury), but they're not limited to liquids alone. Gases also expand with heat, and if confined, their pressure increases with temperature. This principle can be demonstrated by confining hot air in a bottle using a candle, then cooling it in water, observing the pressure change. This experiment illustrates **Boyle's Law**, where pressure is directly proportional to temperature (T) when volume is constant.

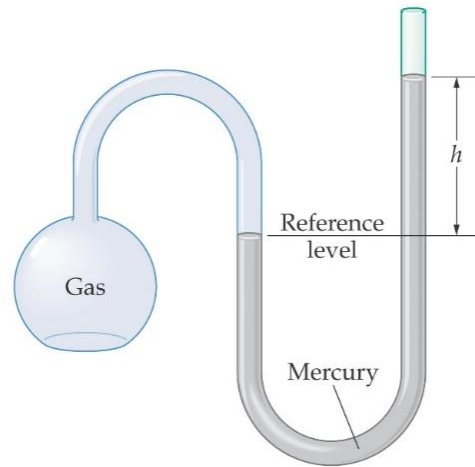


Figure 36. 2. A constant volume gas thermometer that measures temperature by maintaining a fixed gas volume and observing pressure changes using a flexible tube for height adjustment.

A constant volume gas thermometer (fig 36.2) measures temperature of a gas by keeping its volume fixed and observing the pressure changes, which are calculated using the formula:

$$P = \rho \times g \times h$$

where P is pressure, ρ is the density of the liquid, g is the acceleration due to gravity, and h is the height. Here the reference level is kept fixed by raising or lowering the tube on the right side (the tube below is made of rubber or some flexible material). So, the gas volume is fixed, and the pressure is known from measuring h.

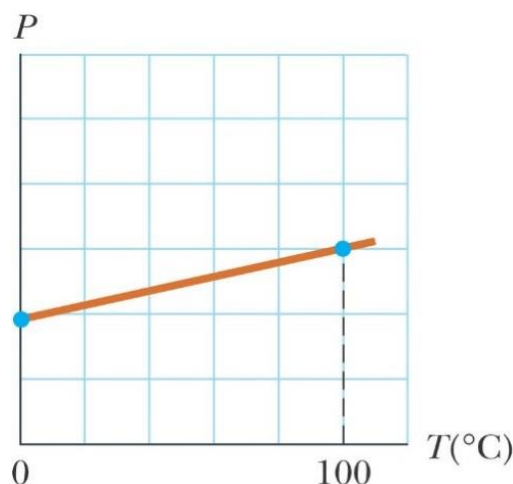


Figure 36. 3. A pressure vs. temperature graph shows a direct linear relationship, forming a straight line from 0 to 100 degrees Celsius.

Absolute Zero

There is a temperature below which it is not possible to go. In other words, if you cool and cool there comes a point after which you cannot cool anymore! If we draw P vs T graph for different gases as illustrated in fig 36.4, we observe that all the gases converge at a single point (-273.15°C) on a temperature graph, known as absolute zero (0°K or zero-degree Kelvin). This is the point where gases theoretically have zero kinetic energy.

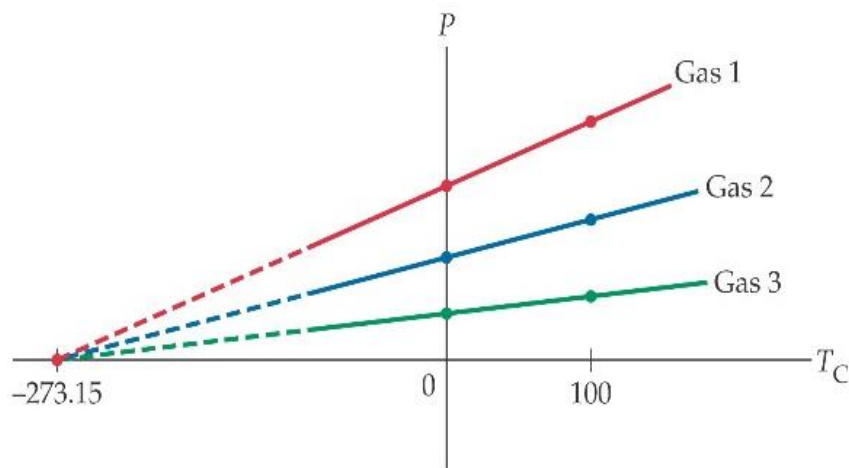


Figure 36. 4. A Pressure vs Temperature graph for different gases shows convergence at -273.15°C , known as absolute zero (0 K).

Temperature Scales

Centigrade, Fahrenheit, and Kelvin are three different temperature scales that use water as a reference due to its global abundance. For instance, water freezes at (0°C), which is equivalent to (273 K) and (32°F), and boils at (100°C), (373 K), or (212°F) as shown in fig 36.5 (a).

Temperature Conversion

Conversion between degrees Celsius and degrees Fahrenheit

$$T_F = \left(\frac{9}{5} F^{\circ} / C^{\circ}\right) T_C + 32^{\circ} F$$

Conversion between degrees Fahrenheit and degrees Celsius

$$T_C = \left(\frac{5}{9} C^{\circ} / F^{\circ}\right) (T_F - 32^{\circ} F)$$

Conversion between Celsius and Kelvin temperatures

$$T = T_C + 273.15$$

Temperature Extremes

While absolute zero is the lowest possible temperature, the maximum temperatures in the universe are found inside stars, reaching up to 10 million Kelvin (see fig 36.5 (b)). Star surfaces, including our Sun, range from 5000 K to 50,000 K, with hydrogen bombs producing even higher temperatures. Metals melt around 1000 °C, and gases like nitrogen, hydrogen, and helium liquefy below 0 °C. The lowest temperature achieved on Earth is approximately 10^{-7} degrees Kelvin (0.00001 K). The **lowest temperatures are achieved using cryogenic refrigeration**, approaching but not reaching absolute zero.

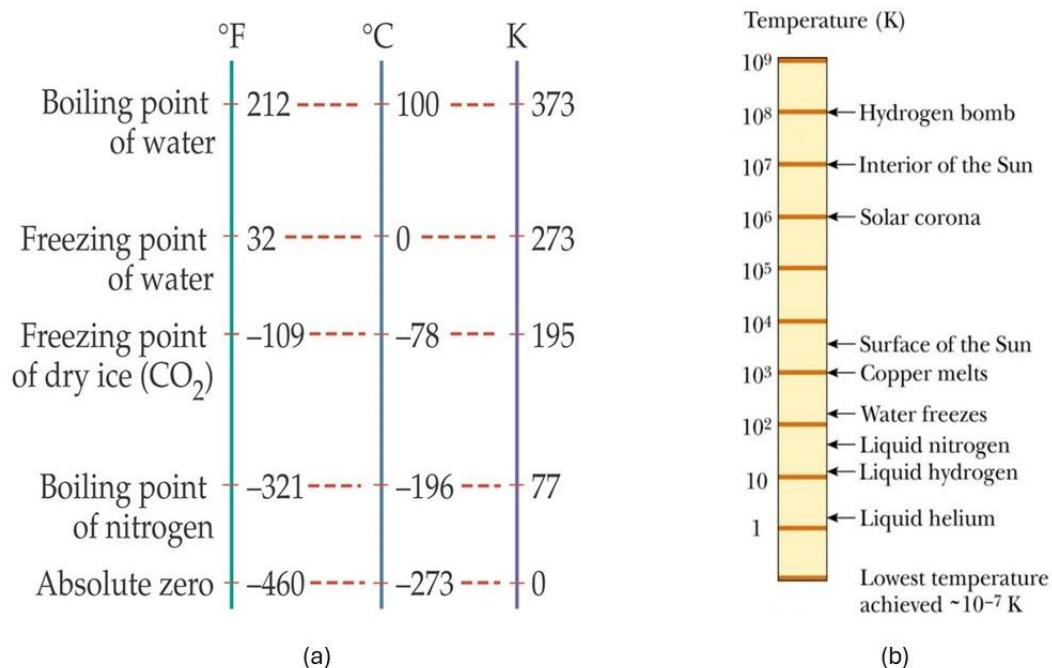


Figure 36. 5. (a) Centigrade, Fahrenheit, and Kelvin scales equivalence. (b) temperature ranges of various phenomena.

Thermometric Properties: Apart from expanding a substance and increasing gas pressure heating affects physical properties like **electrical conductivity and resistivity**, which can also be used to measure temperature. For example, a heated wire's increased resistivity indicates its temperature rise.

Joule's Experiment

A historical experiment demonstrated that *heat is a form of energy generated by work*. In the experiment, a falling weight tied to a rope moved a paddle in water, heating it as shown in fig 36.6. The greater the fall, the hotter the water became. Lifting the weight increased its potential energy, which was then converted to heat energy, not kinetic energy. This disproves the old

belief that heat was a fluid called "phlogiston." Joule's experiment showed that work and heat are equivalent. Similarly, rubbing your hands together generates heat because the work done increases molecular activity.

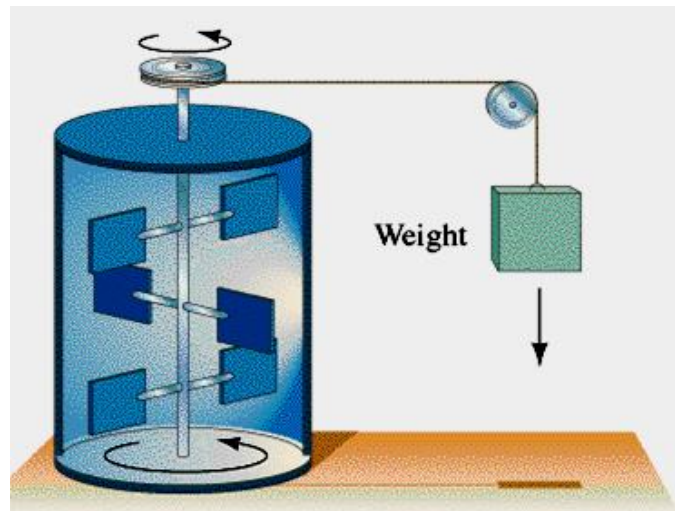


Figure 36. 6. Joule's experimental setup demonstrates that mechanical work converts into heat energy, validating the work-heat equivalence principle.

The unit of heat, the **calorie**, is defined as *the amount of heat required to raise the temperature of 1 gram of water by 1 °C*.

One kilocalorie (1 kcal) = 1000 calories

Joule's experiment also established that **1 calorie = 4.186 joules**, known as the **mechanical equivalent of heat**. This concept helps us understand that what we perceive as heat in an object is actually its internal energy.

Internal Energy:

Internal energy is a measure of the random motion of molecules within a substance, such as water or gas. The faster these molecules move, the higher the internal energy. For gases, this internal energy or kinetic energy is directly proportional to the temperature (T), described by the equation

$$K.E = \frac{3}{2}Nk_B T$$

where N is the number of molecules and k_B is the Boltzmann constant. Joule's experiments showed that work can be converted into heat, increasing temperature.

Microwave ovens are common household appliances that heat food without heating themselves. They work by emitting microwaves at frequencies that resonate with water

molecules, causing them to vibrate and heat up. This selective heating allows water to warm while other materials, like a candle, remain unaffected. The microwaves' frequency is specifically tuned to the natural frequency of water molecules, enabling efficient energy transfer. This shows that understanding heat at a microscopic level reveals many other phenomena.

Thermal Expansion:

When we heat a piece of iron, it expands due to the increased vibration of its molecules, arranged in a lattice structure. Mathematically, the increase in length (ΔL) is proportional to the temperature change (ΔT). Mathematically, this relationship is expressed as

$$L = L_0 [1 + \alpha (T - T_0)]$$

where L_0 is the original length, L is the new length T_0 is the original temperature, and α is the coefficient of linear expansion.

$$L - L_0 = \alpha L_0 (T - T_0)$$

$$\Delta L = \alpha L_0 \Delta T$$

$$\alpha = \frac{\Delta L / L_0}{\Delta T}$$

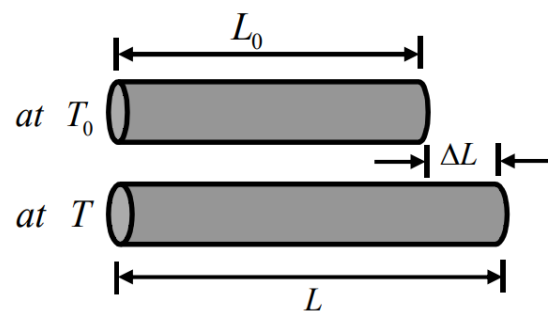


Figure 36. 7. Linear thermal expansion of an iron rod.

Thus, the **coefficient of linear expansion** α is defined as *the ratio of change in length of a material to the original length per degree change in temperature*. The units of α are per degree Celsius $^{\circ}\text{C}^{-1}$ or per Kelvin K^{-1} , as the temperature change can be measured in either Celsius or Kelvin, and the change in length is a ratio, making it dimensionless.

The same principle applies to volume expansion. The coefficient of volume expansion β relates to how much a substance expands in three dimensions. The **coefficient of volumetric expansion**, β , is defined as *the ratio of the change in volume of a material to the original volume per degree change in temperature*.

$$\beta = \frac{\Delta V / V_0}{\Delta T}$$

Relation between α and β

If we consider a cube of volume V_0 as shown in fig 36.8, length L of each side increases due to thermal expansion, and the new volume V' is:

$$V' = (L + \Delta L)^3 = (L + \alpha L \Delta T)^3 \quad \therefore \text{Volume of cube} = (\text{length})^3$$

$$= L^3 + 3\alpha L^3 \Delta T + 3\alpha^2 L^3 \Delta T^2 + \alpha^3 L^3 \Delta T^3 \quad (a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

If ΔT is small, we can ignore ΔT^2 and ΔT^3 . Hence, we get

$$\approx L^3 + 3\alpha L^3 \Delta T$$

$$= V_0 + 3\alpha V_0 \Delta T$$

$$\Rightarrow \Delta V = 3\alpha V_0 \Delta T \quad \therefore \beta = \frac{\Delta V / V_0}{\Delta T}$$

$$\Rightarrow \boxed{\beta = 3\alpha}$$

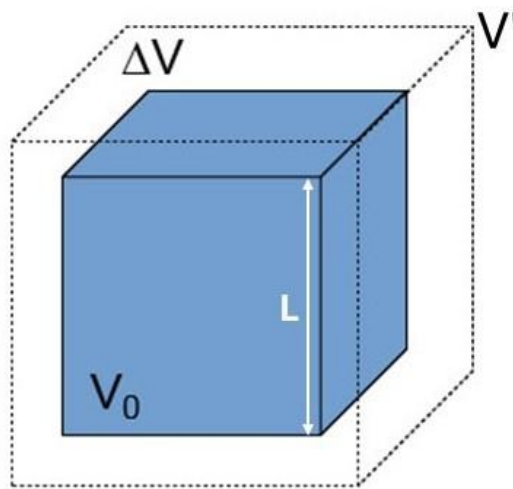


Figure 36. 8. Volume thermal expansion of a cube.

Different materials expand at different rates (fig 36.9). For example, air has a volume expansion coefficient (β) of about 0.001, while iron's is 5×10^{-4} . Materials like glass and quartz expand even less.

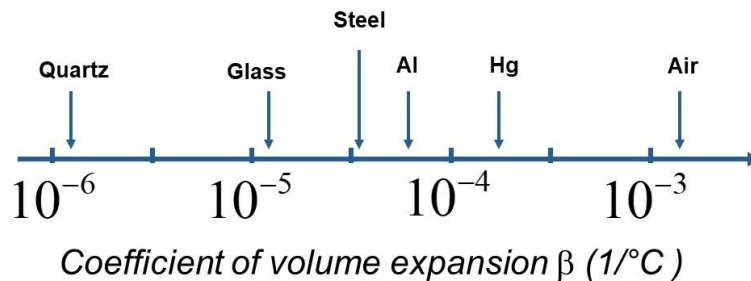


Figure 36. 9. Rates of thermal expansions of different materials.

Application of Thermal Expansion

The property of thermal expansion is utilized in thermostats as shown in fig 36.10, where a bi-metallic strip made of two different metals expands at different rates, breaking the circuit at a certain temperature (25-30 °C) by distorting the shape to control devices like fridges or ovens.

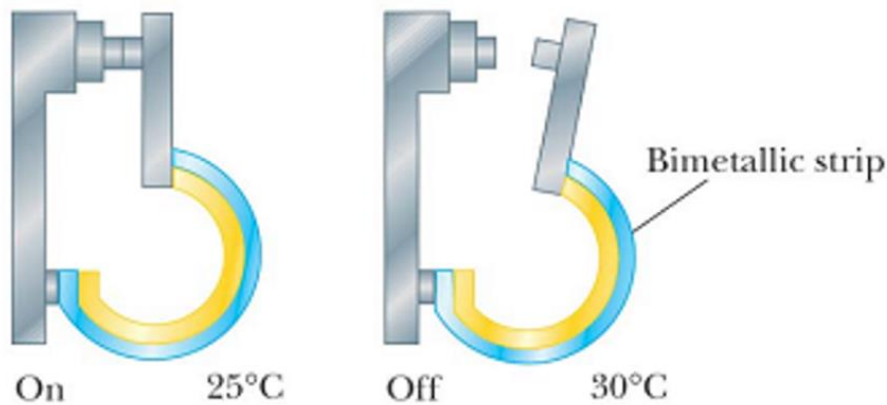


Figure 36. 10. Thermal expansion thermostat bi-metallic strip, bending at specific temperatures to break the circuit.

Anomalous Behavior of Water

When water is heated from 0 °C to 4 °C, it becomes denser, meaning its volume decreases. This anomalous behaviour is due to the complex interactions between H₂O molecules. As water temperature increases beyond 4 degrees, it behaves normally, expanding with temperature (see fig 36.11).

This property explains why, in a frozen lake, the densest water (at 4 degrees) sinks while warmer water rises. Ice, being colder than 0°C, forms on the lake's surface, while the water beneath remains at a relatively stable 4°C providing a safe environment for fish. The constant 4°C temperature below the ice ensures that fish can survive through winter. Both in summer and winter, the water at this depth maintains this temperature, providing a stable habitat. However, if the temperature fluctuates significantly, it can be detrimental to the fish, potentially leading to their demise.

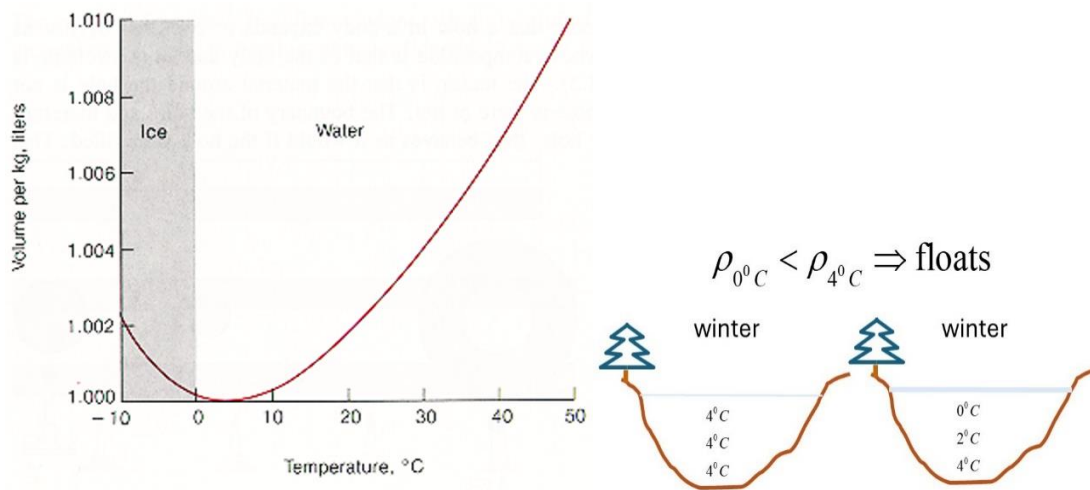


Figure 36. 11. Volume Vs Temperature graph showing anomalous Behavior of Water.

Heat Capacity

Material properties such as conductivity and resistivity are indeed affected by temperature changes. Additionally, materials can expand or contract with temperature variations. **Heat capacity** is another crucial property, representing ‘**the amount of heat required to raise the temperature of 1 gram of a substance by 1 K or 1 °C**’. It's mathematically defined as the heat absorbed (Q) per unit increase in temperature (ΔT), given by the formula:

$$C = \frac{Q}{\Delta T}$$

Unit of heat capacity is $J/K = J/^{\circ}C$. Heat Q and ΔT have the same sign:

- Q is positive if ΔT is positive, heat is added to a system
- Q is negative if ΔT is negative, heat is removed from a system

Problem: A 0.5-kg block of metal with an initial temperature of $30.0^{\circ}C$ is dropped into a container holding 1.12 kg of water at $20.0^{\circ}C$. If the final temperature of the block-water system is $20.4^{\circ}C$, what is the specific heat of the metal?

Solution:

Mass of the water = $m_w = 1.12 \text{ kg}$

Mass of the block = $m_b = 0.5 \text{ kg}$

Specific heat of water = $c_w = 4186 \text{ J / (kg} \cdot \text{K)}$

Initial temperature of the water = $T_w = 20.0^{\circ}C$

Initial temperature of the block = $T_b = 30.0^\circ\text{C}$

Final equilibrium temperature = $T = 20.4^\circ\text{C}$

The heat lost by the metal block = $Q_{\text{block}} = m_b c_b (T_b - T)$

Heat gained by the water = $Q_{\text{water}} = m_w c_w (T - T_w)$

According to law of conservation of energy,

Heat lost = Heat gained

$$\begin{aligned} Q_{\text{block}} &= Q_{\text{water}} \\ m_b c_b (T_b - T) &= m_w c_w (T - T_w) \\ c_b &= \frac{m_w c_w (T - T_w)}{m_b (T_b - T)} \end{aligned}$$

By substituting the given values, we find that the specific heat of the metal block

$$\begin{aligned} &= \frac{(1.12\text{kg})[4186\text{J} / (\text{kg} \cdot \text{K})](20.4^\circ\text{C} - 20.0^\circ\text{C})}{(0.500\text{kg})(30.0^\circ\text{C} - 20.4^\circ\text{C})} \\ &= 391\text{J} / (\text{kg} \cdot \text{K}) \end{aligned}$$

This method allows us to determine the specific heat capacities of various materials.

Specific Heat

Specific heat is a property that indicates a material's heat capacity, quantified in $\text{J}/(\text{kg}\cdot\text{K})$. It reflects the internal energy dynamics of the material, determining its ability to store heat. Here's a table showing the specific heats of different materials at atmospheric pressure.

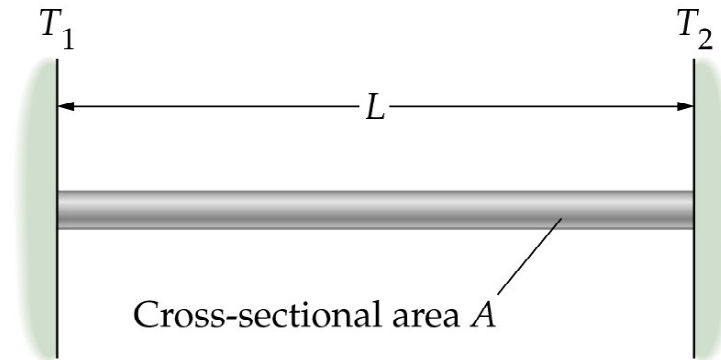
Specific Heats at Atmospheric Pressure

Substance	Specific heat, c [$\text{J} / (\text{kg} \cdot \text{K})$]
Water	4186
Ice	2090
Steam	2010
Beryllium	1820
Air	1004
Aluminum	900
Glass	837
Silicon	703
Iron (steel)	448
Copper	387
Silver	234
Gold	129
Lead	128

Heat Transfer

Heat transfer occurs in three primary ways:

1. **Conduction** involves the transfer of heat through the vibration and collision of atoms within a material, like metal. For instance, in a metal rod heated at one end, the heat travels along the rod as atoms vibrate and collide.



2. **Convection** involves the physical movement of atoms and molecules within fluids, where heated parts move away, allowing cooler parts to take their place. For example, when a match burns, the hot air rises, cools, and then falls, creating a circulation pattern that transfers heat.
3. **Radiation** transfers heat through electromagnetic waves. A microwave oven heats water by emitting microwaves that excite the water molecules, generating heat without any physical contact.

These fundamental concepts are essential for understanding how heat interacts with different materials and environments. The principles of physics are significant due to their broad range of applications. For instance, the concepts of convection and radiation are not only fundamental to understanding heat transfer but also explain stellar phenomena. In the Sun, the core's immense temperature of 10 million degrees kelvin generates heat that rises to the surface through convection currents. Once at the surface, the heat is emitted as radiation, traveling across space to reach us. This is just one example that heat, and thermal energy play a crucial role in the universe.

Lecture 37 – THERMAL PHYSICS II

Introduction to Thermodynamics

Thermodynamics is the study of heat and how it flows. This lecture will cover the laws of thermodynamics, which are essential to physics. Despite the relatively recent discovery of atoms and molecules a century ago, the principles of thermodynamics remain unchanged and deeply rooted in our understanding of matter. Interestingly, we now interpret heat as the movement of atoms and molecules. As previously discussed, exerting work on an object increases its temperature due to the accelerated, random motion of its particles (atoms and molecules). Notably, the foundational thermodynamic laws were established without knowledge of these microscopic constituents, yet they still yield accurate predictions.

System: The specific part of the universe being studied, which can be a quantity of matter or a region in space.

Surroundings: Everything outside the system that can interact with it.

State Variables: The quantities that define the state of a thermodynamic system at equilibrium depending only on the current state of the system, not on how the system reached that state. Examples include pressure (P), volume (V), temperature (T), and internal energy (U).

Thermodynamic Variables: This is a broader term that encompasses state variables and may include other quantities used in thermodynamic analyses, such as heat (Q) and work (W). Unlike state variables, heat and work are path-dependent and not properties of the system's state.

Equation of State: The mathematical relation between these variables. Consider a gas with N atoms or molecules. The gas has a volume V , pressure P , and temperature T . These are thermodynamic variables but not independent of each other. For example, in a container with N molecules, if the volume increases, the pressure decreases (**Boyle's law:** P is inversely proportional to V) with constant temperature. Alternatively, under constant pressure, if the temperature increases, the volume increases (**Charles's law:** V is proportional to T). These relationships form the equation of state, where pressure is a function of volume and temperature, and vice versa.

$$P = f(V, T)$$

$$V = g(P, T)$$

$$T = h(P, V)$$

The equation of state for an ideal gas is given by:

$$PV = Nk_B T$$

where k_B is Boltzmann's constant, T is the absolute temperature measured in Kelvin (Kelvin = Celsius + 273.15).

Ideal Gas: An ideal gas consists of widely spaced atoms or molecules that rarely collide with each other, allowing for mostly free movement. They do collide with the container walls, but inter-particle collisions are minimal. This concept is crucial for understanding thermodynamic relationships.

Internal Energy: It is the average kinetic energy of a system's atoms and molecules (fig 37.1), which increases with temperature. It can be expressed mathematically,

$$E = \frac{3}{2} Nk_B T$$

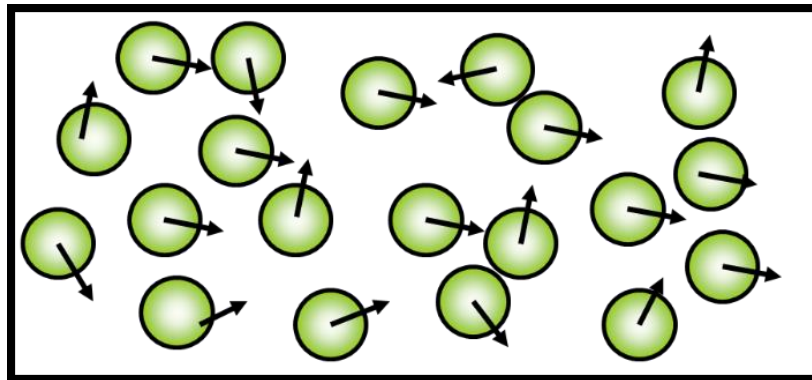


Figure 37. 1. Random motion of gas molecules contributing to the internal energy of the system.

First Law of Thermodynamics

The first law of thermodynamics—a statement of energy conservation—states that when work is done on a body and it is heated, the internal energy of the system increases. Mathematically, the increase in internal energy (ΔE) equals the heat added (ΔQ) plus the work done (ΔW), expressed as

$$\Delta E = \Delta Q + \Delta W$$

- ΔQ and ΔW are positive if heat is added to the system or work is done on a system.
- ΔQ and ΔW are negative if heat is removed from the system or work is done by a system.

This law highlights that while internal energy does not depend on the path but work and heat are path variables – their values depend on the steps leading from one state to another.

Consider a closed system with a piston, shown in fig 37.2: it's sealed except for a movable piston, containing an ideal gas. When the piston moves, it alters the system's volume without allowing gas to escape, demonstrating the relationship between work, heat, and internal energy in a controlled environment.

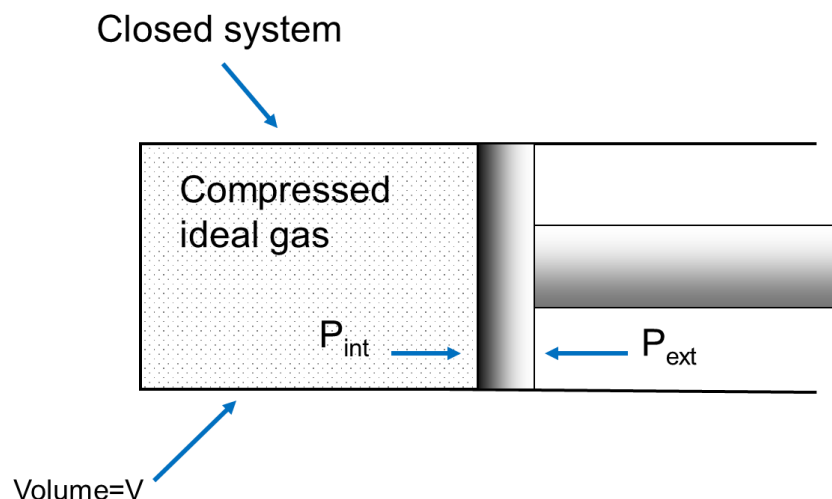


Figure 37. 2. Closed system with a movable piston, changing the volume of an ideal.

To determine the work done by a gas during expansion, If the external pressure is P_{external} , the work done by the gas on the piston, dW , is given by:

$$dw = - P_{\text{external}}. dV \quad \therefore w = F.d = (PA)d = P(Ad) = PV$$

where dV is the change in volume. The negative sign indicates that the work is done by the gas on the surroundings.

- When internal and external pressures are equal, the piston is in equilibrium.
- If $P_{\text{internal}} > P_{\text{external}}$, the gas expands, and
- If $P_{\text{internal}} < P_{\text{external}}$, it compresses.

For total work, we integrate $P_{\text{external}} \cdot dV$ from the initial to the final volume.

$$w = - \int_{V_1}^{V_2} P_{\text{ext}} dV$$

Case I: $P_{\text{external}} = 0$ (a vacuum)

No work is done by the gas on the piston as it expands, since work is a product of force and displacement, and with zero external pressure, there's no force opposing the expansion. Consequently, if no heat is exchanged ($\Delta Q = 0$), the system's internal energy remains unchanged (so, $\Delta E = 0$). This is consistent with the first law of thermodynamics, which relates changes in internal energy to heat and work.

Case II: $P_{\text{external}} = \text{constant}$

With constant external pressure P_{external} , the work done by the gas during expansion is given by

$$W = - P_{\text{external}} \cdot \Delta V$$

where ΔV is the change in volume (i.e., $V_{\text{final}} - V_{\text{initial}}$). The negative sign signifies that the gas is doing work on the surroundings (gas expands against the external pressure).

Case II: $P_{\text{external}} \approx P_{\text{internal}}$

Assume P_{external} and P_{internal} to be the same for calculation purposes. Using the ideal gas law ($PV = NkT$), the work done during an isothermal process (constant temperature) is calculated by integrating the pressure with respect to volume

$$P_{\text{ext}} = \frac{NkT}{V}$$

$$w = - \int_{V_1}^{V_2} \frac{NkT}{V} dV = -NkT \int_{V_1}^{V_2} \frac{dV}{V} = -NkT (\ln V_2 - \ln V_1) = -NkT \ln \frac{V_2}{V_1}$$

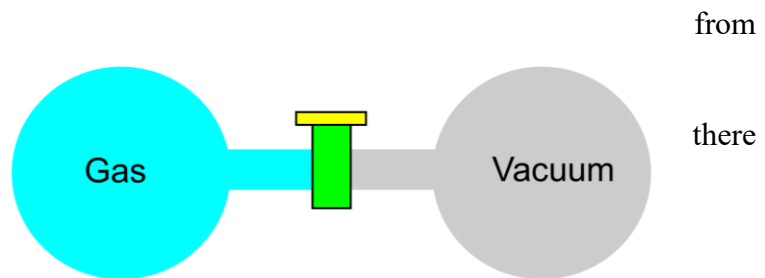
where V_1 and V_2 are the initial and final volumes, respectively.

These cases illustrate that the work done depends on the path taken to reach a particular state of the gas:

1. Free expansion: No work is done ($w = 0$).
2. Constant external pressure: Work is $- P_{\text{external}} \Delta V$.
3. Isothermal Process with $P_{\text{external}} \approx P_{\text{internal}}$: Maximum work is $w = -NkT \ln \frac{V_2}{V_1}$.

Thus, the amount of work is path-dependent, and asking for the work done by the gas without specifying the process is incorrect. The internal energy, however, is a state function and remains the same regardless of the path taken.

When a valve separating a gas from a vacuum is opened, the gas expands without doing work, as there is no external pressure to work against. Since it is an isolated system undergoing free expansion, no external work is done ($\Delta W = 0$) and no heat is exchanged ($\Delta Q = 0$), so the internal energy remains constant ($\Delta E = \Delta Q + \Delta W = 0$). Initially, if the energy is $\frac{3}{2}NkT$, it stays the same after expansion.



PV diagrams

PV diagrams are helpful in visualizing this process, where the area under the curve represents the work done. For a constant pressure process (see fig. 37.3), the work done is the product of the pressure and the change in volume ($W = P \Delta V$).

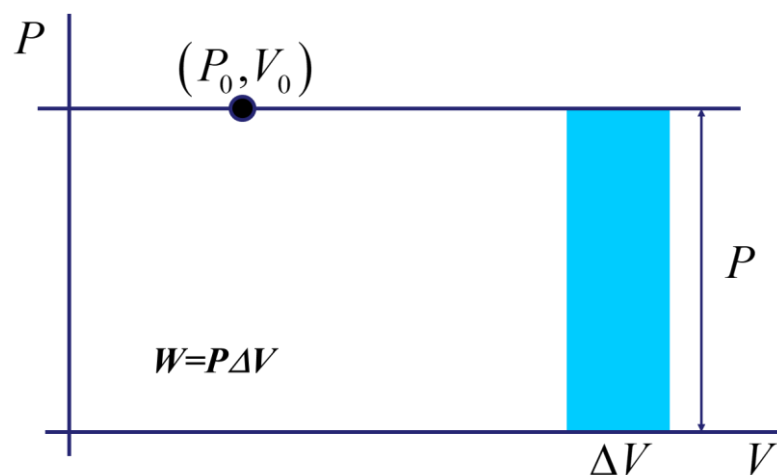


Figure 37. 3. PV diagram for constant pressure process.

Path-Dependence of Work

The work done can be different depending on the path taken. For example, if we move from an initial state P_1V_1 to a final state P_2V_2 via two different paths, the work done will vary. In an isothermal expansion, where temperature remains constant, the work done is determined as follows:

Path A: Firstly, the volume increases from V_1 to V_2 at constant pressure P_1 . Then, the pressure decreases to P_2 at constant volume V_2 , shown in fig 37.4 (path A). Since the temperature remains constant, there's no change in internal energy. Thus, the work done is pressure (starting pressure) times the change in volume.

$$W = P_1(V_2 - V_1)$$

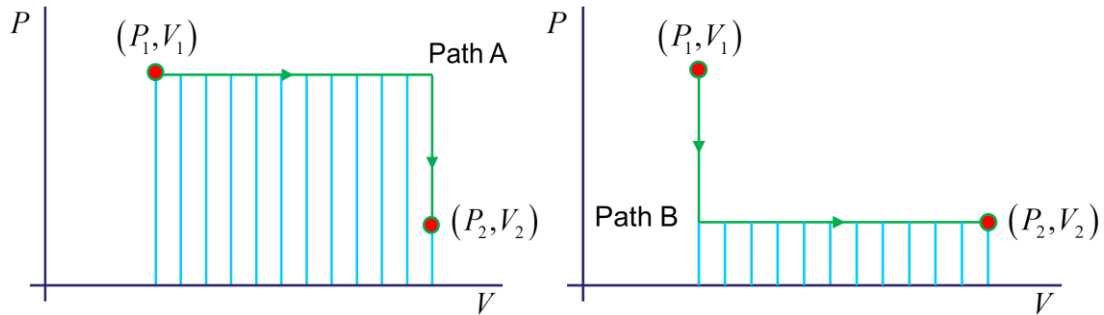


Figure 37. 4. PV diagrams showing path dependence of work

Path B: Exploring alternative paths (Path B in fig 37.4) in thermodynamics demonstrates the path-dependence of work done on or by a gas. For instance, on path B, First, decrease the pressure from P_1 to P_2 at constant volume V_1 . Then, increase the volume from V_1 to V_2 at constant pressure P_2 , the work done is

$$W = P_2(V_2 - V_1)$$

This contrasts with the previous path where the work was $W = P_1(V_2 - V_1)$. This shows that work done is indeed path-dependent, and the signs of work done on or by the gas are opposite.

Path C: For the third path (fig 37.5), which adheres to the ideal gas law $PV = NkT$, the work done is calculated as:

For an isothermal process,

$$PV = Nk_B T$$

$$dW = PdV$$

$$= Nk_B \frac{T}{V} dV$$

$$W = \int_{V_1}^{V_2} PdV = \int_{V_1}^{V_2} \frac{Nk_B T}{V} dV$$

$$W = Nk_B T \ln \frac{V_2}{V_1}$$

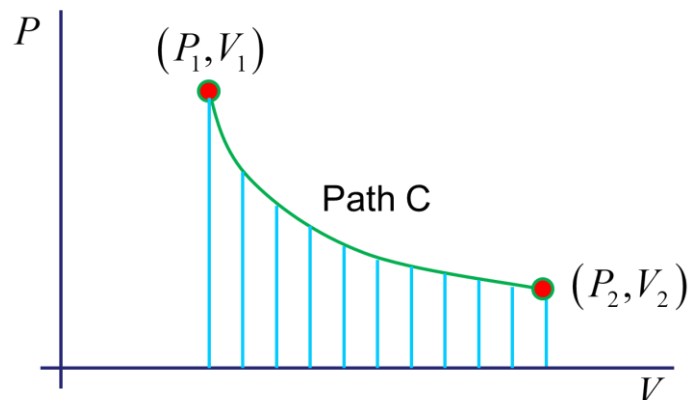


Figure 37. 5. Path C showing variable change in pressure and volume.

Specific Heat

Now, let's calculate the heat capacity of a gas, it's a measure of the amount of heat required to change the temperature of a system. There are two ways to define specific heat: constant volume (C_V) and constant pressure (C_P).

Specific Heat at Constant Volume

It (C_V) is the amount of heat required to raise the temperature of a unit mass of a substance by one degree Celsius (or one Kelvin) while keeping its volume constant. For an ideal gas at constant volume (C_V),

$$C_V = \frac{Q_V}{\Delta T}$$

$$Q_V = C_V \Delta T$$

As the volume remain constant, $dV = 0$, Hence,

$$W = 0$$

$$Q_V = \Delta E + W = \Delta E$$

$$\Delta E = C_V \Delta T$$

As $\Delta T \rightarrow 0$

$$dE = C_V dT$$

$$C_V = \frac{dE}{dT} \quad \therefore E = \frac{3}{2} NkT$$

$$C_V = \frac{d}{dT} \left(\frac{3}{2} NkT \right) = \frac{3}{2} Nk$$

This is a fundamental result in thermodynamics, reflecting the intrinsic energy changes within a gas due to temperature variations.

Specific Heat at Constant Pressure

C_P represents the heat capacity of a gas when it is allowed to expand under constant pressure. This means the system can do work on its surroundings as it heats up. C_P is defined as the heat absorbed Q_P divided by the change in temperature ΔT .

$$C_P = \frac{Q_P}{\Delta T}$$

$$Q_P = C_P \Delta T$$

Let's calculate the specific heat at constant pressure C_P using the first law of thermodynamics,

$$Q_p = \Delta E + W = \Delta E + P\Delta V$$

$$C_p \Delta T = \Delta E + P\Delta V$$

$$\text{As } \Delta T \rightarrow 0$$

$$C_p dT = dE + PdV$$

Expressing dE in terms of C_v

$$C_p dT = C_v dT + PdV$$

$$\text{As } PV = Nk_B T$$

Differentiating w.r.t T

$$PdV + VdP = Nk_B dT \quad \text{At const P (dP=0)}$$

$$PdV = Nk_B dT$$

Substitute into the first law

$$C_p dT = C_v dT + Nk_B dT$$

$$C_p = C_v + Nk_B$$

$$E = \frac{3}{2} Nk_B T$$

$$C_v = \frac{dE}{dT} = \frac{3}{2} Nk_B$$

$$C_p = \frac{3}{2} Nk_B + Nk_B$$

$$C_p = \frac{5}{2} Nk_B$$

$$C_p - C_v = \frac{5}{2} Nk_B - \frac{3}{2} Nk_B = Nk_B$$

This shows that C_p is indeed greater than C_v for an ideal gas, by an amount equal to Nk_B , which is the number of particles multiplied by Boltzmann's constant. The difference arises because at constant pressure, the gas does work on the surroundings as it expands, requiring additional energy input to raise the temperature by the same amount compared to a constant volume scenario.

The behavior of an ideal gas can become complex when considering real molecules that have rotational and vibrational modes. For diatomic or polyatomic molecules, such as water (H_2O), carbon dioxide (CO_2), or hydrogen (H_2), their rotational motions contribute to their energy and, consequently, to their specific heat capacities displayed in fig 37.6.

The specific heat at constant volume C_V for a monatomic gas is $\frac{3}{2} nK$, accounting for translational motion along the x, y, and z axes. For diatomic or polyatomic gases, which can rotate, additional degrees of freedom contribute to the specific heat. For example, a diatomic molecule such as CO_2 has two additional rotational degrees of freedom.

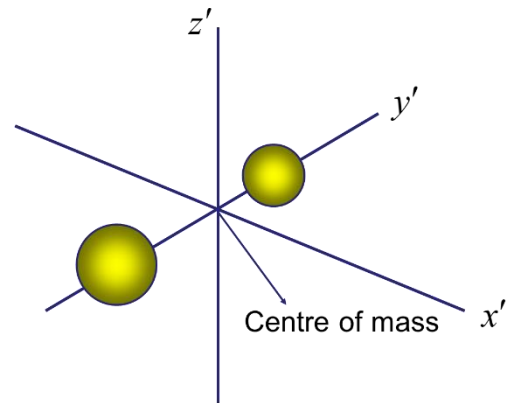


Figure 37. 6. Diatomic molecule having translational and rotational degrees of freedom.

$$K = \frac{1}{2}mv_x^2 + \frac{1}{2}mv_y^2 + \frac{1}{2}mv_z^2 + \frac{1}{2}I_x\omega_x^2 + \frac{1}{2}I_y\omega_y^2$$

$$E = 5 \times \left(\frac{1}{2} Nk_B T \right) = \frac{5}{2} Nk_B T$$

$$C_V = \frac{5}{2} Nk_B$$

Therefore, molecules that can rotate have a higher specific heat.

Adiabatic Process

So far, we've discussed gas expansions under isothermal (constant temperature) and constant pressure conditions. The third type of expansion is adiabatic, where no heat is exchanged with the surroundings ($dQ = 0$). An isolated system executing adiabatic expansion is shown in fig 37.7.

$$dQ = dE + dW = 0 \quad \text{or} \quad dE = -dW$$

This means any work done by the gas reduces its internal energy. For an adiabatic process, the relationship between pressure and volume is given by

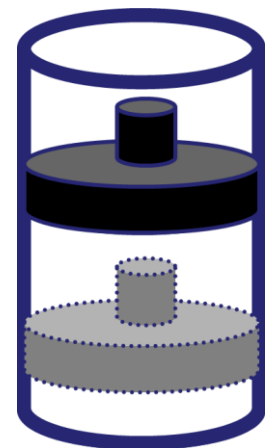


Figure 37. 7. An isolated system in which no heat enters or leave.

$$C_V dT + PdV = 0$$

$$C_V dT + Nk_B T \frac{dV}{V} = 0 \quad \therefore P = \frac{Nk_B T}{V}$$

$$\frac{dT}{T} + \frac{Nk_B}{C_V} \frac{dV}{V} = 0 \dots \dots \dots (1)$$

$$\text{As } C_P - C_V = Nk_B$$

$$\frac{Nk_B}{C_V} = \frac{C_P - C_V}{C_V} = \frac{C_P}{C_V} - 1$$

$$\text{Define } \frac{C_P}{C_V} \equiv \gamma$$

$$\frac{Nk_B}{C_V} = \gamma - 1, \text{ put in eq (1) and integrate}$$

$$\frac{dT}{T} + (\gamma - 1) \frac{dV}{V} = 0$$

$$\ln T + (\gamma - 1) \ln V = \text{Constant}$$

$$\ln(TV^{\gamma-1}) = \text{Constant}$$

$$TV^{\gamma-1} = \text{Constant}$$

$$\frac{PV}{Nk_B} V^{\gamma-1} = \text{Constant} \quad \therefore PV = Nk_B T \rightarrow T = \frac{PV}{Nk_B}$$

$$PV^\gamma = \text{Constant}$$

For an ideal gas undergoing an adiabatic expansion from an initial state $P_1 V_1$ to a final state $P_2 V_2$, the work done is:

$$dQ = dE + dW = 0$$

$$C_V dT + dW = 0 \Rightarrow dW = -C_V dT$$

$$W_{\text{adiabatic}} = dW = -\int C_V dT = -C_V \Delta T$$

$$W_{\text{adiabatic}} = -C_V \Delta T = C_V (T_1 - T_2)$$

$$W_{\text{adiabatic}} = C_V \left(\frac{P_1 V_1}{Nk_B} - \frac{P_2 V_2}{Nk_B} \right) = \frac{P_1 V_1 - P_2 V_2}{\gamma - 1}$$

This is a direct application of the first law of thermodynamics, which is fundamentally the law of conservation of energy. It states that the change in internal energy of a system is equal to the heat added to the system plus the work done on the system.

Second Law of Thermodynamics

The second law of thermodynamics asserts that *there can be no process whose only final result is to transfer thermal energy from a cooler object to a hotter object*. This principle implies that it's impossible to extract work from a system solely by transferring heat from a colder to a hotter body.

Applications of Second Law of Thermodynamics

This principle has numerous applications, particularly in engines. Fig 37.8 shows a **steam engine** in which water is heated in a boiler until it becomes steam. This steam then expands and pushes a piston, which in turn drives a wheel, converting the steam's thermal energy into mechanical work. After doing work, the steam is condensed back into water, which can be heated again, creating a cycle.

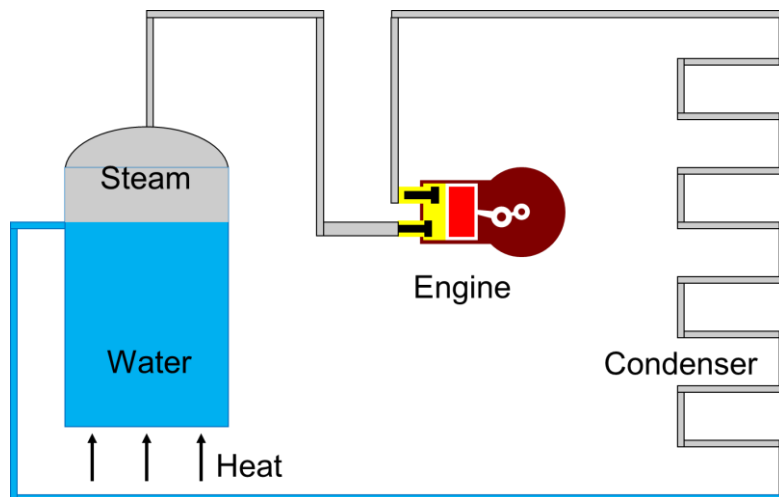


Figure 37. 8. Steam engine

Similarly, **internal combustion engines** (fig 37.9) in motorcycles and cars function on the same basic principle. Gas enters the engine, is compressed, and ignited with a spark (ignition stroke). The piston then moves out (power stroke) and finally, the exhaust stroke occurs.

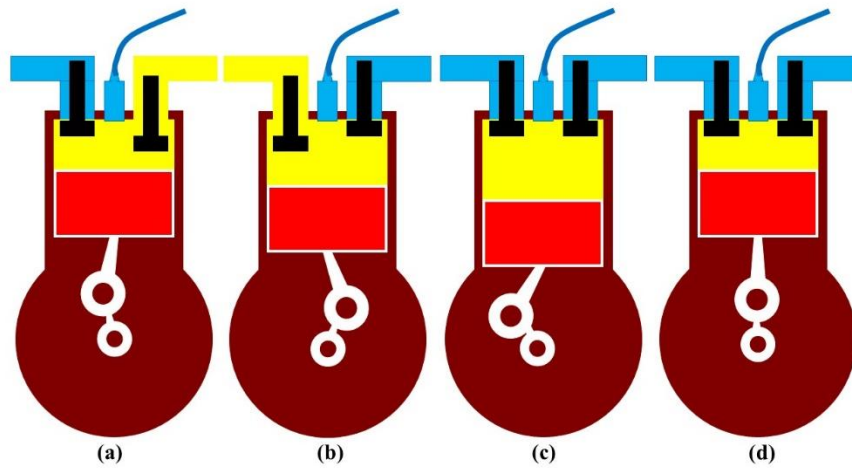


Figure 37. 9. Internal combustion engine in vehicles demonstrating (a) compression stroke (b) ignition stroke (c) power stroke (d) exhaust stroke

These engines operate between a high temperature (over 100°C for steam, higher for diesel or petrol during ignition) and a low temperature ($30\text{--}40^{\circ}\text{C}$ for exhaust release). Heat, Q_h , is taken from the high temperature reservoir T_h , allowing the engine to perform work. Not all heat converts to work; some is rejected as excess heat, Q_c , to the environment, called the cold reservoir at temperature T_c . Fig 37.10 demonstrates the schematics of heat engine.

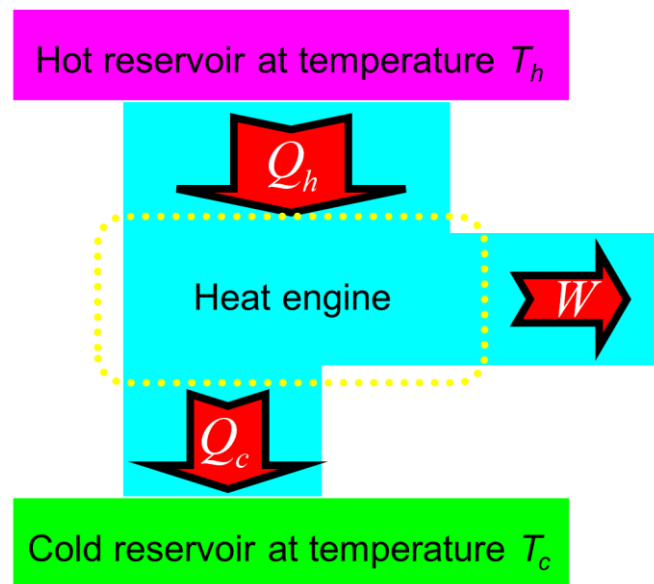


Figure 37. 10. Working mechanism of heat engine.

This follows the first law of thermodynamics:

$$Q = \Delta E + W$$

$$W = Q_{in,h} - Q_{out,c}$$

Efficiency (ϵ) measures the work obtained from heat input: mathematically defined as the work output (W) divided by the heat input (Q_{in}).

$$\epsilon = \frac{W}{Q_{in,h}} = \frac{Q_{in,h} - Q_{out,c}}{Q_{in,h}} = 1 - \frac{Q_{out,c}}{Q_{in,h}}$$

Since $Q_{out,c}$ is proportional to the lower temperature (T_c) and $Q_{in,h}$ to the higher temperature (T_h), efficiency is:

$$\epsilon = 1 - \frac{T_c}{T_h}$$

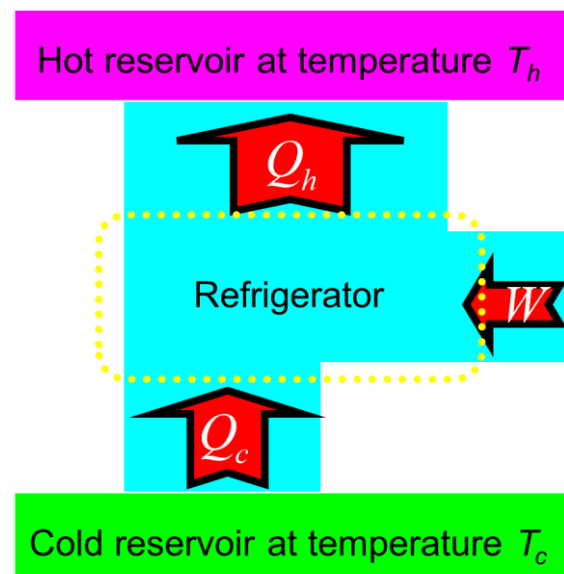
This ensures efficiency is always less than 1, making 100% efficiency impossible.

Refrigerator

A refrigerator operates in reverse compared to a heat engine. While a heat engine converts heat from a high temperature to work and expels some waste heat, a refrigerator removes heat from a low temperature area (like a water bottle inside it) and rejects it at a higher temperature. This process requires external energy, usually electricity.

In a refrigerator, work (W) is input, and heat (Q_c) is removed from inside and rejected as Q_h outside (fig 37.11).

The efficiency of a refrigerator is measured by its coefficient of performance, which is the amount of heat extracted per unit of work input. The second law of thermodynamics states that a refrigerator cannot operate without external work; it is impossible for a refrigerator to transfer heat from a cold to a hot body without additional energy input. Thus, its efficiency will always be less than 1.



Entropy

Entropy, a concept from the second law of thermodynamics, measures disorder in a system.

Mathematically, if you add a small amount of heat (dQ) to a system at temperature (T), the increase in entropy (dS) is:

$$ds = \frac{dQ}{T}$$

Entropy of an ideal gas

For an ideal gas, the state equation is $PV = nKT$. When heat is added, the entropy change can be derived from the first law of thermodynamics:

$$dQ = dE + dW = dE + PdV$$

Substituting P from the ideal gas law,

$$dQ = C_v dT + Nk_B T \frac{dV}{V}$$

$$dS = \frac{dQ}{T} = C_v \frac{dT}{T} + Nk_B \frac{dV}{V}$$

To find the total change in entropy ΔS when the temperature changes from T_1 to T_2 , we integrate:

$$\Delta S = \int \frac{dQ}{T} = C_v \ln \frac{T_2}{T_1} + Nk_B \ln \frac{V_2}{V_1}$$

Physically, as energy is added to a system and temperature increases, molecules move faster (more randomly), increasing disorder and entropy. More order means less entropy, and more disorder means higher entropy.

The second law of thermodynamics states that entropy always increases in an isolated system. Over time, any isolated system, whether gas, liquid, or solid, will naturally become more disordered. For example, imagine gathering all the atoms and molecules scattered in a room and confining them in a small box. This action decreases entropy because the system becomes more ordered. However, if you open the box and allow the atoms and molecules to disperse throughout the room, entropy increases due to the rise in disorder. This concept also links entropy to the flow of time. We never observe atoms and molecules spontaneously gathering in one place, which would decrease entropy. Thus, the increase in entropy is a measure of time's forward progression.

Lecture 38 – THERMAL PHYSICS III

This is the final lecture on heat. Let's review key concepts from the last two lectures. Imagine a hot piece of iron and a cold piece of iron. The atoms in the hot piece are vibrating rapidly, exhibiting more random thermal energy than the cold piece. When combined, heat transfers from the hot piece to the cold piece, not the other way around. This illustrates the second law of thermodynamics, which states that work can only occur when one body is hot and the other is cold, with heat always moving from hot to cold.

Entropy

Entropy is a measure of disorder. When energy transfers from a hot body to a cold one, the system's entropy increases. The change in entropy, dS , is defined as the heat added, dQ , divided by the temperature T ,

$$ds = \frac{dQ}{T}$$

In the previous lecture, we learned to calculate the increase in entropy for an ideal gas. To find the total change in entropy, we integrate from the initial temperature (T_1) to the final temperature (T_2):

$$\Delta S = \int_{T_1}^{T_2} \frac{dQ}{T} = C_v \ln \frac{T_2}{T_1} + Nk_B \ln \frac{V_2}{V_1} \dots\dots\dots(1)$$

Conditions for Entropy Increase:

If the initial and final temperatures or volumes are equal ($T_1 = T_2$ or $V_1 = V_2$), the entropy change is zero because the logarithm of 1 is zero. Thus, entropy increases only when $T_2 > T_1$ or $V_2 > V_1$, or both. The above equation shows that entropy increases with higher temperatures or volumes.

Examples of Increase in Entropy

1. **Matchsticks:** Entropy means disorder, to illustrate this, consider matchsticks in the box. Initially, they are neatly arranged (fig 38.1 (a)), representing low entropy. When scattered, they become disordered, shows increasing entropy (fig 38.1 (b)).

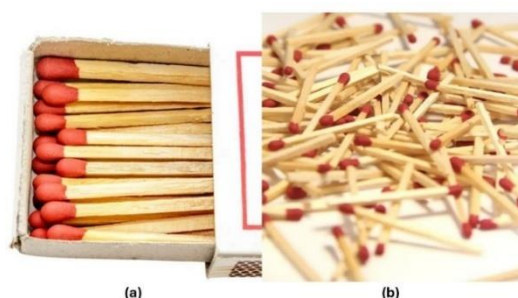


Figure 38. 1. (a) Organized matchsticks (b) scattered matchsticks representing increasing disorder.

2. **Ink in Water:** Another example is dropping ink into water. Initially, the ink is concentrated (low entropy). As it spreads, the disorder increases, raising entropy.
3. **Molecules in a Box:** In the figure 38.2, molecules are confined to one side of a box. They move rapidly, bouncing off walls. When the divider is removed, they spread out, increasing entropy without adding energy showing how entropy naturally increases.

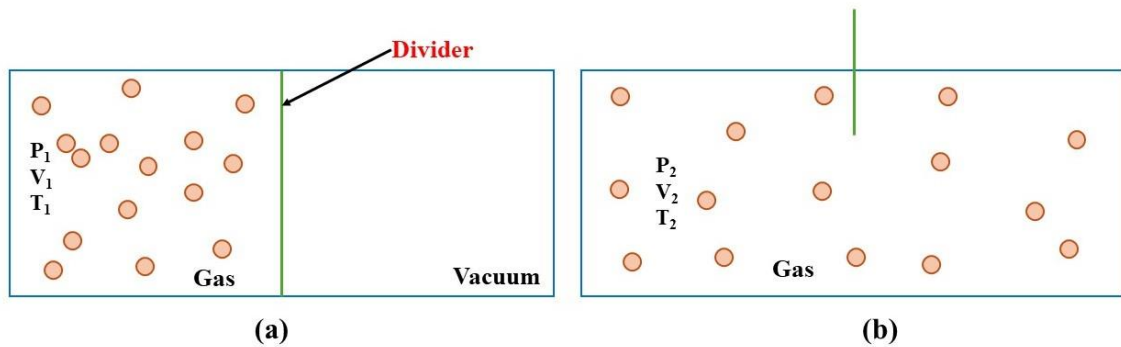


Figure 38. 2. (a) Molecules confined to one side of a box, (b) molecules spread out, illustrating increasing entropy.

The second law of thermodynamics states that in an isolated system, entropy can increase or remain constant but never decrease. For example, when the partition in a box of confined molecules is removed, atoms and molecules spread out, increasing entropy without adding energy. This is reflected as $\ln \frac{V_2}{V_1}$ in the eq (1) where V_2 is greater than V_1 , leading to increased entropy.

Entropy and States of Matter

Now, let's apply entropy in different states of matter:

1. **Solids:** Solids have the least entropy because their atoms are fixed in place, only vibrating slightly around their positions. This high level of order means low entropy.
2. **Liquids:** Liquids have higher entropy than solids because their atoms or molecules can move more freely. The increased movement and ability to flow result in more disorder.
3. **Gases:** Gases have the highest entropy as their atoms or molecules move freely and rapidly in all directions. The high degree of movement and spread leads to the greatest disorder.

A better understanding of these states helps us see how entropy varies with the degree of molecular movement and disorder.

Heat of Transformation

Latent Heat of Fusion

Let's consider a piece of ice. When heat is added to it, its entropy increases as the molecules become more active. If we continue to add heat, the ice will start to melt because the kinetic energy overcomes the intermolecular forces holding the molecules together. This process requires energy known as the **latent heat of fusion (L_F) or latent heat of melting**, which is the **heat absorbed per unit mass of the material to change its state from solid to liquid**.

As the ice melts, the temperature remains constant at 0°C (273.15 K) until all the ice has melted. The heat required to melt m grams of ice is:

$$Q = mL_F$$

Latent Heat of Vaporization

Similarly, heating water increases its temperature until it reaches 100°C , where it begins to boil and undergoes a phase transition from liquid to gas. The heat required for this transition is called the **latent heat of vaporization (L_V)**, which is **defined as the heat needed to vaporize a unit mass of the material**.

$$Q = mL_V$$

How much heat must be added to change ice to water to steam?

1. Heat required to rise the temperature of 1 gram of ice by $1^\circ\text{C} = 0.5 \text{ cal/g}$

Heat required to rise the temperature of m grams of ice is: $\Delta Q = c_i m \Delta T$

2. Heat required melt ice = 80 cal/g $\therefore$ calculated as $\Delta Q = mL_F$
3. As we heat the water, its temperature rises. Heat required to rise the temperature of 1 gram of water by $1^\circ\text{C} = 1 \text{ cal/g} = 4.18 \text{ J}$ $\therefore \Delta Q = c_w m \Delta T$
4. Heat required vaporize water = 540 cal/g $\therefore \Delta Q = mL_V$
5. Heating steam further requires $0.5 \text{ cal/ (g } ^\circ\text{C)}$. $\therefore \Delta Q = c_s m \Delta T$

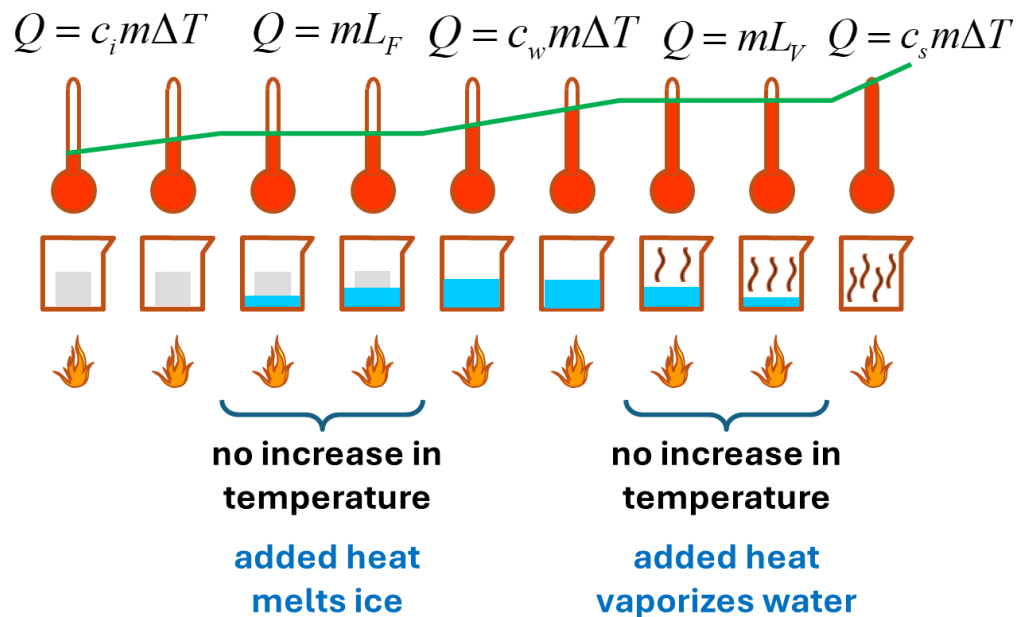


Figure 38. 3. Heat Stages: Ice to steam—temperature rise, melting, boiling, and continued heating.

The process of heating ice involves increasing its temperature to 0 °C, where it melts. The temperature remains constant until all the ice melts, then rises to 100 °C, where water boils and turns to steam. Continued heating increases the steam's temperature (see fig 38.3).

A graph represented in fig 38.4 shows the specific heat of ice, the phase transition at 0 °C, the heating of water, the phase transition at 100 °C, and the heating of steam. If we heat steam excessively, the water molecules dissociate into hydrogen and oxygen atoms. Further heating causes ionization, where electrons separate from protons, requiring several thousand degrees Celsius.

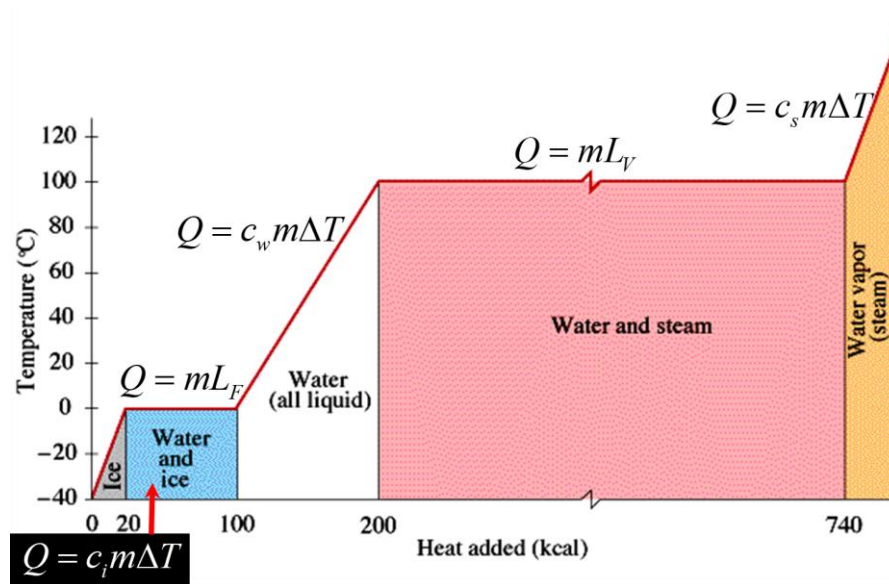


Figure 38. 4. Heating curve illustrating the transitions of ice at -40°C into hot steam in five steps.

Other Types of Phase Transitions

1. There are various types of phase transitions in different materials. Besides gas, liquid, and solid transitions, consider a permanent magnet. When heated beyond a certain point, it loses its magnetism because the magnetic domains become disordered, increasing entropy. This is a **phase transition between magnetized and un-magnetized states**.

2. **Evaporation** is another phase transition. In a liquid, molecules move due to temperature. As temperature rises, molecular motion increases. Fast-moving molecules can escape the liquid, turning into vapor as displayed in fig 38.5. For



Figure 38. 5. Water molecules with high enough KE due to random intermolecular collisions can break free from the surface and escape.

example, water in a glass decreases over time as some molecules evaporate. The temperature of the liquid reflects its average random kinetic energy. When fast-moving molecules leave, the average kinetic energy decreases, cooling the remaining liquid. This principle explains why water stored in porous pots stays cool; evaporation lowers the average kinetic energy, reducing the temperature. Evaporation causes cooling, explaining why we use fans in the summer. Fans help evaporate sweat, which cools the body by removing fast-moving molecules, thus lowering temperature and average kinetic energy.

3. Now, let's discuss **boiling**. Water boils at 100 °C at standard atmospheric pressure. However, this boiling point changes with pressure. In a pressure cooker, water can boil at higher temperatures, like 105 °C or 110 °C. Conversely, at higher altitudes, water boils at lower temperatures, such as 95°C or 98 °C, due to reduced atmospheric pressure.

For a molecule to escape from a liquid, it must possess enough energy to overcome the attractive forces of other molecules. This principle underlies evaporation and boiling, showcasing how molecular energy and external conditions affect phase transitions.

Statistical Thermodynamics

Statistical thermodynamics is the *microscopic* or *molecular* approach to understand and predict the macroscopic phenomena and the calculation of macroscopic properties from the properties of individual molecules. For example, knowing the molecules in a system (like oxygen, carbon, hydrogen) allows us to determine properties such as pressure, entropy, boiling point, and melting point.

In contrast, **classical thermodynamics** doesn't consider individual molecules but focuses on macroscopic properties, like the energy required to change the temperature or state of a substance.

Statistical thermodynamics emerged about 100 years after classical thermodynamics and offers a deeper understanding of heat. As substances are heated, their atoms and molecules exhibit more random motion and vibration, which we interpret as heat. Understanding randomness is crucial, as it applies to many systems.

To better understand this concept, think about randomness: variations are always present in any system like cars moving at different speeds on a road.

To create a histogram (fig 38.6) of car speeds, we divide the speed range into intervals (bins) and count the number of cars in each bin. For example, if there are 11 cars with speeds between 1 and 2 miles per hour. The height of each bar in the histogram corresponds to the number of cars within each speed interval.

Average speed can be calculated as:

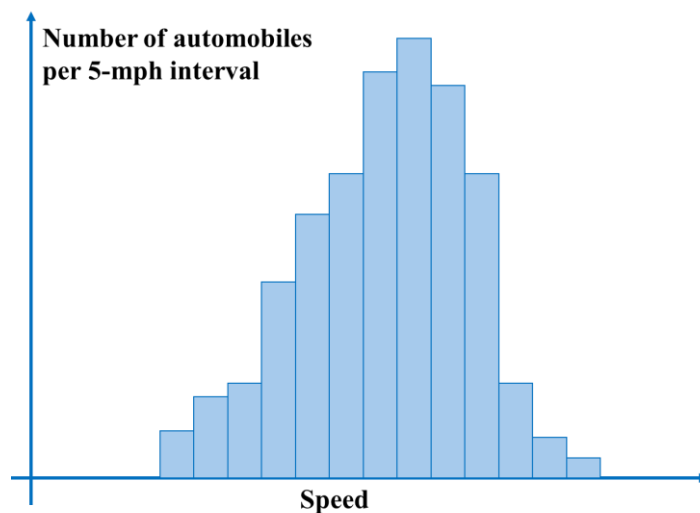


Figure 38. 6. Histogram of Car Speeds: bar height represents the number of cars in each speed interval.

$$N = \sum_i n(v_i)$$

$$\bar{v} = \frac{\text{Sum of all observations}}{\text{Total number of all observations}} = \frac{\sum_i v_i n(v_i)}{\sum_i n(v_i)} \quad \text{Where } n(v_i) \text{ is the number of cars moving with velocity } v_i.$$

with velocity v_i .

The **probability** $P(v_i)$ of finding a car with speed v_i represents the likelihood that a randomly selected car from a given set will have a velocity exactly equal to v_i . We can express it as:

$$P(v_i) \equiv \frac{n(v_i)}{\sum_i n(v_i)} = \frac{n(v_i)}{N}$$

Where N is the total number of cars. Using this probability, the average speed can be expressed as:

$$\bar{v} = \sum_i v_i P(v_i)$$

This approach allows us to statistically analyse and understand the distribution of car speeds based on empirical data gathered from the histogram. To calculate averages, we consider that some cars move faster and some slower than the average speed. This concept applies to gases as well, where different molecules move at different velocities, but there is an average speed.

Mean Free Path

When examining gas molecules, they often pass by each other but occasionally collide. **The average distance between collisions is called the mean free path (λ).**

To calculate this, we need the average speed of the molecules v and the number of molecules per unit volume n . Two molecules collide when their distance is equal to the sum of their radii ($d = r_1 + r_2$), shown in fig 38.7 (a).

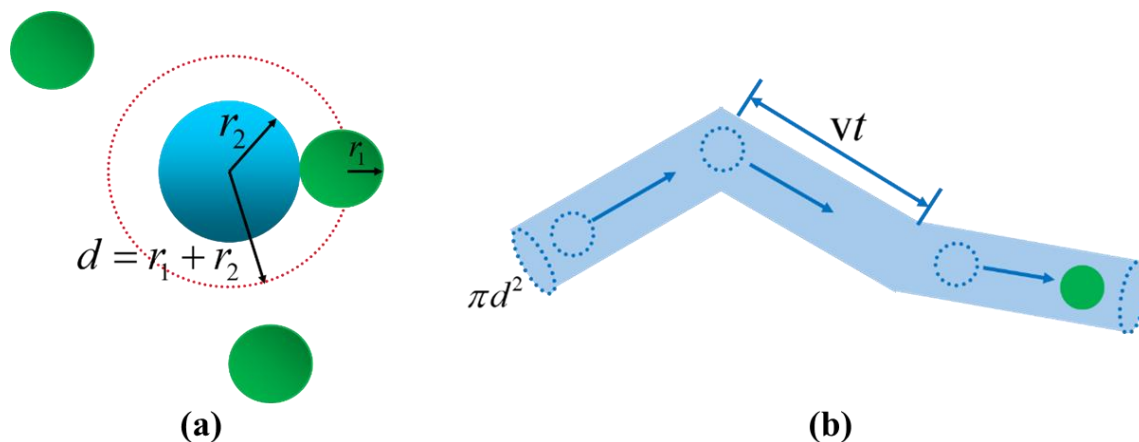


Figure 38. 7. (a) Molecules collide when the $d = r_1 + r_2$ (b) In a tube, mean free path λ is calculated using tube dimensions and mean free time t , where area = πd^2 and length = vt .

Imagine a tube, depicted in fig 38.7 (b), where molecules are moving with the average speed v . The area and length of the tube covered in time t are πd^2 and vt . t is the average time it takes for a molecule to collide with another after a collision known as mean free time. We can calculate the mean free path λ using the formula:

$$\begin{aligned} \lambda(n_v \pi d^2 vt) &\approx vt & \therefore d &= R_1 + R_2 \\ \therefore \lambda &\approx \frac{1}{n_v \pi d^2} \\ \lambda &= \frac{1}{\sqrt{2} n_v \pi d^2} \quad (\text{more accurate}) \end{aligned}$$

Where n_v is the number of molecules per unit volume ($V = \pi d^2 vt$) or number density, d is the effective diameter used to determine collision.

In denser gases, the mean free path is shorter because collisions are more frequent. Conversely, in less dense gases, the mean free path is longer, meaning collisions are less frequent and the distance between them is greater.

In very dilute gases, where collisions are rare, we refer to them as ideal gases. Before discussing ideal gases, let's define a mole. A **mole** is the amount of gas (or any other substance) containing 6.02×10^{23} molecules, known as Avogadro's number (N_A). Different gases have different types of atoms with varying masses. The **molar mass** (M) is the mass of one mole of a gas, measured in grams. The **number of moles** (n) is the number of atoms divided by Avogadro's number ($n = N / N_A$).

The ideal gas law can be written as

$$PV = Nk_B T$$

where P is the pressure, V is the volume, n is the number of moles, k is the Boltzmann constant, and T is the temperature. Alternatively, using the universal gas constant R , it is

$$PV = nRT$$

where $R = 8.315 \text{ J/(mol K)}$ and n is the number of moles in a gas. It's crucial to use kelvin for temperature because pressure varies linearly with it as illustrated in fig 38.8, converging to zero at absolute zero.

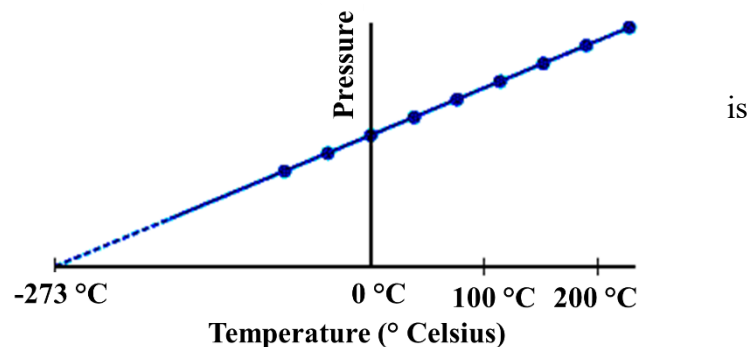


Figure 38. 8. Graph illustrating linear relation between temp in kelvin and pressure, converging to zero at absolute zero.

In an ideal gas, molecules move independently without collisions, and their velocities vary. The speed distribution graph (fig 38.9 (a)) shows this randomness: some move fast, others slow, with an average speed. The distribution of molecular speeds changes with temperature: in a cold gas, molecules move slower, while in a hot gas, they move faster. This results in different speed distributions for different temperatures as shown in fig 38.9 (b). For example, at $T = 80 \text{ K}$, the distribution peak is lower compared to $T = 300 \text{ K}$, where molecules move faster and collide more frequently, pushing against the container walls.

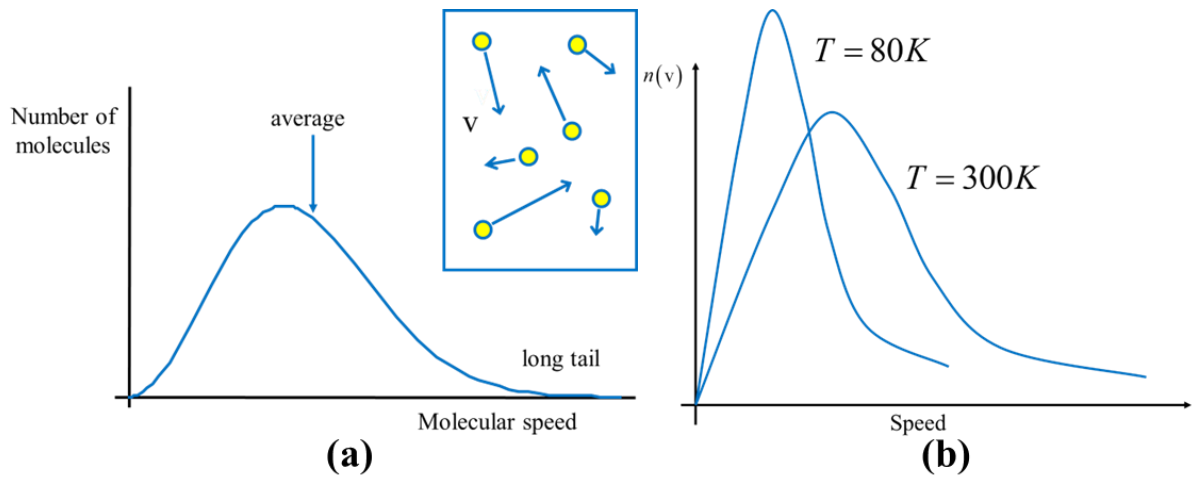


Figure 38. 9. (a) Speed distribution for the molecules of a gas. (b) A comparison of the speed distribution at two different temperatures, with lower average speeds at lower temperatures, but equal areas due to the same total number of molecules.

Pressure in a Gas

Pressure in a gas is created by molecules colliding with the walls of a container. When a molecule hits the wall, its momentum changes direction, creating a force according to Newton's law (force equals the rate of change of momentum).

To calculate this pressure, consider the number of molecules N per unit volume V and their average speed v_x . The distance a molecule travels in time Δt is $(v_x \Delta t)$. The volume it covers is $V_x \Delta t A$, where A is the area of the wall. Only half the molecules move in the positive x -direction, so the number of collisions is:

$$\text{Molecules that hit the wall} = \frac{1}{2} \frac{N}{V} v_x \Delta t A$$

$$\text{The change in momentum for one molecule} = mv_x - (-mv_x) = 2mv_x$$

where m is the mass. The total change in momentum is:

$$\begin{aligned} \Delta p &= (2mv_x) \times \left(\frac{1}{2} \frac{N}{V} v_x \Delta t A \right) \\ &= \frac{N}{V} m v_x^2 A \Delta t \\ P &= \frac{F}{A} = \frac{1}{A} \frac{\Delta p}{\Delta t} = \frac{N}{V} m v_x^2 \\ PV &= N m v_x^2 = 2N \left(\frac{1}{2} m v_x^2 \right)_{av} \end{aligned}$$

This formula shows how pressure relates to the average kinetic energy of gas molecules (through $\left(\frac{1}{2}mv_x^2\right)$ and their number density $\left(\frac{N}{V}\right)$. It helps us understand how molecular collisions create pressure in a gas.

The Molecular Interpretation of Temperature

Considering the kinetic theory of gases, the average kinetic energy per molecule can be related to the temperature as:

$$PV = Nk_B T = 2N \left(\frac{1}{2}mv_x^2 \right)_{av}$$

$$\left(\frac{1}{2}mv_x^2 \right)_{av} = \frac{1}{2}k_B T$$

Considering all directions (x, y, z), the average velocity squared is:

$$\left(v_x^2 \right)_{av} = \left(v_y^2 \right)_{av} = \left(v_z^2 \right)_{av}$$

$$\left(v^2 \right)_{av} = \left(v_x^2 \right)_{av} + \left(v_y^2 \right)_{av} + \left(v_z^2 \right)_{av} = 3 \left(v_x^2 \right)_{av}$$

The average kinetic energy of a molecule is:

$$K_{av} = \left(\frac{1}{2}mv^2 \right)_{av} = \frac{3}{2}k_B T$$

Thus, the total kinetic energy of N molecules is:

$$K = N \left(\frac{1}{2}mv^2 \right)_{av} = \frac{3}{2}Nk_B T$$

This shows that the average kinetic energy of a molecule is proportional to the temperature T.

The root mean square velocity v_{rms} is:

$$\left(v^2 \right)_{av} = \frac{3k_B T}{m}$$

$$v_{rms} = \sqrt{\left(v^2 \right)_{av}} = \sqrt{\frac{3k_B T}{m}}$$

This means the mean squared speed is proportional to the temperature, and as temperature increases, the average speed of molecules increases. The root mean square speed is equal to $\sqrt{T}$.

In this lecture, we explored how statistical thermodynamics, specifically kinetic theory simplifies complex problems. In physics, solving interactions between multiple bodies (like the Sun, Earth, and Moon) is challenging. With more bodies, it becomes even harder, often requiring computers. However, statistical thermodynamics handles many-body problems more easily because it focuses on average properties rather than detailed interactions. For example, when calculating pressure, we consider the average effect of many molecules, not the specific actions of each one. This approach allows us to ask different, more manageable questions, making statistical thermodynamics a powerful tool.

Lecture_39

Let's dig into the significant changes that have shaped our world. When Alexander the Great, made his mark on the inhabited regions of the world, it brought about tremendous changes. History has seen the rise and fall of countries, sultanates, and capitals over thousands of years. However, in my opinion, the revolution that occurred in the year 1905 stands out distinctly.

The Imaginary Revolution

This revolutionary thought troubled many minds for a long time, presenting a strange yet fascinating picture. It was certainly an imaginary revolution, but its implications were profound. This revolution was about breaking the atom and harnessing infinite amounts of energy from it. This discovery revealed how energy is released from the sun and stars, leading to the development of atomic power plants and even the atomic bomb.

This breakthrough allowed us to uncover the secrets hidden inside the atom. On one hand, the atom is the smallest unit, while on the other, it is as if the universe itself started from an atom. Einstein contributed significantly to this understanding. From the Theory of Relativity, we learned that before Newton's time, people had a specific understanding of time and space.

The Concept of Space-Time

Previously, time was perceived as a flowing river, moving in one direction with a constant flow. Space was considered something that could only be measured, and it was viewed as a fixed entity. However, Einstein changed this picture, introducing the concept of space-time. He explained that space and time are not separate but rather a single, interconnected entity known as space-time.

The Nature of Time

Physicists are concerned with how we can measure time and space. Understanding time involves recognizing periodic processes. For example, a pendulum swings from one side to the other, and this periodic motion can be used to measure time.

However, pendulums are not perfect clocks. Factors like temperature changes can alter the length of the pendulum, affecting its time period. A more accurate clock is the Earth's rotation, which completes one revolution in 24 hours. The Earth's orbit around the sun, completing one revolution in a year, also serves as a reliable timekeeper. Even more precise are atomic clocks, which measure time with such accuracy that we can observe the slowing rotation of the Earth.

Measuring Space

To measure space, we can use various methods. One way is to use standardized lengths, although creating identical units of measure can be challenging due to slight differences. Modern methods use atomic properties to determine length accurately. The distance between atoms can be used to measure total length.

Inertial Frames

To grasp Einstein's Theory of Special Relativity, we must first understand the concept of an inertial frame. Imagine sitting in a train that moves smoothly without any vibrations or noise. In such a situation, you cannot tell if the train is stationary or moving. This describes an inertial frame—a frame of reference where there is no acceleration.

Newton's First Law of Motion states that an object will remain at rest or continue in uniform motion unless acted upon by an external force. This law holds true in an inertial frame. For example, if a body is at rest or moving with constant velocity, it will continue to do so in the absence of external forces.

Non-Inertial Frames

If the train you are in starts to accelerate or decelerate, you will feel a force acting on you, indicating a non-inertial frame. In such frames, external forces cause changes in motion. Newton defined an inertial frame as one where his First Law is valid. In non-inertial frames, forces such as acceleration can affect objects' motion.

The Earth as an Inertial Frame

While we often consider the ground as a reference frame, it is not truly inertial. The Earth rotates on its axis and orbits the sun, causing slight acceleration. Newton approximated the Earth as an inertial frame for practical purposes, noting that objects on Earth experience minimal acceleration.

The Universal Frame and Time keeping

Newton also proposed a universal frame, fixed with respect to distant stars, providing an exact inertial frame. He imagined a clock in the universe that could tell the exact time, with innumerable reference frames and inertial frames.

Observations and Events in Different Frames

In any frame of reference, we can observe events. For example, a person traveling in a car or on a ship experiences different reference frame. Observations made in these frames are considered events. For instance, striking a match could be an event, and its occurrence can be noted in different frames of reference.

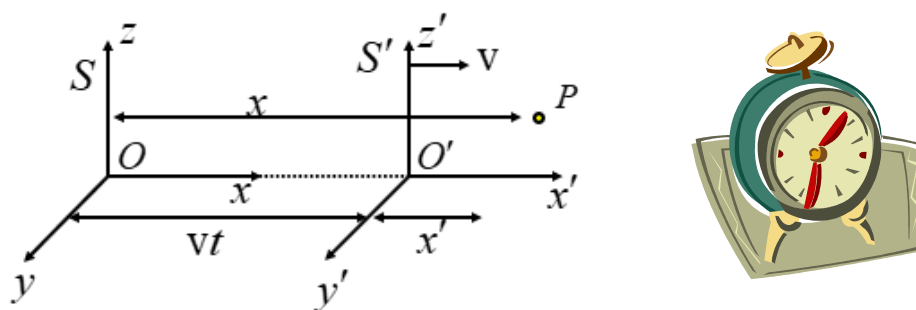


Figure 39.1: An event at time $t = t'$ in S frame of reference.

In one frame, as shown in figure 39.1, we describe an event by specifying its coordinates, such as (x, y, z, t) . The same event in another frame will have different spatial coordinates but the same time coordinate, according to Newton's view of absolute time. If we denote the second frame's coordinates with primes, we get transformations like:

$$\begin{aligned}
 x' &= x - vt, \\
 y' &= y, \\
 z' &= z, \\
 t' &= t
 \end{aligned}$$

These transformations are known as Galilean transformations, which give common sense results. Here is an example of a rod, as shown in figure 39.2, that is at rest, lies on x-axis.



Figure 39.2: A rod AB is placed along x-axis.

$$\text{length in S-frame} = x_B - x_A$$

$$\text{length in S'-frame} = x'_B - x'_A$$

Galilean transformation gives:

$$x'_B = x_B - vt_B \text{ and } x'_A = x_A - vt_A$$

Thus, length in S'-frame

$$x'_B - x'_A = x_B - x_A - v(t_B - t_A)$$

Since the two end points, A and B are measured at the same instant $t_B = t_A$,

$$x'_B - x'_A = x_B - x_A$$

$$\text{length in S-frame} = \text{length in S'-frame}$$

This shows that the length of the object remains the same in both frames. Similarly, the velocities and accelerations also transform predictably.

Velocity transformation

Galilean transformation of position along x-axis is

$$x' = x - vt$$

differentiate wrt time:

$$\frac{dx'}{dt} = \frac{dx}{dt} - v \frac{dt}{dt}$$

$$\frac{dx'}{dt} = \frac{dx}{dt} - v \quad \text{eq(1)}$$

we know

$$t = t'$$

$$\Rightarrow \frac{dx'}{dt} = \frac{dx'}{dt'}$$

eq(1) becomes

$$\frac{dx'}{dt'} = \frac{dx}{dt} - v$$

similarly, y and z components are

$$\frac{dy'}{dt'} = \frac{dy}{dt}, \quad \frac{dz'}{dt'} = \frac{dz}{dt}$$

$\frac{dx'}{dt'} = u'_x$ = the x-component of the velocity
measured in S'-frame

$\frac{dx}{dt} = u_x$ = the x-component of the velocity
measured in S-frame

Therefore, the velocity transformation is

$$u'_x = u_x - v \quad u'_y = u_y \quad u'_z = u_z$$

In general, the vector form is

$$\vec{u}' = \vec{u} - \vec{v}$$

Acceleration transformation

Galilean transformation of velocity along x-axis is

$$u'_x = u_x - v$$

differentiate wrt time:

$$\frac{du'_x}{dt'} = \frac{d}{dt}(u_x - v)$$

$$\frac{du'_x}{dt'} = \frac{du_x}{dt} - \frac{dv}{dt}$$

$$\frac{du'_x}{dt'} = \frac{du_x}{dt} \quad \text{Q } \frac{dv}{dt} = 0 \text{ because } v = \text{constant}$$

similarly, y and z components are

$$\frac{du'_y}{dt'} = \frac{du_y}{dt}, \quad \frac{du'_z}{dt'} = \frac{du_z}{dt}$$

Therefore, the acceleration transformation is

$$a'_x = a_x, \quad a'_y = a_y, \quad a'_z = a_z$$

In general, the vector form is

$$\vec{a}' = \vec{a},$$

Thus, the acceleration in the S' frame is the same as in the S frame, and Newton's laws apply equally in all inertial frames.

Newton's second law: $\vec{F} = m\vec{a}$ in S-frame

But in S'-frame: $\vec{F}' = m'\vec{a}'$

But: $m' = m$ $\vec{a} = \vec{a}'$

$$\therefore \vec{F}' = m'\vec{a}' = m\vec{a} = \vec{F}$$

This is known as the principle of Newtonian relativity, which states that the laws of physics are the same in all inertial frames.

The Speed of Light and Ether Theory

Now, let's consider the speed of light. Suppose you measure the speed of light in a laboratory and find it to be 3×10^{10} centi-meters per second or 3×10^8 meters per second. If you place a

light source on a moving car, according to Newtonian mechanics, the speed of light should be affected by the car's speed. Scientists once believed that light, like sound, needed a medium to travel through, called ether.

However, the Michelson-Morley experiment, which aimed to detect the presence of ether, failed to find any evidence of it. This experiment showed that the speed of light is constant, regardless of the motion of the source or the observer. This finding puzzled scientists because it contradicted the notion of ether.

The Dawn of New Physics

Einstein's journey into the realm of relativity began with deep questions about the nature of electromagnetism and light. He wondered why Maxwell's equations, which describe electromagnetic waves, would hold true in one frame and not another. This curiosity led to the development of his two fundamental postulates of Special Relativity, which have been tested and validated countless times. Let's delve into the concepts that revolutionized our understanding of space and time.

The Two Pillars of Special Relativity

Einstein proposed two main principles:

1. **The Principle of Relativity:** The laws of physics are the same in all inertial frames of reference. This means that no matter how fast you are moving, if you are not accelerating, the same physical laws apply to you as to someone who is stationary.
2. **The Constancy of the Speed of Light:** The speed of light in a vacuum is constant and independent of the motion of the light source or the observer. This means that no matter how fast you are moving, you will always measure the speed of light as 3×10^{10} centi-meters per second or 3×10^8 meters per second.

These principles seem straightforward, but they lead to some surprising and counterintuitive results. These two postulates led to a profound shift in our understanding of space and time. They implied that measurements of space and time are relative to the observer's frame of reference. This realization paved the way for the development of Special Relativity, fundamentally changing our perception of the universe.

Synchronizing Clocks in Different Frames

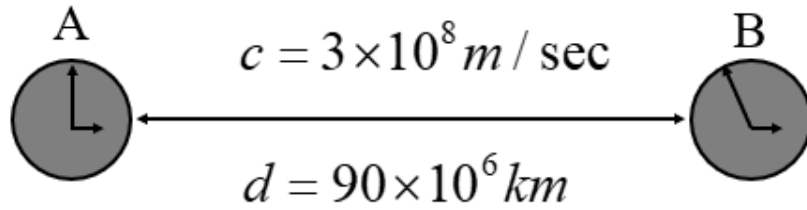


Figure 39.3: Two clocks in a single frame of reference which are far apart.

To understand how time and space are related, let's consider synchronizing clocks, in a single frame of reference. Suppose we have two clocks, one at point A and another at point B, 90 million kilometers apart, as shown in figure 39.3. If a light signal is sent from B to A, it will take 5 minutes to travel this distance (since light travels at 3×10^8 meters per second).

$$\text{Time taken by light reaching from B to A} = t = \frac{d}{c}$$

$$t = \frac{90 \times 10^6 \text{ km}}{3 \times 10^8 \text{ m/sec}}$$

or

$$t = \frac{9 \times 10^{10} \text{ m}}{3 \times 10^8 \text{ m/sec}} = 300 \text{ sec} = 5 \text{ min}$$

When the light signal reaches clock A, it shows 3:00 PM. Since it took 5 minutes for the light to travel, clock B must have shown 2:55 PM when the light left. Thus, we can synchronize the clocks by accounting for the time it takes for light to travel between them. This method ensures that all clocks in the frame show the same time, considering the finite speed of light.

Measuring Time in Moving Frames

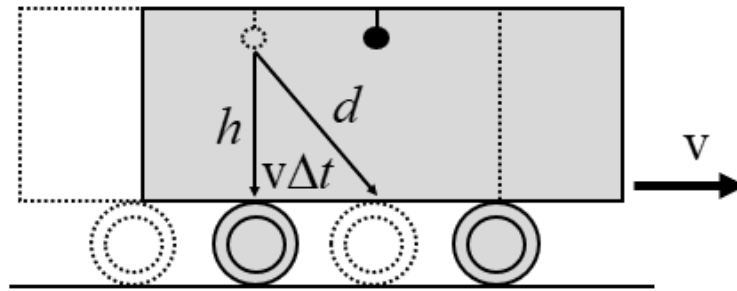


Figure 39.4: A moving train with a flashlight fixed at ceiling pointing to the floor.

Now, imagine you are on a train moving smoothly and rapidly, as shown in figure 1.4. You have a flashlight pointing from the ceiling to the floor. If the distance from the ceiling to the floor is h , and the speed of light is c , the time t it takes for the light to travel from the ceiling to the floor, according to your clock, is:

$$\text{Time it take the light to reach at floor} = \Delta t' = \frac{h}{c} \quad \text{eq(i)}$$

However, an observer standing on the ground sees the train moving. For this observer, the light also has to cover the horizontal distance that the train travels while the light is moving from the ceiling to the floor. If the train's speed is v and the light takes time Δt , the horizontal distance covered is $v\Delta t$.

The total distance the light travels, according to the ground observer, forms the hypotenuse of a right triangle with sides h and $v\Delta t$. Using the Pythagorean theorem:

$$d^2 = h^2 + (v\Delta t)^2$$

$$d = \sqrt{h^2 + (v\Delta t)^2}$$

as we know $d = c\Delta t$

$$c\Delta t = \sqrt{h^2 + (v\Delta t)^2}$$

$$\Delta t = \frac{\sqrt{h^2 + (v\Delta t)^2}}{c}$$

Solving for Δt , taking square on both sides

$$(c\Delta t)^2 = h^2 + (v\Delta t)^2$$

$$(c\Delta t)^2 - (v\Delta t)^2 = h^2$$

$$(\Delta t)^2(c^2 - v^2) = h^2$$

$$(\Delta t)^2 = \frac{h^2}{(c^2 - v^2)}$$

taking squareroot on both sides

$$\Delta t = \frac{h}{\sqrt{c^2 - v^2}}$$

taking c^2 common

$$\Delta t = \frac{h}{\sqrt{c^2 \{1 - (v^2 / c^2)\}}}$$

$$\Delta t = \frac{h}{c\sqrt{1 - (v^2 / c^2)}}$$

or

$$\Delta t = \frac{h}{c} \frac{1}{\sqrt{1 - (v / c)^2}}$$

using eq(i)

$$\Delta t = \Delta t' \frac{1}{\sqrt{1 - (v / c)^2}}$$

Therefore

$$\Delta t' = \Delta t \sqrt{1 - (v / c)^2}$$

$$\text{or } \Delta t' = \frac{\Delta t}{\gamma} \text{ where } \gamma = \frac{1}{\sqrt{1 - (v / c)^2}}$$

$$\text{for } v < c, \gamma > 1 \Rightarrow \Delta t' < \Delta t$$

This time Δt is longer than the time t measured by the observer on the train. The difference is due to the fact that the light has to cover more distance when the train is moving.

Gamma (γ) is always greater than or equal to 1. When the speed v is much less than the speed of light c , γ is very close to 1, and we do not notice time dilation in our everyday experiences. However, as v approaches c , γ increases significantly, and time dilation becomes noticeable.

For example, if you travel in a fast rocket, the time measured on your clock (moving clock) would appear to run slower compared to a clock on Earth (stationary clock). This effect is more pronounced at speeds close to the speed of light.

Example:

A muon is traveling through the laboratory at three-fifths the speed of the light. How long does it last?

Solution:

$$\text{Speed of muon} = v = \frac{3}{5}c$$

$$\Delta t' = \frac{\Delta t}{\gamma} \quad \text{where } \gamma = \frac{1}{\sqrt{1 - (v/c)^2}}$$

lets calculate γ

$$\gamma = \frac{1}{\sqrt{1 - (v/c)^2}}$$

$$\gamma = \frac{1}{\sqrt{1 - \left(\frac{3c}{5c}\right)^2}}$$

$$\gamma = \frac{1}{\sqrt{1 - (3/5)^2}} = \frac{5}{4}$$

So, it lives longer (than at rest) by a factor of 5/4

The Twin Paradox: An Intriguing Comparison

To grasp the nuances of Special Relativity, let's delve into the Twin Paradox, a fascinating thought experiment that challenges our understanding of time and motion.

The Setup

Imagine two identical twins, A and B, born at the same time. Twin A remains on Earth while Twin B embarks on a journey into space, traveling at speeds close to that of light. After many years, Twin B returns to Earth and finds that Twin A has aged significantly, while Twin B has aged much less.

The Paradox

At first glance, this seems paradoxical. According to the Theory of Relativity, all inertial frames are equivalent. Thus, Twin A could argue that Twin B should age faster, and vice versa. However, in practice, the traveling twin (B) is younger upon return.

Breaking the Symmetry

The resolution lies in understanding that the traveling twin (B) was not in a purely inertial frame. Twin B experienced acceleration and deceleration phases during the journey, breaking the symmetry of inertial frames. Thus, Twin B's frame was not equivalent to Twin A's purely inertial frame on Earth.

Lorentz Contraction: Objects Shrink in Motion

Now, let's explore another fundamental aspect of Special Relativity: Lorentz Contraction. This phenomenon states that objects moving at high speeds appear shorter in the direction of motion to an outside observer.

The Thought Experiment

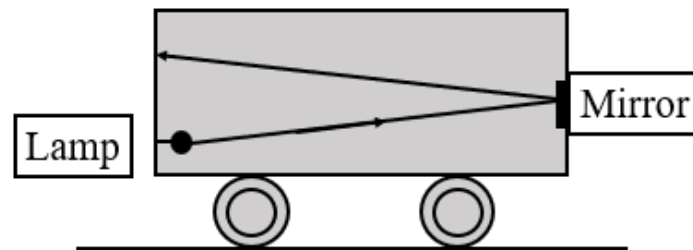


Figure 39.5: A high-speed train having a flashlight and mirror at its ends.

Imagine you're on a high-speed train with a flashlight, as shown in figure 39.5. You stand at one end of the train car, and a mirror is placed at the other end. You shine the light towards the mirror. According to your frame of reference, the light travels a distance $2\Delta x'$ (to the mirror and back) in a time $2\Delta x'/c$, as shown in figure 39.6.

$\Delta x' =$ length of the car with respect to car observer

$$\Delta t' = \frac{2\Delta x'}{c} \quad \text{eq(I)}$$

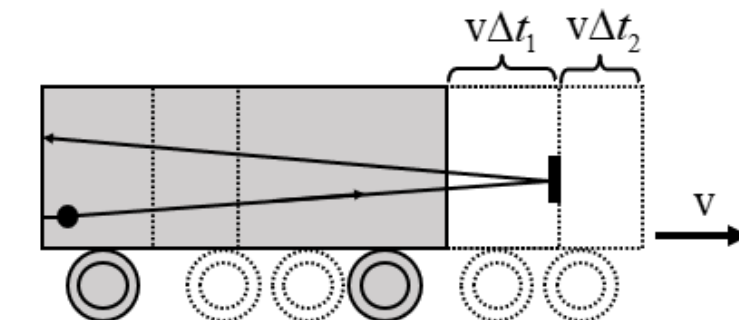


Figure 39.6: In fast-speed train light travels an extra distance $2\Delta x'$

$\Delta x =$ length of the car according to ground observer

$\Delta t_1 =$ time for the signal to reach the front end

$$\Delta t_2 = \text{return time}$$

However, for an observer on the ground, the train is moving. Thus, they perceive the light traveling a longer distance because the mirror is moving away from the light when it is emitted and moving towards the light on its return journey.

Mathematical Explanation

For the ground observer, the total distance covered by light includes the forward distance (increased by the train's movement) and the backward distance (decreased by the train's movement).

Let Δt_1 be the time for the light to reach the mirror and Δt_2 be the time for the light to return.

1. For the forward journey, the distance covered by light is $\Delta x + v\Delta t_1$:

$$c\Delta t_1 = \Delta x + v\Delta t_1$$

$$\text{solving for } \Delta t_1$$

$$c\Delta t_1 - v\Delta t_1 = \Delta x$$

$$\Delta t_1(c - v) = \Delta x$$

$$\Delta t_1 = \frac{\Delta x}{(c - v)}$$

2. For the return journey, the distance covered by light is $\Delta x - v\Delta t_2$:

$$c\Delta t_2 = \Delta x - v\Delta t_2$$

$$\text{solving for } \Delta t_2$$

$$c\Delta t_2 + v\Delta t_2 = \Delta x$$

$$\Delta t_2(c + v) = \Delta x$$

$$\Delta t_2 = \frac{\Delta x}{(c + v)}$$

The total time Δt for the round trip is:

$$\begin{aligned}
 \Delta t &= \Delta t_1 + \Delta t_2 = \frac{\Delta x}{(c-v)} + \frac{\Delta x}{(c+v)} \\
 \Delta t &= \Delta x \left(\frac{1}{c-v} + \frac{1}{c+v} \right) \\
 \Delta t &= \Delta x \left(\frac{c+v+c-v}{c^2-v^2} \right) \\
 \Delta t &= \Delta x \left(\frac{c+c}{c^2-v^2} \right) \\
 \Delta t &= \Delta x \frac{2c}{c^2(1-v^2/c^2)} \\
 \Delta t &= \frac{2\Delta x/c}{(1-v^2/c^2)} \quad \text{eq(II) (w.r.t ground observer)}
 \end{aligned}$$

From time dilation formula:

$$\begin{aligned}
 \Delta t' &= \sqrt{1-v^2/c^2} \Delta t \\
 &\text{using eq(II)} \\
 \Delta t' &= \sqrt{1-v^2/c^2} \left(\frac{2\Delta x/c}{(1-v^2/c^2)} \right) \\
 \Delta t' &= \sqrt{1-v^2/c^2} \left(\frac{2\Delta x/c}{\sqrt{(1-v^2/c^2)^2}} \right) \\
 \Delta t' &= \sqrt{1-v^2/c^2} \left(\frac{2\Delta x/c}{\sqrt{(1-v^2/c^2)} \sqrt{(1-v^2/c^2)}} \right) \\
 \Delta t' &= \frac{2\Delta x/c}{\sqrt{1-v^2/c^2}}
 \end{aligned}$$

$$\text{using eq(I) } \therefore \Delta t' = \frac{2\Delta x'}{c}$$

$$\frac{2\Delta x'}{c} = \frac{2\Delta x/c}{\sqrt{1-v^2/c^2}}$$

$$\Delta x' = \frac{\Delta x}{\sqrt{1-v^2/c^2}}$$

Conclusion of Lorentz Contraction

The length of the moving object (train) appears contracted by a factor of $\sqrt{1 - v^2 / c^2}$. For example, a one-meter rod moving at high speed will appear shorter to a stationary observer. The faster the object moves, the greater the contraction.

Practical Implications and Observations

In our daily lives, the effects of Special Relativity are negligible because the speeds involved are much less than the speed of light. However, in high-energy physics and astronomy, these effects are significant. For instance, particles accelerated to near-light speeds in colliders exhibit both time dilation and length contraction, validating Einstein's predictions.

The Evolution of Transformations: Galilean vs. Lorentz

In classical mechanics, we often use Galilean transformations to relate the coordinates of events between different reference frames. However, these transformations fall short when dealing with speeds close to the speed of light. In Einstein's theory of relativity, Galilean transformations are replaced by Lorentz transformations, which accurately describe the relationships between coordinates in different frames at high velocities.

Galilean Transformations

Galilean transformations work well for everyday speeds. They assume that time is absolute and the same for all observers, while positions and velocities transform simply with the relative motion between frames.

The Need for Lorentz Transformations

When dealing with velocities approaching the speed of light, the Galilean transformations are no longer valid. Instead, we use Lorentz transformations, which account for the relativistic effects predicted by Einstein's theory of relativity. These transformations show how time and space coordinates of an event change between two frames moving at a constant velocity relative to each other.

Lorentz Transformations

Consider two frames of reference, as shown in figure 39.7, one is with coordinates (x, y, z, t) and other has coordinates (x', y', z', t') , moving with relative speed of v , as given below.

Now, an event E occurs. Let's calculate Lorentz transformation from moving frame to stationary and vice versa.

where:

- x and t are the coordinates of the event in the original frame.
- x' and t' are the coordinates in the moving frame.
- v is the relative velocity between the frames.
- c is the speed of light.
- γ (gamma) is the Lorentz factor, defined as $\gamma = \frac{1}{\sqrt{1 - v^2 / c^2}}$.

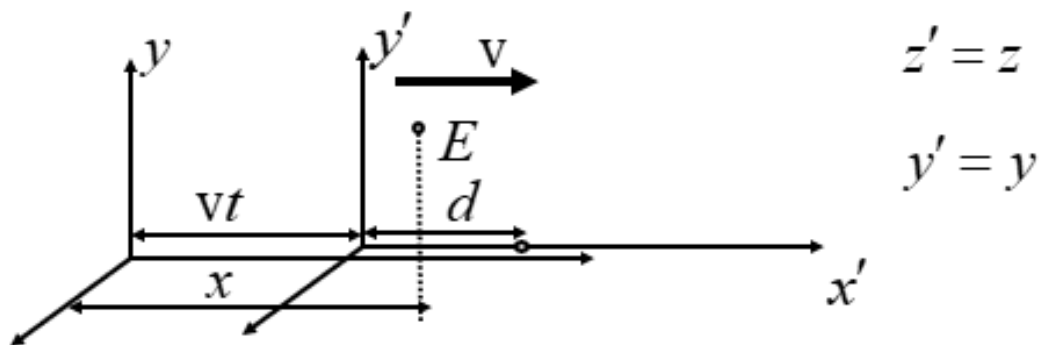


Figure 39.7: For an event E , frame of reference with prime coordinates has a relative speed v .

$$x = d + vt$$

But

$$d = \frac{x'}{\gamma}$$

$$\Rightarrow x' = \gamma(x - vt)$$

because x' is moving with respect to O – frame

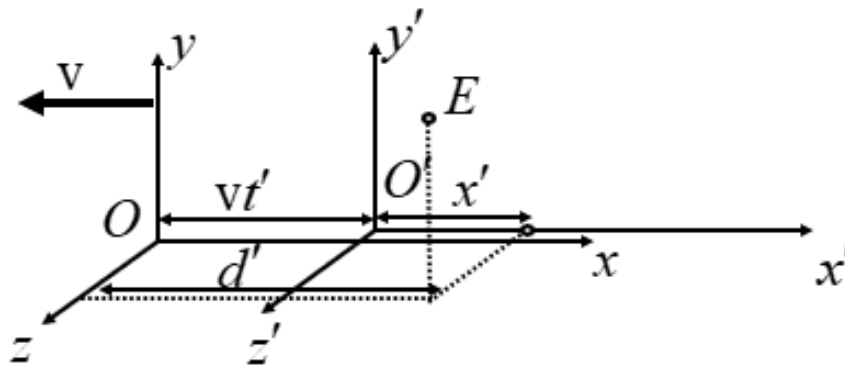


Figure 39.8: For an event E, frame of reference with coordinates (x, y, z, t) has a relative speed v.

$$x' = d' - vt'$$

But

$$d' = \frac{x}{\gamma}$$

$$\Rightarrow x = \gamma(x' + vt')$$

$$\begin{aligned}
\therefore x &= \gamma(x' + vt') \\
x &= \gamma(\gamma(x - vt) + vt') \\
x &= \gamma^2(x - vt) + \gamma vt' \\
x &= \gamma^2 x - \gamma^2 vt + \gamma vt' \\
\text{dividing with } \gamma^2 \text{ on both sides, we get} \\
\frac{x}{\gamma^2} &= x - vt + \frac{vt'}{\gamma} \\
x(1 - v^2/c^2) &= x - vt + \frac{vt'}{\gamma} \quad \therefore \gamma^2 = \frac{1}{1 - v^2/c^2} \\
x - xv^2/c^2 &= x - vt + \frac{vt'}{\gamma} \\
-xv^2/c^2 &= -vt + \frac{vt'}{\gamma} \\
vt - xv^2/c^2 &= \frac{vt'}{\gamma} \\
v(t - xv/c^2) &= \frac{vt'}{\gamma} \\
(t - xv/c^2) &= \frac{t'}{\gamma} \\
t' &= \gamma(t - xv/c^2) \\
\therefore t' &= \gamma\left(t - \frac{v}{c^2}x\right)
\end{aligned}$$

So, collectively, all Lorentzian equations are given below:

$$\begin{array}{ll}
x' = \gamma(x - vt) & x = \gamma(x' + vt) \\
y' = y & y = y' \\
z' = z & z = z' \\
t' = \gamma\left(t - \frac{v}{c^2}x\right) & t = \gamma\left(t' + \frac{v}{c^2}x'\right)
\end{array}$$

Simultaneity:

Let's test this Lorentz transformation for a specific situation. Consider we have clocks in S-frame (stationary) and in S'-frame (moving), as shown in figure 1.9. Now at $t=0$ check the value of t' .

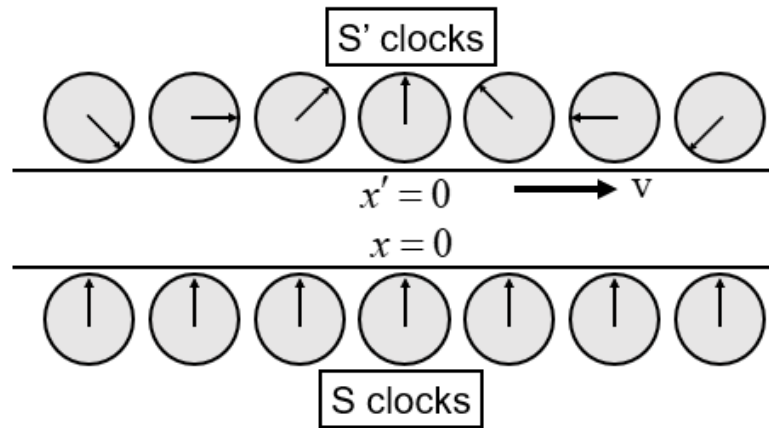


Figure 39.9: For a fixed position, clocks for two frames of references are presented.

As we know

$$t' = \gamma \left(t - \frac{v}{c^2} x \right)$$

put $t=0$

$$t' = \gamma \left(0 - \frac{v}{c^2} x \right)$$

$$t' = -\gamma \frac{v}{c^2} x$$

Clearly, t is not equal to t' .

Time Dilation and Length Contraction

The Lorentz transformations reveal that time and space are intertwined, and their measurements depend on the observer's frame of reference. This intertwining leads to two significant effects: time dilation and length contraction.

Time Dilation

Time dilation refers to the phenomenon where time appears to pass slower for an object moving relative to an observer's frame.

From the Lorentz transformation, if we consider an event occurring at the same location in the moving frame ($\Delta x = 0$), the relationship between time intervals in the two frames can be calculated.

We know

$$t = \gamma \left(t' + \frac{v}{c^2} x' \right)$$

$$\Rightarrow \Delta t' = \gamma \left(\Delta t - \frac{v}{c^2} \Delta x \right)$$

putting $\Delta x = 0$, we get

$$\Delta t' = \gamma \Delta t$$

This equation shows that the time interval t' in the moving frame is longer than the time interval t in the stationary frame, indicating that moving clocks run slower.

Length Contraction

Length contraction states that objects moving relative to an observer will appear shortened along the direction of motion.

If we measure the length of an object moving with velocity v , the contracted length $\Delta x'$ can be calculated as:

We know

$$x' = \gamma (x - vt)$$

$$\Rightarrow \Delta x' = \gamma (\Delta x - v \Delta t)$$

putting $\Delta t = 0$, we get

$$\Delta x' = \gamma \Delta x$$

Here, Δx is the length of the object in its rest frame. As the speed increases, γ increases, making $\Delta x'$ shorter.

Einstein velocity addition rule:

The Lorentz transformation also helps us relate the velocities of objects in different frames. If a particle moves with velocity u in frame S , its velocity u' in frame S' moving with velocity v relative to S is derived below.

Suppose a particle moves a distance dx in a time dt . Its velocity u is then

$$u = \frac{dx}{dt} \text{ (in S-frame)}$$

In S' -frame, meanwhile, it has moved a distance

$$dx' = \gamma(dx - vdt)$$

in time

$$dt' = \gamma\left(dt - \frac{v}{c^2}dx\right)$$

The velocity in S' -frame is therefore

$$u' = \frac{dx'}{dt'}$$

putting values of dx' and dt' , we get

$$u' = \frac{\gamma(dx - vdt)}{\gamma\left(dt - \frac{v}{c^2}dx\right)}$$

$$u' = \frac{dx - vdt}{dt - \frac{v}{c^2}dx}$$

$$u' = \frac{dt(dx/dt - v)}{dt\left(1 - \frac{v}{c^2}dx/dt\right)}$$

$$u' = \frac{dx/dt - v}{1 - \frac{v}{c^2}dx/dt}$$

$$u' = \frac{u - v}{1 - \frac{uv}{c^2}} \quad \text{eq(a)} \quad \therefore u = dx / dt$$

lets calculate for u , we have

$$u' = \frac{u - v}{1 - \frac{uv}{c^2}}$$

$$u' \left(1 - \frac{uv}{c^2} \right) = u - v$$

$$u' - u' \frac{uv}{c^2} = u - v$$

$$u' + v = u + u' \frac{uv}{c^2}$$

$$u' + v = u \left(1 + \frac{u'v}{c^2} \right)$$

$$\frac{u' + v}{\left(1 + \frac{u'v}{c^2} \right)} = u$$

or

$$u = \frac{u' + v}{1 + \frac{u'v}{c^2}} \quad \text{eq(b)}$$

Eq(a) and eq(b) are Einstein velocity addition rule.

Example:

Suppose, instead of particle with speed u we have light itself then u becomes c . What will be its speed in prime coordinates?

Solution:

If $u = c$ then $u' = ?$

Using the Einstein velocity addition rule, we get

$$u' = \frac{u - v}{1 - \frac{uv}{c^2}}$$

putting $u = c$

$$u' = \frac{c - v}{1 - \frac{cv}{c^2}}$$

$$u' = \frac{c - v}{1 - \frac{v}{c}}$$

$$u' = c \frac{c - v}{c - v} = c$$

$$\Rightarrow u' = c$$

Speed of light same in all reference frames

Example:

Suppose an ordinary particle of speed u and relative speed of v have very low speed as compared to speed of light c . What will be its speed in prime coordinates?

Solution:

If $u, v \ll c$ then $u' = ?$

Using the Einstein velocity addition rule, we get

$$u' = \frac{u - v}{1 - \frac{uv}{c^2}} \rightarrow u - v \quad \because uv \ll c^2 \text{ so } 1 - \frac{uv}{c^2} \approx 1$$

$$u' = u - v$$

For ordinary speeds, Einstein velocity addition rule reduces to Galilean velocity addition rule

The Famous Equation: $E=mc^2$

Einstein's most famous equation, $E=mc^2$, illustrates the equivalence of mass and energy. It implies that mass can be converted into energy and vice versa. This principle is fundamental to nuclear reactions and particle physics, where tiny amounts of mass convert into vast amounts of energy.

Lecture_40

Introduction

Physics is a field where expressing ideas using the least number of words or formulas is highly appreciated. If an idea is interesting and can be explained concisely, it stands out. In this book, we will delve into the fascinating theory of Special Relativity, as introduced by Albert Einstein. Special Relativity revolves around two main postulates which we will explore in detail, along with their implications.

Key Postulates of Special Relativity

1. The Speed of Light is Constant

The first postulate of Special Relativity is that light travels at the same speed in every inertial frame of reference, regardless of the motion of the light source or the observer. Whether the frame is moving, stationary, or running very fast in any direction, the speed of light remains constant.

2. The Laws of Physics are the Same in All Inertial Frames

The second postulate states that the laws of physics have the same form in every inertial frame of reference. Whether you are on Earth, on the moon, or in a spaceship, the physical laws you discover and verify remain consistent.

These two postulates, although simple and brief, have profound implications and have led to significant discoveries.

Time Dilation and Length Contraction

Time Dilation

One of the consequences of Special Relativity is time dilation. If an observer is moving very fast relative to you in an inertial frame, all the clocks in that observer's frame will appear to be running slower from your perspective. This means that everything in that frame seems to happen more slowly.

Lorentz / Length Contraction

Another consequence is length contraction. If an object is moving towards or away from you, its length will appear shorter. If the original length is L , the observed length L' is given by:

$$L' = \frac{L}{\gamma}$$

where γ (gamma) is the Lorentz factor defined as:

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Here, v is the relative velocity of the moving frame, and c is the speed of light.

Lorentz Transformation

Understanding Frames of Reference

Consider two frames of reference, as shown in fig 40.1: S and S' . The origin of S is at O , and the origin of S' is at O' . If something is at a certain place at a certain time, it has coordinates in space and time. For simplicity, let's consider one-dimensional space with coordinates (t, x) in frame S and (t', x') in frame S' .

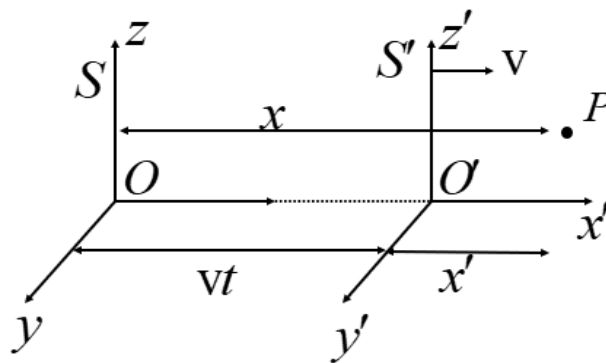


Figure 40.1: Two frame of references with origin O and O'

Lorentz Transformation Equations

The relationship between these coordinates is given by the Lorentz transformation. Without deriving it fully here, the transformation equations are:

$$x' = \gamma(x - vt) \quad x = \gamma(x' + vt')$$

$$t' = \gamma\left(t - \frac{v}{c^2}x\right) \quad t = \gamma\left(t' + \frac{v}{c^2}x'\right)$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

These equations show how space and time coordinates transform between two inertial frames moving relative to each other at a constant velocity v .

Invariant Interval

The distance between two events in spacetime is defined by the interval, which remains invariant under Lorentz transformation. For events separated by Δt in time and Δx in space, the interval I is given by:

$$I = c^2\Delta t^2 - \Delta x^2$$

This interval remains the same in all inertial frames, ensuring consistency of physical laws.

Consequences of Invariant Interval

To understand the significance, consider a light pulse emitted in frame S' . The light pulse spreads at the speed of light c . In frame S , the same pulse is observed moving at c . The interval for this light pulse is zero:

$$I' = (c\Delta t')^2 - (\Delta x')^2$$

Putting $\Delta t'$ and $\Delta x'$ from Lorentz transformation

$$= c^2 \gamma^2 \left(\Delta t - \frac{v}{c^2} \Delta x \right)^2 - \gamma^2 (\Delta x - v \Delta t)^2$$

$$Q(a-b)^2 = a^2 + b^2 - 2ab$$

$$= c^2 \gamma^2 \left((\Delta t)^2 + \left(\frac{v}{c^2} \Delta x \right)^2 - 2\Delta t \frac{v}{c^2} \Delta x \right) - \gamma^2 ((\Delta x)^2 + (v\Delta t)^2 - 2\Delta x v \Delta t)$$

$$= \gamma^2 \left(c^2 (\Delta t)^2 + c^2 \left(\frac{v}{c^2} \Delta x \right)^2 - c^2 2\Delta t \frac{v}{c^2} \Delta x \right) - \gamma^2 ((\Delta x)^2 + (v\Delta t)^2 - 2\Delta x v \Delta t)$$

$$= \gamma^2 \left((c\Delta t)^2 + \frac{v^2}{c^2} (\Delta x)^2 - 2v\Delta t \Delta x \right) - \gamma^2 ((\Delta x)^2 + (v\Delta t)^2 - 2\Delta x v \Delta t)$$

taking γ^2 common

$$= \gamma^2 ((c\Delta t)^2 + (\Delta x)^2 v^2 / c^2 - 2v\Delta t \Delta x - (\Delta x)^2 - (v\Delta t)^2 + 2v\Delta x \Delta t)$$

$$\begin{aligned} I' &= \gamma^2 ((c\Delta t)^2 + (\Delta x)^2 v^2 / c^2 - \cancel{2v\Delta t \Delta x} - (\Delta x)^2 - (v\Delta t)^2 + \cancel{2v\Delta t \Delta x}) \\ &= \frac{1}{(1 - v^2 / c^2)} ((c^2 - v^2)(\Delta t)^2 - (\Delta x)^2 (1 - v^2 / c^2)) \quad Q \gamma^2 = \frac{1}{(1 - v^2 / c^2)} \\ &= \frac{1}{(1 - v^2 / c^2)} (c^2 (1 - v^2 / c^2)(\Delta t)^2 - (\Delta x)^2 (1 - v^2 / c^2)) \\ &= \frac{(1 - v^2 / c^2)}{(1 - v^2 / c^2)} ((c\Delta t)^2 - (\Delta x)^2) \\ &= (c\Delta t)^2 - (\Delta x)^2 = I \end{aligned}$$

Thus,

$$(c\Delta t)^2 - (\Delta x)^2 = (c\Delta t')^2 - (\Delta x')^2$$

Time-Like and Space-Like Intervals

Time-Like Intervals

When the time difference between two events is greater than the spatial distance

$(c^2 \Delta t^2 > \Delta x^2)$, the interval is time-like. This means the events are separated more by time than by space.

Space-Like Intervals

When the spatial distance is greater than the time difference ($\Delta x^2 > c^2 \Delta t^2$), the interval is space-like. This means the events are separated more by space than by time.

Consistency Across Frames

If an interval is time-like or space-like in one frame, it remains so in all frames, ensuring the consistency of cause and effect.

World Lines and Light Cones

World Lines

A world line represents the trajectory of a particle through spacetime, as shown in fig 40.2. For a particle at rest, the world line is vertical, indicating no change in spatial position over time. For a moving particle, the world line is slanted.

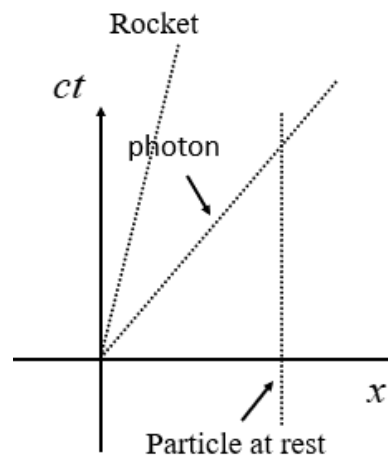


Figure 40.2: A world line graph where position on x-axis and ct on y-axis.

Light Cones

The light cone represents the possible positions of a light pulse emanating from an event, as shown in fig 40.3. The past light cone includes all events that could have influenced the event, while the future light cone includes all events that the event could influence.

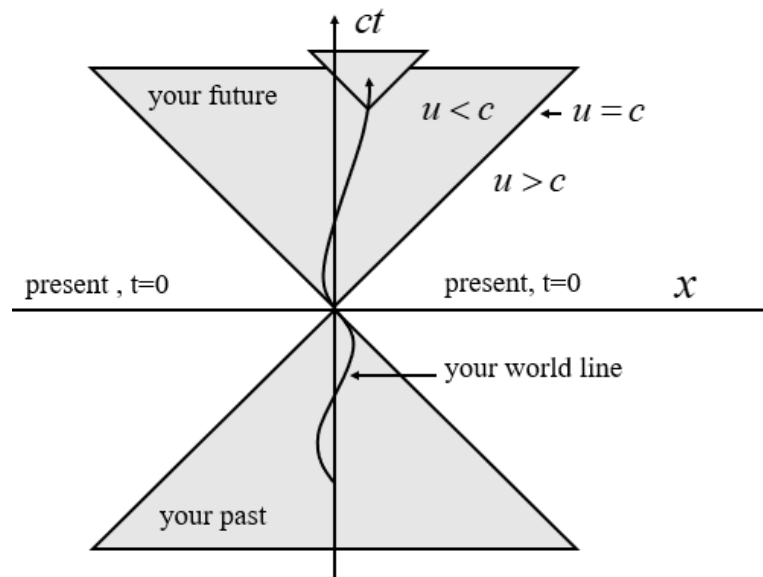


Figure 40.3: The light cone that represents future, past and present.

Particle Movement

A particle can only move within its light cone, as it cannot travel faster than the speed of light. This constraint shapes the particle's past, present, and future possibilities.

Preservation of Causality

Cause and Effect

In Special Relativity, the order of cause and effect is preserved. The invariant interval ensures that if one event causes another, this relationship remains consistent in all frames. For instance, if a bullet is fired or a ball is thrown, as shown in fig 40.4, and later hits a target, this sequence remains the same in all frames.

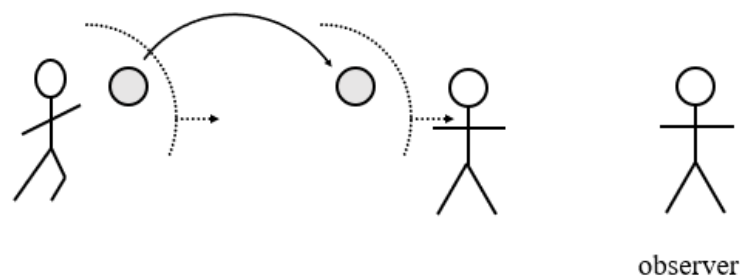


Figure 40.4: A ball is thrown from one man to another with an observer

No Reversal of Order

It is impossible for the order of events to reverse in any frame. The invariance of the interval guarantees that cause always precedes effect, maintaining the logical sequence of events.

The Doppler Effect and Relativity

In previous discussions, we have talked about sound and the Doppler effect. The Doppler effect is observed when there is a change in the frequency of sound as the source moves relative to an observer. For instance, if a car is honking its horn while moving, a person standing still will hear the frequency change as the car approaches and then passes by. This change in frequency is called the Doppler effect.

The Doppler Effect for Light

The Doppler effect also applies to light, but here, we must use relativity to understand it fully. Imagine a vehicle moving towards a person with its headlights on, as shown in fig 40.5. The frequency of the light from the headlights, as seen by a person standing still, will change. We denote the frequency in the source rest frame (the frame of the moving vehicle) as ν_0 , and its time period as T_0 , so:

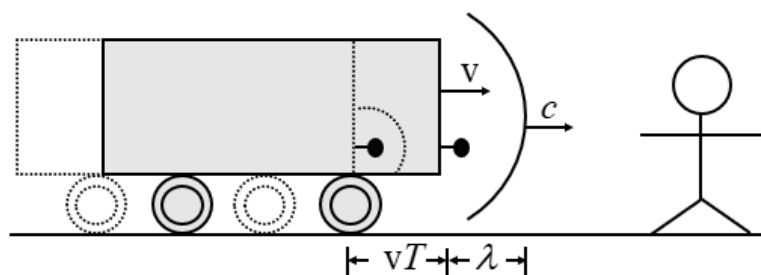


Figure 40.5: A moving vehicle that generates a light pulse

$$\nu_0 = \frac{1}{T_0} = \text{frequency measured in source rest frame}$$

ν = frequency measured with respect to observer = ?

λ = distance between two successive crests

Now, we want to determine the frequency that the person standing still (the observer) will see. Consider that one pulse of light is emitted from the source, and then another pulse is emitted after a time T_0 . The distance between two successive pulses or crests of the light wave is λ . The speed of light is c , so the distance covered by the light in time T_0 is cT_0 .

In the frame of the observer, the vehicle covers a distance vT_0 in the same time period, where v is the velocity of the vehicle. Thus, the distance λ between successive pulses of light is given by:

$$\lambda = c \cdot T_0 - v \cdot T_0 = T_0(c - v)$$

From this, we can determine the frequency ν that the observer will see:

$$\nu = \frac{c}{\lambda} = \frac{c}{T_0(c - v)}$$

However, T_0 is the time period measured in the rest frame of the source. The observer measures the time period T , which is related to T_0 through the time dilation formula in relativity:

$$T = \gamma T_0 = \frac{T_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Therefore, the observed frequency ν becomes:

$$\begin{aligned}
 \nu &= \frac{c}{\lambda} = \frac{c}{T(c-v)} \\
 \nu &= \frac{c}{(c-v) \frac{T_0}{\sqrt{1-\frac{v^2}{c^2}}}} \\
 \nu &= \frac{c}{(c-v)T_0} \sqrt{1-\frac{v^2}{c^2}} \\
 \nu &= \left(\frac{c}{c-v} \right) \frac{\sqrt{c^2-v^2}}{c} \nu_0 \\
 \nu &= \left(\frac{c}{c-v} \right) \frac{\sqrt{(c+v)(c-v)}}{c} \nu_0 \quad \because a^2-b^2 = (a+b)(a-b) \\
 \nu &= \frac{\sqrt{(c+v)(c-v)}}{\sqrt{(c-v)(c-v)}} \nu_0 \quad \because a-b = \sqrt{(a-b)^2} = \sqrt{(a-b)(a-b)} \\
 \Rightarrow \nu &= \sqrt{\frac{c+v}{c-v}} \nu_0
 \end{aligned}$$

Applications of the Doppler Effect in Relativity

This Doppler effect for light has many practical applications. For example, police use Doppler radar to measure the speed of vehicles. An electromagnetic pulse (often of a lower frequency than visible light) is emitted from the radar gun, hits the moving vehicle, and reflects back. The change in frequency of the reflected pulse allows the radar to calculate the speed of the vehicle.

Another interesting application is in astronomy. By observing the light from distant stars and galaxies, scientists can determine their velocities. If a star is moving away from us, the light we receive is redshifted (its frequency decreases). If it is moving towards us, the light is blue shifted (its frequency increases). This observation has led to significant discoveries about the universe.

Edwin Hubble and the Expanding Universe

About 100 years ago, Edwin Hubble observed that light from distant galaxies is redshifted, indicating that these galaxies are moving away from us. The farther away a galaxy is, the faster it is receding. This observation led to the hypothesis that the universe began with a massive explosion, now known as the Big Bang, and has been expanding ever since.

Energy and Momentum in Special Relativity

In special relativity, the concepts of energy and momentum are intertwined with space and time. According to Newton's laws, momentum is defined as the product of mass and velocity ($p=mv$). In non-relativistic situations, momentum is conserved in collisions.

However, in relativistic contexts, where speeds are close to the speed of light, this definition of momentum does not hold. Instead, relativistic momentum is defined as:

$$p = \gamma mv$$

Where,

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

This ensures that momentum is conserved even at high velocities. To understand this, consider two particles colliding in a non-relativistic context, as shown in fig 40.6. Before the collision, particle A has velocity u_A and particle B has velocity u_B . After the collision, their velocities change to u_C and u_D , respectively. In S-frame, conservation of momentum gives:

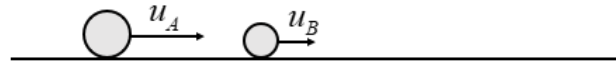
$$m_A u_A + m_B u_B = m_C u_C + m_D u_D$$

and

$$m_A + m_B = m_C + m_D$$

Consider the collision of two particles...

Before collision



Collision



After collision

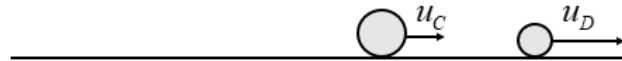


Figure 40.6: Collision between two particles in non-relativistic speed

In S'-frame which is moving with speed v with respect to S-frame. Using Galilean velocity addition rule, we get:

$$u' = u - v \quad \text{or} \quad u = u' + v$$

Conservation of momentum becomes

$$m_A(u'_A + v) + m_B(u'_B + v) = m_C(u'_C + v) + m_D(u'_D + v)$$

$$m_A u'_A + m_B u'_B = m_C u'_C + m_D u'_D$$

Now using Einstein velocity addition rule

$$u' = \frac{u - v}{1 - \frac{uv}{c^2}} \quad u = \frac{u' + v}{1 + \frac{u'v}{c^2}}$$

Conservation of momentum becomes

$$\begin{aligned} m_A \left(\frac{u'_A + v}{1 + u'_A v/c^2} \right) + m_B \left(\frac{u'_B + v}{1 + u'_B v/c^2} \right) \\ = m_C \left(\frac{u'_C + v}{1 + u'_C v/c^2} \right) + m_D \left(\frac{u'_D + v}{1 + u'_D v/c^2} \right) \end{aligned}$$

Now momentum is not conserved. So, momentum with old definition will not be conserved relativistically. Can we find some suitable new definition of momentum? However, in relativistic mechanics, we must adjust the definition of momentum to ensure it remains conserved at high velocities.

The new definition must have two properties

- 1) It must yield a law of conservation of momentum that satisfies the principle of relativity.
- 2) At low speeds, the new definition must reduce to

$$\vec{p} = m\vec{u}$$

Relativistic momentum

The relativistic formula for the momentum of a particle of mass m moving with velocity $\vec{u}$ is

$$\vec{p} = \frac{m\vec{u}}{\sqrt{1-u^2/c^2}} = \gamma m\vec{u}$$

In S-frame, conservation of momentum implies that

$$m_A \gamma_A u_A + m_B \gamma_B u_B = m_C \gamma_C u_C + m_D \gamma_D u_D$$

In S'-frame, conservation of momentum becomes

$$m_A \left(\frac{1}{\gamma} \gamma'_A u'_A + \gamma_A v \right) + m_B \left(\frac{1}{\gamma} \gamma'_B u'_B + \gamma_B v \right) = m_C \left(\frac{1}{\gamma} \gamma'_C u'_C + \gamma_C v \right) + m_D \left(\frac{1}{\gamma} \gamma'_D u'_D + \gamma_D v \right)$$

$$m_A \gamma'_A u'_A + m_B \gamma'_B u'_B = m_C \gamma'_C u'_C + m_D \gamma'_D u'_D$$

Thus, momentum is conserved in both S and S' frames

In the limit of low speeds:

$$\vec{p} = \frac{m\vec{u}}{\sqrt{1-u^2/c^2}} \rightarrow m\vec{u} \text{ as } u/c \rightarrow 0$$

Relativistic Energy

Just as momentum is redefined in relativity, so is energy. The relativistic kinetic energy K is not simply

$$K = \frac{1}{2} mu^2$$

as in Newtonian mechanics.

Let's calculate the expression for energy.

$$K = \int F dx$$

$$K = \int \frac{dp}{dt} dx \quad \text{Q } F = \frac{dp}{dt}$$

$$K = \int \frac{dx}{dt} dp$$

$$K = \int u dp \quad \text{eq(1)}$$

we know

$$p = m \left(\frac{u}{\sqrt{1 - u^2 / c^2}} \right)$$

so,

$$dp = md \left(\frac{u}{\sqrt{1 - u^2 / c^2}} \right)$$

Using product rule of derivative,

$$dp = \frac{mdu}{\sqrt{1 - u^2 / c^2}} + \frac{m(u^2 / c^2) du}{(1 - u^2 / c^2)^{3/2}} = \frac{mdu}{(1 - u^2 / c^2)^{3/2}}$$

Substituting dp in eq(1), we get

$$K = \int_0^u u dp$$

$$K = m \int_0^u \frac{u du}{(1 - u^2 / c^2)^{3/2}}$$

$$K = \frac{mc^2}{\sqrt{1 - u^2 / c^2}} - mc^2$$

Or,

$$K = E - E_0$$

Where total relativistic energy E is defined as

$$E = \frac{mc^2}{\sqrt{1 - u^2 / c^2}}$$

and rest energy E_0 is defined as

$$E_0 = mc^2$$

In an isolated system of particles, the total relativistic energy remains constant.

In the limit of low speeds:

$$K = \frac{mc^2}{\sqrt{1 - u^2 / c^2}} - mc^2$$

$$\frac{1}{\sqrt{1 - u^2 / c^2}} = (1 - u^2 / c^2)^{-1/2} = 1 + \frac{u^2}{2c^2} + \dots$$

$$K = mc^2 \left(1 + \frac{u^2}{2c^2} + \dots \right) - mc^2 \rightarrow \frac{1}{2} mu^2$$

as $u / c \rightarrow 0$

Thus, relativistic kinetic energy expression reduced into old kinetic energy expression for low speeds.

From

$$E = \frac{mc^2}{\sqrt{1-u^2/c^2}} = \gamma mc^2 \quad \text{and} \quad \vec{p} = \frac{m\vec{u}}{\sqrt{1-u^2/c^2}} = \gamma m\vec{u}$$

We get

$$\vec{u} = \frac{\vec{p}}{\gamma m} = \frac{\vec{p}c^2}{E}$$

$$\text{or } pc = E \left(\frac{u}{c} \right)$$

$$\begin{aligned} (pc)^2 &= E^2 \left(\frac{u}{c} \right)^2 = \gamma^2 m^2 c^4 (1 - 1/\gamma^2) \\ &= \gamma^2 m^2 c^4 - m^2 c^4 = E^2 - m^2 c^4 \end{aligned}$$

$$E^2 = p^2 c^2 + m^2 c^4$$

A particle may have energy and momentum even when it has no mass

$$E = pc$$

$$\text{or } c = \frac{pc^2}{E}$$

Rest Energy

Even when a particle is at rest ($v = 0$), it possesses energy due to its mass, known as rest energy: $E_0 = mc^2$

This equation reveals that mass can be converted into energy, and vice versa, a concept fundamental to nuclear physics and cosmology.

Nuclear Fission and Fusion

In nuclear reactions, such as fission (splitting of heavy atoms) and fusion (combining of light atoms), mass is converted into energy. These principal powers nuclear reactors and atomic bombs.

Particle-Antiparticle Creation

Energy can also create particles. For example, a high-energy photon can produce an electron and its antimatter counterpart, the positron. This particle creation is observed in high-energy physics experiments.

Lecture#41

The Reality of Matter: A Physics Perspective

This lecture aims to elucidate the debate about the nature of matter and light, specifically focusing on whether they exist as particles or waves. This exploration is rooted in physics, a field known for its logical approach to understanding and explaining the world through arguments and experiments.

Introduction

For centuries, humanity has pondered the nature of reality, leading to profound questions about the fundamental makeup of the universe. This lecture will delve into one such debate: the reality of matter, particularly through the lens of physics. The focus will be on the dual nature of light and matter, exploring the concepts of particles and waves.

The Nature of Light and Matter

Physics is a logical discipline, where theories and concepts are proven or disproven through systematic experimentation. However, some findings in physics might seem counterintuitive in our everyday experience. For instance, if someone tells you that matter exists in a lump sum form or as individual particles, you might accept it without question. But if the same person claims that matter exists as waves, this might seem nonsensical. Yet, this is the very crux of the debate we will explore.

The Dual Nature of Light

In previous discussions, we explored the question of whether light is composed of particles or waves. Experiments have shown that light exhibits properties of both. When light interacts with matter, it often behaves as if it consists of particles, known as photons. Photons are packets of energy.

The Photoelectric Effect

The photoelectric effect is a phenomenon where light striking a metal surface causes the ejection of electrons. Here's a simplified explanation:

1. **Incident Light:** Light of a certain frequency (ν) hits a metal plate.
2. **Electron Ejection:** The light energy is absorbed by electrons in the metal, causing them to be ejected.

3. **Current Flow:** These ejected electrons move towards a collecting plate, creating a current in the circuit.

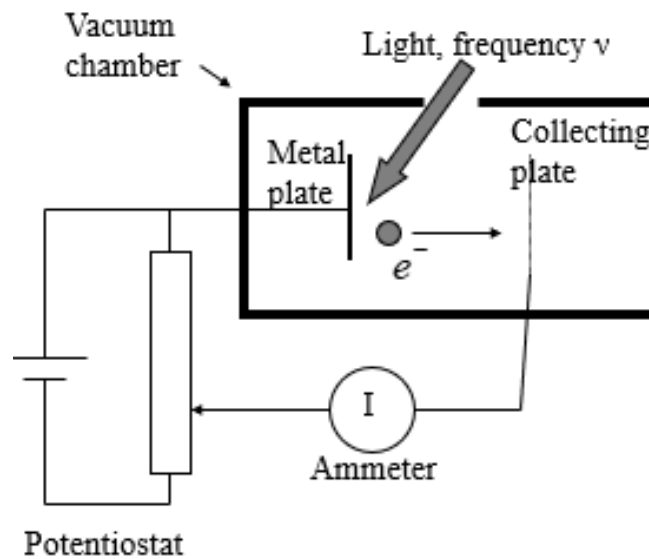


Fig 41.1: Schematic of the photoelectric effect experiment: Light of frequency ν is incident on a metal plate inside a vacuum chamber, causing the emission of electrons (e^-). These electrons are collected by a collecting plate, and the resulting current is measured by an ammeter. A potentiostat is used to control the potential difference between the metal and collecting plates.

Increasing the voltage using a potentiometer increases the energy of the ejected electrons. This observation leads to the conclusion that the energy of photons is proportional to their frequency.

$$E = h\nu$$

Here, E is the energy of the photon, h is Planck's constant, and ν is the frequency of the light.

Einstein's Photon Theory

Albert Einstein proposed that light consists of photons, and the energy of each photon is proportional to its frequency. This was revolutionary because it explained why increasing light intensity alone does not cause electron ejection unless the light has a certain minimum frequency. This minimum frequency corresponds to the work function of the metal.

Work Function (W)

The work function is the minimum energy required to eject an electron from the metal surface.

$$K_{\max} = h\nu - W$$

This equation explains that the kinetic energy of the ejected electron ($K_{\max}$) is equal to the energy of the incoming photon minus the work function.

Compton Scattering

Another critical experiment proving the particle nature of light is Compton scattering. This experiment involves X-rays colliding with a target material, resulting in a change in the wavelength of the scattered X-rays.

1. **Incident Photon:** An X-ray photon strikes an electron in the target.
2. **Scattering:** The photon transfers some of its energy to the electron, causing the electron to recoil.
3. **Shift in Wavelength:** The scattered photon has a longer wavelength (lower energy) than the incident photon.

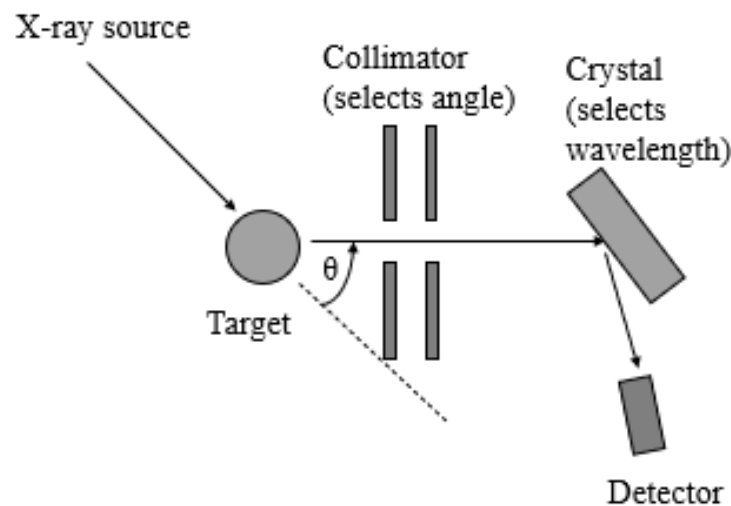


Fig 41.2: Schematic of an X-ray diffraction experiment: X-rays from an X-ray source are directed onto a target. The collimator selects the angle (θ) of the X-ray beam, and a crystal is used to select specific wavelengths of the diffracted X-rays. The intensity of the diffracted X-rays is measured by a detector.

$$\lambda' - \lambda = \frac{h}{m_e c} (1 - \cos \theta)$$

Here, λ is the wavelength of the scattered photon, λ is the initial wavelength, h is Planck's constant, m_e is the electron mass, c is the speed of light, and θ is the scattering angle.

Result: peak in scattered radiation shifts to longer wavelength than source. Amount depends on θ (but not on the target material).

Classical picture: Re-radiation is at *same frequency and wavelength* as incident radiation. Change in wavelength of scattered light is completely unexpected classically

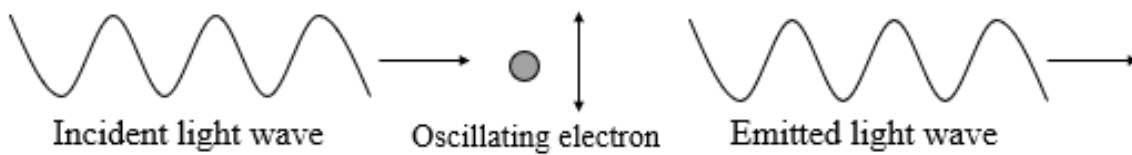


Fig 41.3: Representation of light scattering: An incident light wave interacts with an electron, causing it to oscillate. The oscillating electron then emits a light wave, resulting in scattered light.

Compton's explanation: “billiard ball” collisions between particles of light (X-ray photons) and electrons in the material.

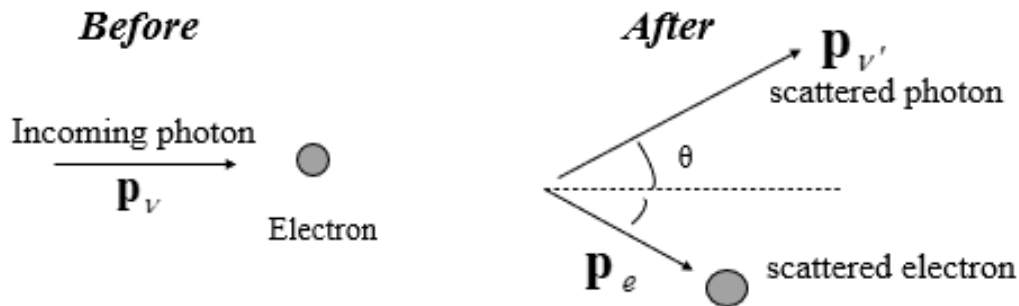


Fig 41.4: Diagram of Compton Scattering: Before the collision, an incoming photon with momentum p_v interacts with a stationary electron. After the collision, the photon is scattered with momentum $p_{v'}$, and the electron is scattered with momentum p_e in a different direction.

Conservation of energy

$$h\nu + m_e c^2 = h\nu' + (p_e^2 c^2 + m_e^2 c^4)^{1/2}$$

Conservation of momentum

$$\mathbf{p}_\nu = \mathbf{p}_{\nu'} + \mathbf{p}_e$$

$$\frac{h}{\lambda} \hat{\mathbf{i}} = \mathbf{p}_{\nu'} + \mathbf{p}_e$$

From this Compton derived the change in wavelength

$$\lambda' - \lambda = \frac{h}{m_e c} (1 - \cos \theta)$$

$$= \lambda_c (1 - \cos \theta) \geq 0$$

$$\lambda_c = \text{Compton wavelength} = \frac{h}{m_e c} = 2.4 \times 10^{-12} \text{ m}$$

Relativity and Photon Properties

Relativity also plays a crucial role in understanding photons. According to Einstein's theory of relativity, the energy (E) and momentum (p) of a photon are related.

Relativistic Energy-Momentum Relation:

$$E^2 = (pc)^2 + (m_e c^2)^2$$

For a photon, since its rest mass (m) is zero: $E = pc$ and $c = \lambda \nu$

Given that the momentum p is related to the wavelength (λ) of the photon:

$$p = \frac{h}{\lambda} = \frac{h\nu}{c}$$

$$E = \hbar \omega \quad p = \hbar k \quad \omega = 2\pi \nu$$

$$k = \frac{2\pi}{\lambda} \quad \hbar \equiv \frac{h}{2\pi}$$

This indicates that the energy of a photon is inversely proportional to its wavelength.

The dual nature of light and matter as both particles and waves is a cornerstone of modern physics. Experiments like the photoelectric effect and Compton scattering provide substantial evidence for the particle nature of light, leading to the understanding that light can behave as discrete packets of energy called photons.

The exploration of these concepts not only enriches our understanding of physics but also offers profound insights into the nature of reality. This duality challenges our everyday notions and encourages us to view the universe through a more nuanced lens.

By adhering to the principles of logic and experimentation, physics continues to unveil the intricate and often surprising fabric of the cosmos.

The Reality of Wave-Particle Duality: Understanding Light and Matter

This book explores the complex nature of light and matter, focusing on the intriguing concept of wave-particle duality. The discussion is grounded in physics, a field characterized by its logical approach and reliance on experiments to prove or disprove theories.

Wave-Particle Duality

The concept of wave-particle duality explains that light and matter exhibit both wave-like and particle-like properties. This duality is essential for understanding phenomena such as interference and diffraction.

Interference and Diffraction

If light were only particles, we could not explain interference and diffraction. For example, when light from two sources meets, we can observe areas of constructive interference (where light intensity is doubled) and destructive interference (where light intensity is nullified). This behavior is characteristic of waves.

De Broglie Hypothesis

The concept of matter waves was first introduced by the French scientist Prince Louis de Broglie in the 1920s. De Broglie proposed that, like light, particles of matter also exhibit wave-like properties. This idea is encapsulated in the de Broglie hypothesis, which states that the wavelength (λ) of a particle is given by:

$$\lambda = \frac{h}{p}$$

Here, h is Planck's constant (6.63×10^{-34} Js) and p is the particle's momentum ($p=mv$, where m is mass and v is velocity).

Example 1: Cricket Ball

Let's calculate the de Broglie wavelength of a cricket ball with a mass of 0.5 kg moving at 2 m/s:

$$\lambda = \frac{h}{p} = \frac{6.63 \times 10^{-34}}{0.5 \times 2} = 6.63 \times 10^{-34} \text{ m}$$

This wavelength (10^{-34}) is *extremely small* compared to the size of the cricket ball ($\sim 10^{-2}$ m). For macroscopic objects like cricket balls, the wave-like behavior is negligible.

Example 2: Electron with Kinetic Energy of 50 eV

Now consider an electron with a kinetic energy of 50 eV. The relationship between momentum and kinetic energy is:

$$K = \frac{p^2}{2m_e} = \frac{h^2}{2m_e \lambda^2} \Rightarrow \lambda = \frac{h}{\sqrt{2m_e K}} = 1.7 \times 10^{-10} \text{ m}$$

This wavelength (1.7×10^{-10} m) is comparable to the size of an atom ($\sim 10^{-10}$ m). Therefore, for electrons, the wave nature becomes significant and can lead to observable phenomena, such as diffraction and interference.

Key Takeaway

- For macroscopic objects (e.g., cricket balls), the de Broglie wavelength is negligibly small, and wave-like properties are irrelevant.
- For microscopic particles (e.g., electrons), the de Broglie wavelength is significant and comparable to atomic dimensions, making their wave-like behavior observable.

Electron Diffraction Experiment (1927)

This experiment demonstrates how electrons behave like waves and can exhibit interference patterns, a key idea in quantum mechanics.

The Setup:

- Electrons are accelerated using a voltage, which determines their energy.
- These electrons are directed onto a crystal, which acts as a reflective surface.
- The electrons collide with the crystal and are reflected at various angles.

Key Concept:

The behavior of the reflected electrons is analysed by measuring the **intensity** of the reflected waves at different angles. To understand this, we consider the **path difference** between the incoming and reflected electron waves.

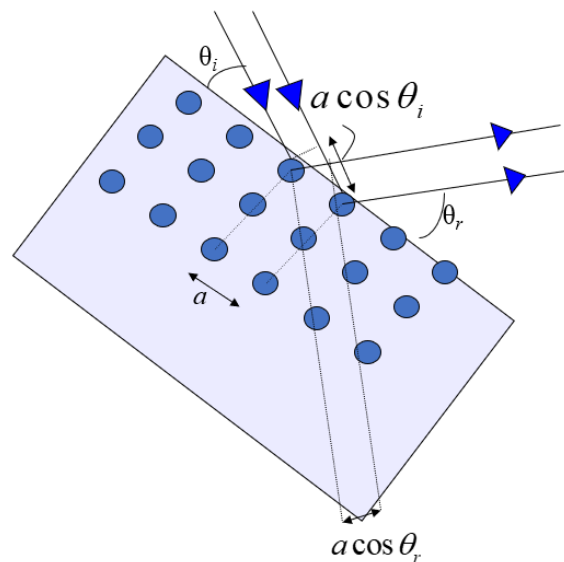


Fig 41.5: Diagram illustrating Bragg's diffraction from a crystal lattice. Incident waves at an angle θ_i interact with lattice planes separated by distance a . The path difference between reflected waves at an angle θ_r is determined by the geometry, enabling constructive interference when the Bragg condition $2a \cos \theta = n\lambda$ is satisfied, where n is an integer and λ is the wavelength of the incident wave.

Path Difference:

The path difference between the incoming and reflected waves is given by:

$$\Delta \text{Path} = a(\cos \theta_r - \cos \theta_i)$$

Where:

- a is the distance between atomic layers in the crystal.

- θ_i is the angle of incidence.
- θ_r is the angle of reflection.

Interference:

- When the path difference (ΔPath) is an integer multiple of the wavelength (λ), i.e., $\Delta\text{Path} = n\lambda$ (where $n=1,2,3,\dots$), **constructive interference** occurs, and the intensity of the reflected electrons is maximized.
- When the path difference is a half-integer multiple of the wavelength ($\Delta\text{Path} = (n+0.5)\lambda$), **destructive interference** occurs, and the intensity is minimized.

Observations:

- The wavelength of electrons is given by the **de Broglie equation**: $\lambda = h/p = h/mv$. Here, h is Planck's constant, m is the mass of the electron, and v is its velocity.
- By changing the accelerating voltage, the energy of the electrons (and hence their wavelength) can be controlled.
- When the electrons reflect off the crystal, their interference creates a **diffraction pattern**, which depends on the wavelength of the electrons and the angles of incidence and reflection.

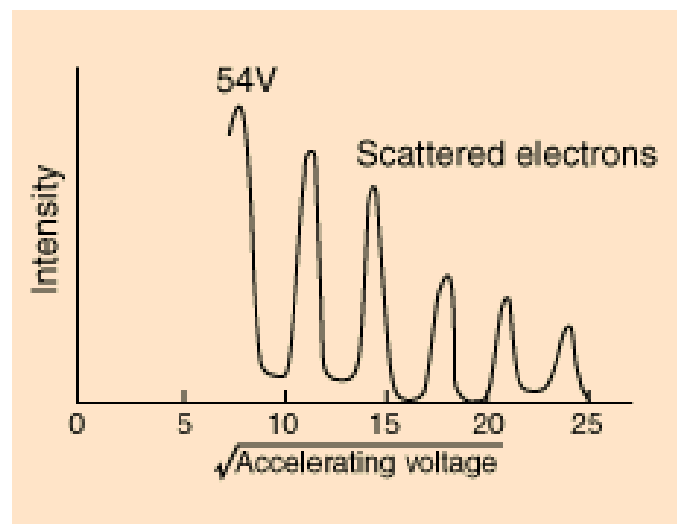


Fig 41.6: Graph showing the variation of intensity of scattered electrons as a function of the square root of the accelerating voltage. The prominent peak at 54V corresponds to resonance, indicative of inelastic scattering, where electrons lose energy by exciting atoms. This behavior is associated with electron diffraction and the quantized nature of energy exchange in atomic systems.

Evidence for Wave Nature:

- If we fix the observation angle (θ_r) and vary the accelerating voltage, the intensity of the reflected electrons shows oscillations: it increases, decreases, and increases again.
- These oscillations indicate the presence of an interference pattern, proving that electrons behave like waves.

Double-Slit Experiment with Particles

This experiment, originally done with light, demonstrates the wave-like behavior of particles such as electrons and helium atoms through interference patterns.

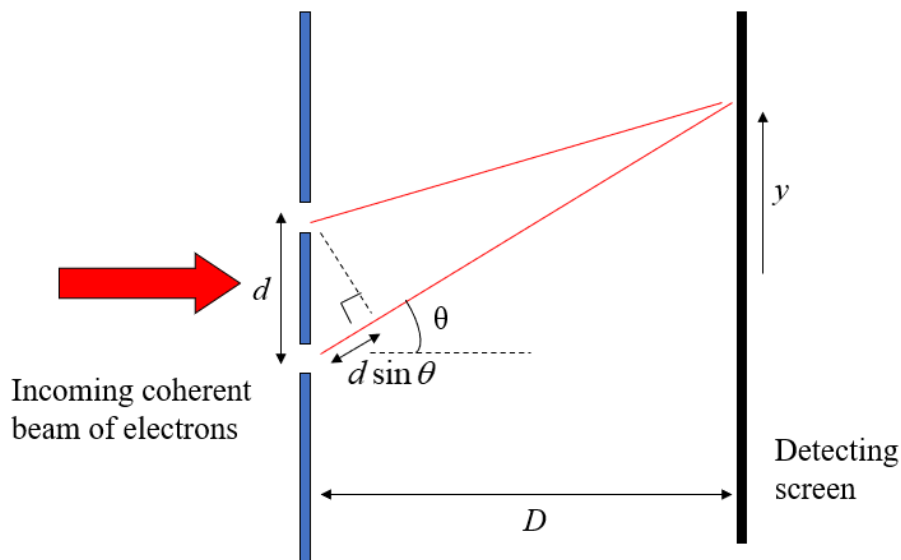


Fig 41.7: Schematic of a double-slit electron diffraction experiment. A coherent beam of electrons passes through two slits separated by distance d , producing an interference pattern on the detecting screen placed at a distance D . The angle θ relates to the path difference $d \sin \theta$, which determines the constructive and destructive interference on the screen at position y .

The Setup:

- A beam of particles (e.g., electrons or helium atoms) is directed toward a plate with **two very small holes (slits)**.
- After passing through the slits, the particles hit a screen, where their positions are recorded using a detector.
- Since particles like electrons are charged, the detector can measure their positions by counting their charges.

Key Idea: Path Difference and Interference

When particles pass through the slits, they follow different paths. The **path difference** between the two paths is:

$$\Delta\text{Path} = d \sin \theta$$

Where:

- d is the distance between the two slits.
- θ is the angle of deflection.

If particles behave like waves, interference occurs based on the following conditions:

- **Constructive interference** (bright spots): $\Delta\text{Path} = n\lambda$ ($n=1,2,3,\dots$).
- **Destructive interference** (dark spots): $\Delta\text{Path} = (n+0.5)\lambda$.

This creates a pattern of alternating bright and dark regions on the screen, indicating where more or fewer particles are detected.

Simplifying the Math:

When the screen is far away from the slits, θ is very small, so $\sin\theta \approx \theta$. The angular position of the interference maxima is:

$$\begin{aligned}\theta &\approx \frac{n\lambda}{d} \\ \Rightarrow \Delta\theta &\approx \frac{\lambda}{d}\end{aligned}$$

The position of the maxima on the screen (y) is related to θ by:

$$y = D \tan \theta \approx D\theta$$

Substituting for θ , we get:

$$\Delta y \approx D\Delta\theta$$

$$\Rightarrow \Delta y = \frac{\lambda D}{d}$$

Where:

- D is the distance from the slits to the screen.
- λ is the wavelength of the particles.
- n indicates the order of the maxima.

Example: Helium Atoms

For helium atoms:

- Distance between the slits (d): $8 \mu\text{m}$.
- Distance to the screen (D): 64 cm .
- De Broglie wavelength (λ): 10^{-10} .

Using $\Delta y = \lambda D / d$, the calculated spacing between maxima is about $8.2 \mu\text{m}$, which matches experimental observations ($8.2\text{--}8.4 \mu\text{m}$). This consistency confirms that helium atoms, like electrons and light, exhibit interference.

Universal Wave Behavior:

This experiment shows that **wave-like interference** is a universal phenomenon:

- All particles, regardless of size (e.g., electrons, helium atoms, or heavier particles), have a wavelength determined by the **de Broglie relation**: $\lambda = h/p = h/mv$ (Where p is the momentum, m is mass, and v is velocity.)
- Even larger particles exhibit interference, though their wavelengths are much smaller.

A Strange Observation: Single-Particle Interference

The most surprising result is that **a single particle can interfere with itself**:

- If only one particle is sent at a time, an interference pattern still forms after many repetitions.
- If one slit is blocked or if an attempt is made to measure which slit the particle passes through, the interference pattern disappears.

This suggests that particles behave as waves when unobserved but as particles when measured. **Observation itself changes their behavior**, a phenomenon that highlights the mysterious nature of quantum mechanics.

Interference and Observation: A Strange Connection

Modern experiments show that a single particle, whether it's a photon, electron, or helium atom, can **interfere with itself** as a wave. However, this wave-like behavior disappears the moment we try to **observe or measure the particle's path**. This phenomenon might seem strange, but it reveals a fundamental principle of quantum mechanics: **the act of measurement changes the state of a system**.

Why Does Measurement Change the System?

Let's break it down with an analogy:

Imagine you're blindfolded and trying to locate two flowers or balls (let's say lilies) in front of you. You don't know exactly where they are, so you stretch out your hand or use a stick to feel around. When you touch a flower, you change its position or bend it slightly. No matter how gently you touch it, your action will alter its original state, even if only a little. The same principle applies in the quantum world.

Now think of the particles:

- If you want to determine the position of an electron, you need to interact with it somehow. For instance, you might use a **photon** (a particle of light) to detect it.
- However, photons carry energy and momentum. When a photon interacts with the electron to "observe" it, this interaction transfers some energy or momentum to the electron, changing its behavior.

Heisenberg's Uncertainty Principle

In the 1920s, **Werner Heisenberg** introduced a fundamental principle in quantum mechanics known as the **Uncertainty Principle**. This principle states:

- **You cannot precisely know both the position and the momentum (speed $\times$ mass) of a particle at the same time.**
 - If you pinpoint the position of a particle, its momentum becomes uncertain.
 - If you know the momentum exactly, you lose information about its position.

This isn't because of a limitation in our measurement tools; it's a fundamental property of nature itself.

Why Does This Happen?

Imagine an experiment where an electron passes through a tiny hole (a slit) and hits a screen with a detector:

1. When the electron passes through the hole, its position becomes relatively well-defined (we know it's somewhere within the hole).
2. However, as a result of this confinement, the electron's **momentum in the perpendicular direction (y-direction)** becomes uncertain. This is because confining the particle's position increases the range of possible momenta, as explained by quantum mechanics.

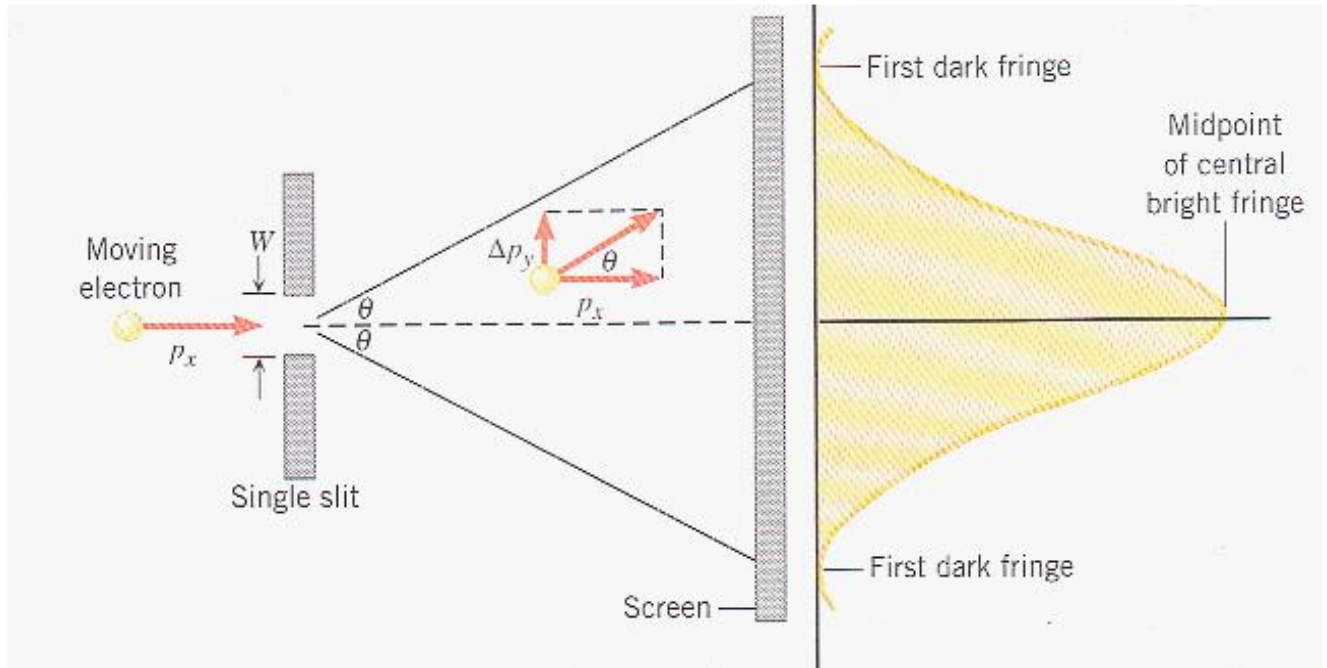


Fig 41.8: Diagram illustrating single-slit diffraction of moving electrons. An electron with momentum p_x passes through a slit of width W , resulting in an angular spread θ due to diffraction. The uncertainty in transverse momentum, Δp_y , corresponds to the spread of the interference pattern on the screen, which includes a central bright fringe and adjacent dark fringes. This setup demonstrates the uncertainty principle.

$$\sin \theta = \frac{\lambda}{W} \Rightarrow \theta \approx \frac{\lambda}{W}$$

But, on the other hand,

$$\tan \theta = \frac{\Delta p_y}{p_x} \Rightarrow \theta \approx \frac{\Delta p_y}{p_x}$$

$$\frac{\Delta p_y}{h / \lambda} = \frac{\lambda}{W}$$

W = uncertainty in y position = Δy

$$\therefore \Delta p_y \Delta y \approx h$$

Here:

- Δy : Uncertainty in the position of the particle in the y -direction.
- Δp_y : Uncertainty in the particle's momentum in the y -direction.
- h : Planck's constant, a tiny but universal number.

This equation tells us that the more precisely we know one value (e.g., Δy), the less precisely we can know the other (Δp_y).

Heisenberg's Microscope

This is an **imaginary thought experiment** designed to illustrate the **Uncertainty Principle** in action.

1. **The Setup:** Imagine using a microscope to observe a particle. A photon (light particle) comes from a source, hits the particle, and scatters. The scattered photon then reaches the microscope lens and forms an image of the particle on a screen.
2. **The Challenge:**
 - To locate the particle's position accurately (Δy), we need photons with a very small wavelength (λ) because shorter wavelengths provide better resolution (the ability to distinguish small details).
 - However, shorter wavelengths mean photons carry more energy and momentum (since $p = h / \lambda$, where h is Planck's constant). When such a high-energy photon collides with the particle, it disturbs the particle's momentum significantly (Δp_y).

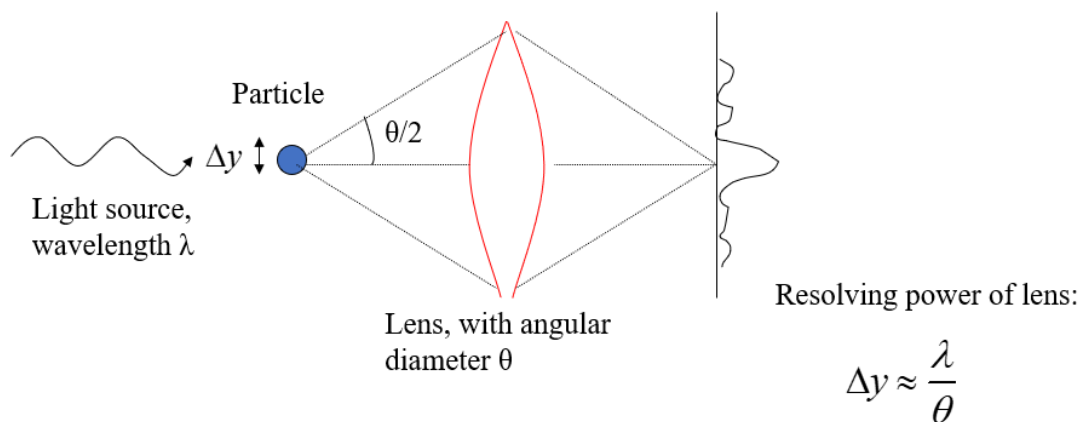


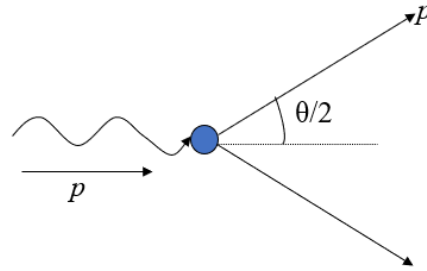
Fig 41.9: Schematic representation of Heisenberg's microscope. A light source emits photons with wavelength λ , which scatter off a particle. The scattered light is focused by a lens with angular aperture θ . The uncertainty in the particle's position, Δy , is related to the resolving power of the lens: $\Delta y \approx \lambda / \theta$. This demonstrates the relationship between measurement precision and the wavelength of the probing light.

Photons transfer momentum to the particle when they scatter.

Magnitude of p is the same before and after the collision.

Uncertainty in **photon** y-momentum
= Uncertainty in **particle** y-momentum

$$-p \sin(\theta/2) \leq p_y \leq p \sin(\theta/2)$$



Small angle approximation

$$\Delta p_y = 2p \sin(\theta/2) \approx p\theta$$

de Broglie relation gives $p = h/\lambda$ and so $\Delta p_y \approx \frac{h\theta}{\lambda}$

From before $\Delta y \approx \frac{\lambda}{\theta}$ hence $\Delta p_y \Delta y \approx h$

Key Insights

1. **Uncertainty Is Fundamental:** Even if we don't actively measure the particle, its position and momentum inherently have uncertainties. This is not due to measurement errors but is a property of nature.
2. **Energy and Time:** A similar uncertainty exists between **energy** (ΔE) and **time** (Δt):

$$\Delta E \Delta t \geq \hbar / 2$$

This means that the more precisely we measure energy, the less precisely we know the time it's associated with, and vice versa.

Why Is This Important?

1. The Uncertainty Principle laid the foundation of **quantum mechanics**, which replaced classical mechanics as the framework for understanding the behavior of particles at very small scales.
2. Quantum mechanics is essential for modern technologies:
 - **Transistors** (used in computers and smartphones), **Semiconductors**, **Lasers**, and much more.

Without the development of quantum mechanics, much of the technology we rely on today would not exist.

Lecture# 42

Introduction to Quantum Mechanics

Introduction

Quantum mechanics is a branch of physics that studies the behavior of atoms and particles smaller than atoms. It explains phenomena that classical physics, which works well for larger objects, cannot describe. This chapter will introduce the need for quantum mechanics, its fundamental concepts, and its implications.

Scale of the Universe: Big and Small

Let us first understand what we mean by 'big' and 'small.' Humans are about 1–2 meters tall. The Earth is roughly 10^7 meters across, while our galaxy is 10^{21} meters across. On the smaller side, cells are about 10^{-5} meters, DNA molecules 10^{-6} meters, and atoms 10^{-10} meters. At the atomic scale and smaller, the rules of classical physics no longer apply.

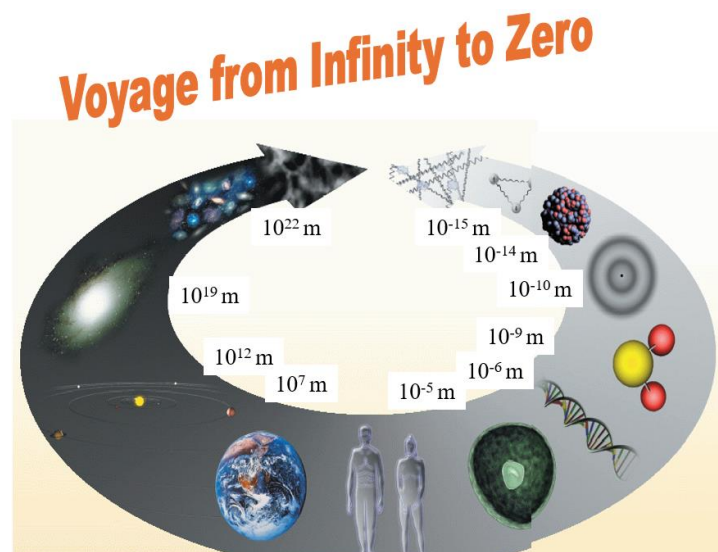


Fig 42.1: A journey through the universe, from the largest scales like galaxies (10^{22} m) to the tiniest particles (10^{-15} m), showing how everything is connected across different sizes.

Quantization in Quantum Mechanics

Quantum mechanics is based on the idea that energy comes in fixed packets or bundles, which are called *quanta*. Imagine you are climbing a ladder: you must climb step by step. You can't be at a height of 1.5 steps or 2.3 steps, it's either one step, two steps, etc. Similarly, in quantum mechanics, energy levels are like steps. You can only gain or lose energy in specific amounts, not in between.

For example, when an electron in an atom absorbs energy, it can only absorb certain quantities of energy—like moving up specific steps on the ladder. The electron cannot exist in a state with an energy level in between these steps. When it drops back down, it releases energy in the same fixed quantity, in discrete packets (quanta).

Similarly, molecules, like water (H₂O) or carbon dioxide (CO₂), also have energy in specific quantities. For instance, a carbon dioxide molecule cannot oscillate with an energy of 0.23 units. Its energy must be in certain fixed amounts, like 1 unit, 2 units, or 3 units.

Quantum mechanics tells us that energy is quantized, meaning it exists only in specific, discrete amounts, not in any arbitrary value.

Why Do We Need Quantum Mechanics?

Classical physics, including Newton's laws of motion, Maxwell's equations for electromagnetism, and thermodynamics, successfully explained most phenomena by the 19th century. However, three significant problems remained unsolved:

1. **Blackbody Radiation:** Heated objects emit electromagnetic radiation. Classical physics predicted an incorrect distribution of energy, known as the ultraviolet catastrophe.
2. **Photoelectric Effect:** Light shining on metal ejects electrons, but only if the light's frequency exceeds a certain threshold. This could not be explained by classical theories.
3. **Atomic Spectra:** Atoms emit light at specific frequencies, forming a discrete spectrum that classical physics failed to explain.

Planck's Hypothesis

In 1900, Max Planck proposed that energy is quantized and comes in packets called 'quanta.'

The energy (E) of a quantum is proportional to its frequency (ν) $E = h\nu$

where h is Planck's constant (6.626×10^{-34} J·s). This idea solved the blackbody radiation problem by correctly predicting the energy distribution.

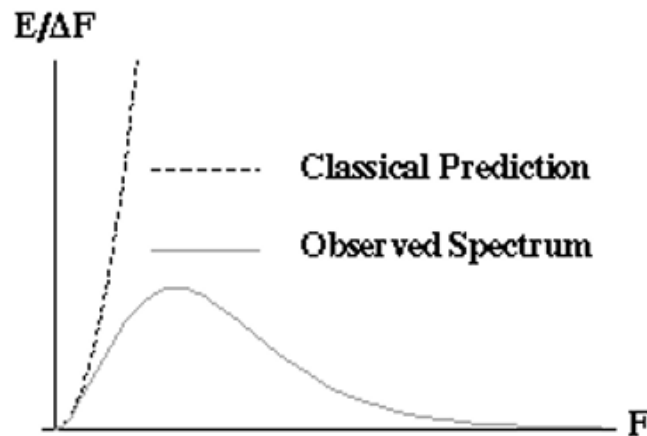


Fig 42.2: Comparison of the classical prediction and the observed spectrum, showing how classical physics fails to explain the actual behavior at higher frequencies, leading to the ultraviolet catastrophe

This hypothesis resolved a significant issue in classical physics concerning blackbody radiation. Classical theories predicted infinite energy at high frequencies, which did not align with experimental data. Planck's quantization of energy matched observations, explaining why energy levels are distributed as they are and laying the foundation for quantum mechanics.

Einstein and the Photoelectric Effect

In 1905, Albert Einstein expanded on Planck's idea, proposing that light itself is made of quanta, called photons. Each photon's energy is given by $E = h\nu$. The photoelectric effect showed that electrons are ejected only if the photon's energy exceeds the material's threshold energy.

Einstein's contribution to the photoelectric effect explained how light interacts with matter on a quantum level. He proposed that light is composed of discrete packets of energy called photons, building on Planck's earlier ideas about quantization.

The photoelectric effect occurs when light shines on a metal surface and ejects electrons. Classical physics predicted that the intensity of light (its brightness) should determine whether electrons are emitted, but this was incorrect. Instead, Einstein showed that only the light's frequency mattered. If the frequency of the light was above a certain threshold, photons carried enough energy ($E = h\nu$) to eject electrons from the metal. If the frequency was too low, no electrons were emitted, regardless of intensity.

This insight not only explained the photoelectric effect but also demonstrated the particle-like nature of light, a key concept in quantum mechanics. It earned Einstein the Nobel Prize in 1921 and helped establish the foundation of quantum theory.

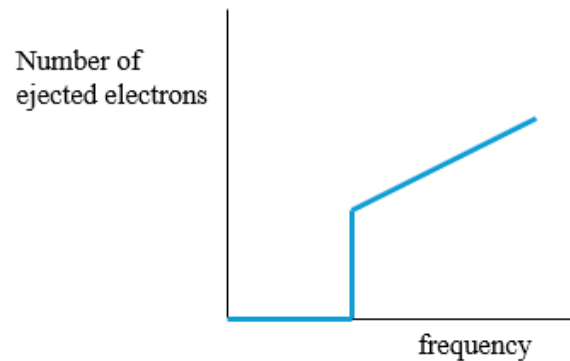


Fig 42.3: The graph shows how the number of ejected electrons depends on the frequency of light, with no electrons ejected below a threshold frequency and an increase above it, demonstrating the photoelectric effect

Mathematically,

Energy: $E = pc = h\nu$ (and $c = \lambda\nu$)

Momentum: $p = \frac{h}{\lambda}$

Also commonly written as:

$E = \hbar\omega$ where $\omega = 2\pi\nu$

$p = \hbar k$ where $k = \frac{2\pi}{\lambda}$, $\hbar \equiv \frac{h}{2\pi}$

Wave-Particle Duality

Louis de Broglie proposed in 1923 that matter, like light, has both wave and particle properties. The wavelength (λ) of a particle is related to its momentum (p):

de Broglie relation $\lambda = \frac{h}{p}$

de Broglie wavelength λ

Planck's constant $h = 6.63 \times 10^{-34} \text{ Js}$

Particles of momentum $p = mv$ have associated

$$\text{wavelength, } \lambda = \frac{h}{mv}.$$

This wave-particle duality was confirmed experimentally in the Davisson-Germer experiment, which showed that electrons produce interference patterns like waves.

Wave-Particle Duality and Bohr's Atomic Model

Matter can behave both as particles and waves, a concept known as *wave-particle duality*. Niels Bohr used this idea to take the first step toward solving the mystery of the atom. However, his solution was not complete at the time, but it marked a significant breakthrough.

The Problem with Classical Physics

In classical physics, if an electron orbits a proton, its velocity is constantly changing direction. This means the electron is accelerating. According to Maxwell's theory, any accelerating charge emits electromagnetic radiation, losing energy in the process. As the electron loses energy, it would spiral into the nucleus, causing the atom to collapse. Classical physics could not explain why this didn't happen, leaving the structure of the atom unresolved.

Bohr's Solution

In 1923, Bohr proposed that the electron behaves like a wave and can only orbit the nucleus if an integer number of *de Broglie waves* fit perfectly within its orbit, as shown in fig 42.4. Imagine the electron as a wave wrapped around the nucleus in a circular shape. If the wave doesn't "close" properly, the orbit is not allowed.

When the wave forms a closed loop, the atom becomes stable, and the electron can stay in that orbit without losing energy. This idea introduced the concept of quantized orbits, where only certain energy levels are allowed for the electron.

This insight was a major step toward understanding atomic structure and laid the foundation for modern quantum mechanics.

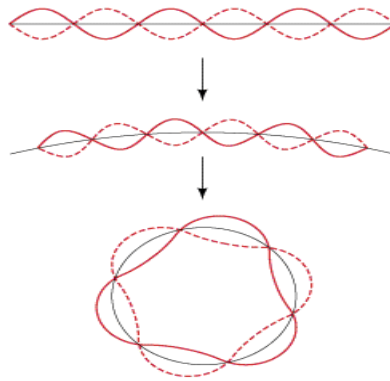


Fig 42.4: This figure illustrates a visualization from the Bohr model of the atom. It shows how an electron's wave can be represented. At the top, the electron behaves like a wave along a straight path. As we move down, the wave shortens and becomes more complex, eventually forming a closed loop, representing the electron orbiting the nucleus in a circular motion. This helps explain the quantized nature of electron orbits in the Bohr model.

Heisenberg's Uncertainty Principle

Imagine measuring the length of an object. Even with the best tools, errors can arise due to external factors like temperature or imperfections in the instrument. These errors can be minimized but never eliminated entirely. Similarly, when measuring tiny particles like electrons, there is always an inherent uncertainty.

Werner Heisenberg stated that it is impossible to measure both the position (Δx) and momentum (Δp) of a particle with perfect accuracy. The principle is expressed as:

$$\Delta x \Delta p_x \geq \hbar / 2$$

$$\Delta y \Delta p_y \geq \hbar / 2$$

$$\Delta z \Delta p_z \geq \hbar / 2$$

where $\hbar$ is the reduced Planck's constant. This principle challenges classical concepts of precision and determinism.

Exploring the Uncertainty Principle in a Confined System

Imagine a single particle, like a ball, trapped between two walls, as shown in fig 42.5. The ball moves back and forth, colliding with the walls endlessly because there is no friction or

energy loss in this ideal setup. In classical physics, we might imagine stopping the ball completely, making it stationary, which means its velocity and momentum would be zero.

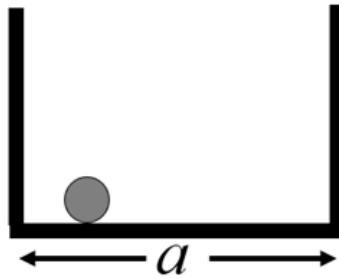


Fig 42.5: This figure represents the particle in a well problem in quantum mechanics. The box has width a , and a particle confined within this space. The particle is restricted to move only inside the box, which illustrates the concept of quantized energy levels that arise due to the confinement.

However, in quantum mechanics, the **Uncertainty Principle** makes this impossible.

The Role of the Uncertainty Principle

The Uncertainty Principle states that we cannot simultaneously know both the exact position (Δx) and exact momentum (Δp) of a particle. These uncertainties are related by the inequality:

$$\Delta x \Delta p \geq \hbar / 2$$

$\therefore a$ is the length of the box

$$\Delta p a \approx \frac{\hbar}{2}$$

$$\Delta p \approx \frac{\hbar}{2a} \quad \text{eq(1)}$$

As we know

$$\text{Kinetic energy} = \frac{1}{2}mv^2$$

$$\therefore p = mv \text{ or } v = \frac{p}{m}$$

$$\text{Kinetic energy} = \frac{1}{2}m \left(\frac{p}{m}\right)^2$$

$$\text{Kinetic energy} = \frac{p^2}{2m}$$

Similarly, we can write

$$\text{Kinetic energy} = \frac{(\Delta p)^2}{2m}$$

using eq(1), we get

$$\text{Kinetic energy} \geq \frac{\hbar^2}{8ma^2}$$

Why the Ball Can't Be Stationary

In classical physics, a stationary ball has zero kinetic and potential energy, and we can pinpoint its position. But in quantum mechanics:

1. If the particle's momentum is exactly zero ($\Delta p=0$), its position becomes completely uncertain, violating the Uncertainty Principle.
2. This means the particle cannot remain perfectly still; there is always some "quantum motion" due to this fundamental limitation.

The Uncertainty Principle demonstrates that even in simple systems, like a ball confined between two walls, the behavior of particles is fundamentally different in quantum mechanics than in classical physics. Particles always exhibit a degree of uncertainty in their position and momentum, preventing perfect stillness or precision.

Application of the Uncertainty Principle to a Harmonic Oscillator

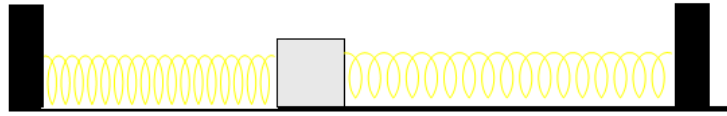


Fig 42.6: This figure illustrates a mass-spring system, a common example of a harmonic oscillator. As the spring compresses and stretches, the mass moves back and forth, demonstrating simple harmonic motion.

The Heisenberg Uncertainty Principle also applies to systems like a **harmonic oscillator**, a fundamental concept in physics. A common example of a harmonic oscillator is a mass attached to a spring as shown in fig 42.6, oscillating back and forth due to the restoring force of the spring.

Classical View of a Harmonic Oscillator

In classical physics, it is possible to envision a scenario where the mass comes to rest at the equilibrium position, eliminating all kinetic energy. In this state, the potential energy is also minimized, and the system is stationary. This view assumes that both the position and velocity of the mass can be known precisely.

Quantum View of a Harmonic Oscillator

In quantum mechanics, however, this stationary state is not possible due to the **Uncertainty Principle**:

$$\Delta x \Delta p \geq \hbar / 2$$

Here, Δx is the uncertainty in position, and Δp is the uncertainty in momentum (related to velocity).

- If we attempt to eliminate all kinetic energy, making the momentum (p) exactly zero ($\Delta p = 0$), the uncertainty in position (Δx) becomes infinite. This means the particle's position cannot be precisely determined, violating the principle.
- Conversely, confining the mass to a very specific position increases the uncertainty in its momentum, ensuring that some motion always exists.

Minimum Energy

The quantum harmonic oscillator inherently possesses a **minimum energy** also called the **zero-point energy**, even in its lowest energy state. This energy is given by:

$$\text{Minimum energy (classical)} = 0$$

$$\text{Minimum energy (quantum)} = \frac{1}{2} \hbar \omega \quad \text{where } \omega = \sqrt{\frac{k}{m}}$$

where ω is the angular frequency of the oscillator. The zero-point energy reflects the fact that the particle cannot be completely stationary due to the Uncertainty Principle.

In both confined systems, like a particle between two walls, and oscillatory systems, like the harmonic oscillator, quantum mechanics imposes fundamental limits on how precisely we can define a particle's position and momentum. These limits give rise to phenomena such as zero-point energy and ensure that no system can be perfectly still, regardless of classical expectations.

The Energy-Time Uncertainty Principle

The **energy-time uncertainty principle** is a fundamental concept in quantum mechanics. It is expressed as:

$$\Delta E \Delta t \geq \hbar / 2$$

where ΔE represents the uncertainty in energy measurement, and Δt represents the uncertainty in the time interval over which the energy is measured.

Explanation of the Principle

This principle implies that:

1. If we want to measure the energy of a particle with absolute certainty ($\Delta E=0$), we would require an infinitely long time interval ($\Delta t \rightarrow \infty$).
2. In reality, measurements are made over finite time intervals (Δt), resulting in an inherent uncertainty in the energy measurement (ΔE).
3. The product of these uncertainties is always greater than or equal to $\hbar/2$, highlighting a fundamental limit in nature.

Implications of the Energy-Time Uncertainty Principle

This principle explains several quantum phenomena that cannot be understood using classical physics.

Quantum Tunnelling

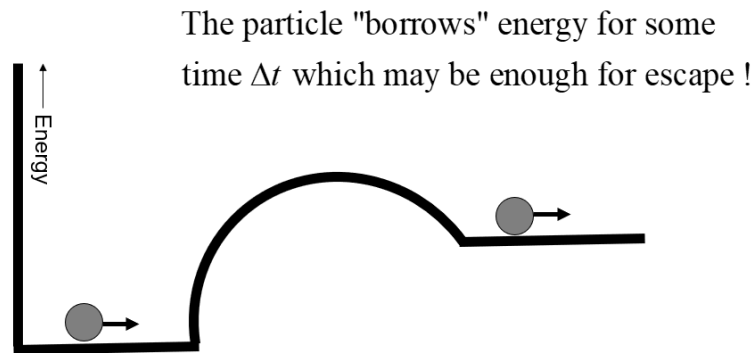


Fig 42.7: Illustration of quantum tunneling: A particle 'borrows' energy temporarily to escape a potential barrier it couldn't classically overcome.

One significant consequence is **quantum tunnelling**, where particles can "tunnel" through energy barriers, as shown in fig 42.7, that would be insurmountable according to classical physics.

- For a very short time (Δt), the particle can "borrow" energy (ΔE) from the uncertainty relationship, allowing it to cross the barrier.
- This process does not violate the conservation of energy because the borrowed energy is "returned" after the particle has tunneled.

Real-World Applications

Quantum tunneling has profound implications, such as:

- **Nuclear Fusion in Stars:** Tunneling enables protons in the cores of stars to overcome the Coulomb barrier, leading to fusion reactions that power stars.
- **Radioactive Decay:** Tunneling allows particles trapped within a nucleus to escape, resulting in radioactive decay.

The energy-time uncertainty principle reveals a fascinating interplay between energy and time in the quantum world. It not only imposes limits on precision but also enables processes

like quantum tunneling, which have critical applications in explaining natural phenomena and technological advancements.

Quantum States and Wave Functions

Probability and Two-State Systems in Quantum Mechanics

Quantum mechanics fundamentally differs from classical mechanics by introducing a probabilistic understanding of particles. The likelihood of finding a particle at a specific location or in a particular state is described by a **probability distribution**, a concept that challenges classical notions of certainty.

In classical mechanics, we can predict the exact state of a system if all initial conditions are known. However, in quantum mechanics this is not possible.

Two-State Systems

A two-state system is one where a particle, such as an electron, can exist in only two possible states. For example, the spin of an electron can be either **up** or **down**, analogous to a coin having either heads or tails.

Representation of States

To describe the spin states:

$|\uparrow\rangle$ is a Dirac "ket" that denotes an electron with spin up.

$|\downarrow\rangle$ is a Dirac "ket" that denotes an electron with spin down.

Example Calculation

Suppose the state of the electron is given by:

$$|\Psi\rangle = \sqrt{\frac{2}{3}}|\uparrow\rangle + \sqrt{\frac{1}{3}}|\downarrow\rangle \quad \because \text{denotes an electron with Dirac ket } |\rangle$$

$$P(\uparrow) = \frac{2}{3} \text{ and } P(\downarrow) = \frac{1}{3}$$

Where

$P(\uparrow)$ means probability of having an electron in spin-up state

$P(\downarrow)$ means probability of having an electron in spin-up state

More generally a state can also be written as : $|\Psi\rangle = c_1|\uparrow\rangle + c_2|\downarrow\rangle$

with probabilities as $P(\uparrow) = |c_1|^2$ and $P(\downarrow) = |c_2|^2$.

$|\Psi\rangle = c_1|\uparrow\rangle + c_2|\downarrow\rangle$ is called a *quantum state* c_1 and c_2 are called *quantum amplitudes*.

If a_1 and a_2 are the amplitudes of the two possibilities for a particular event to take place, then amplitude for the total event is:

$$A = a_1 + a_2.$$

The probability for the event to occur is given by

$$P = |A|^2 = |a_1 + a_2|^2$$

In daily experience we add *probabilities*:

$$P = P_1 + P_2$$

But in quantum mechanics we add *amplitudes*:

$$\begin{aligned} P = |a_1 + a_2|^2 &= a_1^* a_1 + a_2^* a_2 + a_1^* a_2 + a_2^* a_1 \\ &= P_1 + P_2 + a_1^* a_2 + a_2^* a_1 \end{aligned}$$

Two-state systems, such as the spin of an electron, are fundamental examples that demonstrate how quantum states can be probabilistically distributed, leading to measurable outcomes when observed.

The Schrödinger Equation

- In 1926 Schrödinger proposed a wave equation that describes the way matter waves change in space and time.
- Schrödinger's wave equation is a key element in quantum mechanics.
- Schrödinger's wave equation is solved for the *wave function*, Ψ

- The wavefunction $\Psi(x, t)$ of a particle is the amplitude to be at position x at time t .
- The probability of finding the particle at position x between x and $x + dx$ (at time t) is $|\Psi(x, t)|^2 dx$.
- $\Psi(x, t)$ is determined by solving the "Schrodinger equation".

What does it mean?

- Whole subject in itself
- We will look at how can be interpreted
- ψ is the amplitude of the matter wave.
- The end of certainty!

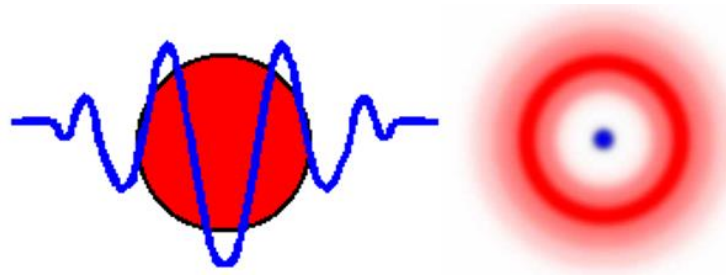


Fig 42.8: Visualization of quantum behavior: On the left, a particle exhibits wave-like oscillations; on the right, the probability distribution of its location is shown as a red intensity pattern

Double-Slit Experiment: Evidence of Wave-Particle Duality

The Experiment Setup

In the famous double-slit experiment, electrons are passed through two small slits in a metal plate, and their positions are observed on a screen behind the slits. The surprising outcome is the formation of an interference pattern, similar to what is seen when light passes through slits. This result provides direct evidence that electrons—and matter in general—exhibit wave-like behavior.

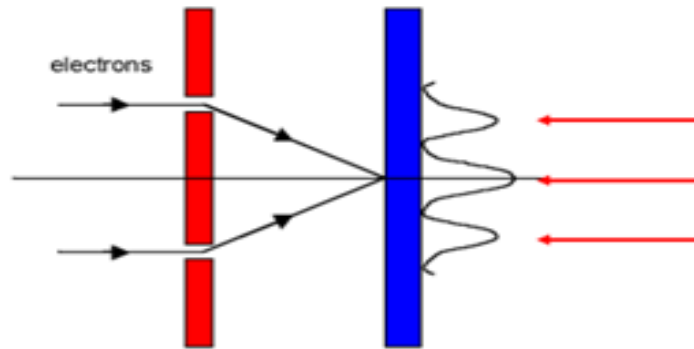


Fig 42.9: Electron double-slit experiment: Electrons pass through two slits (red), creating an interference pattern (blue) that shows their wave-like nature.

Observing the Interference Pattern

- On the screen, the interference pattern appears as alternating regions of high intensity (bright spots) and low intensity (dark spots).
- The high-intensity regions correspond to areas where the value of $|\psi|^2$, the square of the wave function, is large, indicating a higher probability of finding an electron.
- Conversely, the low-intensity regions correspond to areas where $|\psi|^2$ is small, meaning the probability of finding an electron is low.

If a detector is placed on the screen and moved across the interference pattern, it will record fluctuating probabilities: high at some points and low at others. This pattern is evidence that the electrons behave like waves and interfere with each other.

The Wave Nature of Electrons

- The double-slit experiment confirms the wave-particle duality of matter, a cornerstone of quantum mechanics.
- Electrons, which are particles, also behave like waves. These waves can interfere constructively (creating maxima) or destructively (creating minima).

Self-Interference and the de Broglie Wavelength

- When the number of electrons passing through the slits is reduced to the point that only one electron passes through at a time, the interference pattern still forms over time.

- This suggests that an individual electron behaves as a wave and interferes with itself as it passes through both slits simultaneously.
- The wave-like behavior is determined by the electron's de Broglie wavelength, which is related to its momentum:

Wave-Particle Duality and Measurement

If we attempt to measure which slit the electron passes through:

- The act of measurement disrupts the wave function.
- The interference pattern disappears, and the electrons behave purely as particles.

This highlights a fundamental principle of quantum mechanics: the outcome depends on whether or not we observe the system.

Key Takeaways

1. Wave-Particle Duality: Electrons and other particles behave as both particles and waves.
2. Interference Pattern: The wave nature of electrons causes them to interfere, creating observable patterns.
3. Measurement Effect: Attempting to observe the particle's path destroys the interference pattern, emphasizing the probabilistic nature of quantum mechanics.
4. de Broglie Wavelength: The wavelength of matter waves depends on the particle's momentum and determines its wave-like properties.

This experiment profoundly altered our understanding of the nature of matter and laid the foundation for the principles of quantum mechanics.

Understanding the Electron Cloud and Probability Distribution in the Atom

The Role of the Wave Function in Locating Electrons

In quantum mechanics, the **wave function** (ψ) provides information about the probability distribution of an electron in an atom. While it is impossible to pinpoint the exact location of an electron due to the **Heisenberg Uncertainty Principle**, the wave function helps us determine where the electron is most likely to be found.

Electron Probability Distribution in a Hydrogen Atom

Consider a hydrogen atom, which consists of a **proton** at the center and an electron surrounding it. The electron's position is described by a probability distribution:

- Near the proton, the probability of finding the electron is highest.
- As we move farther from the proton, the probability decreases.

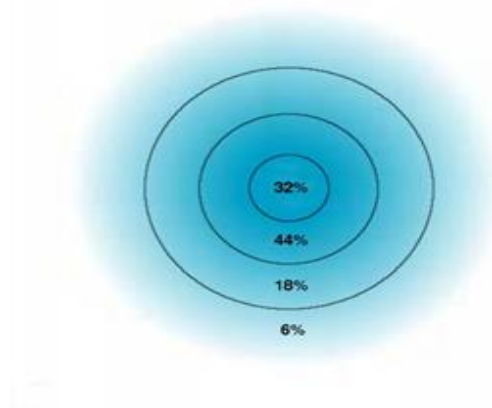


Fig 42.10: This figure illustrates a series of concentric circles, each representing different percentage values (32%, 44%, 18%, and 6%), as a probability distribution of finding an electron at various distances from a central point, illustrating the likelihood of an electron's location in a quantum system.

For example, in a hydrogen atom:

- The probability of finding the electron in the **first region (closest to the proton)** is approximately **32%**.
- In the next outer region, the probability increases to **44%**, indicating that this is the most likely region for the electron to be found.
- Beyond that, the probabilities decrease to **18%**, **6%**, and so on.

While it is possible for the electron to be very far from the proton, the chances of this are extremely small.

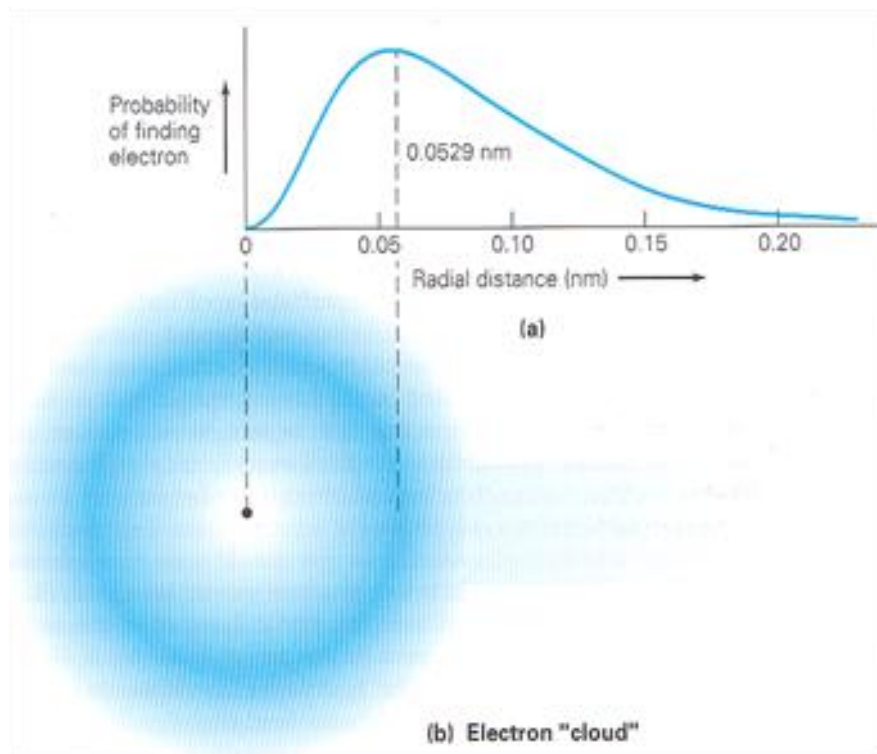


Fig 42.11: The figure shows (a) the probability distribution of finding an electron as a function of radial distance from the nucleus, peaking at 0.0529 nm, and (b) the corresponding electron 'cloud,' representing the regions where the electron is most likely to be found

The 1s State and Electron Cloud

The **1s state** (or the ground state of hydrogen) represents the lowest energy level where the electron is most likely to be found. In this state:

- The **angular momentum** of the electron ($L = 0$) means it has no rotational motion around the nucleus.
- The electron forms a **cloud-like distribution** around the nucleus.

This electron cloud is denser closer to the nucleus and becomes thinner as we move farther away, reflecting the changing probabilities of the electron's location.

A New Picture of the Atom

This modern understanding of the atom differs significantly from the classical **Bohr model**, in which the electron was thought to revolve around the nucleus in fixed orbits with defined angular momentum.

In quantum mechanics:

- The nucleus (proton) forms the dense core of the atom.
- Surrounding the nucleus is a **probability cloud** of electrons, resembling a smoke-like structure.
- The **density of the cloud** indicates the likelihood of the electron's presence at any given location.

Some Insights

1. **Electron Cloud:** The electron does not follow a specific path but exists as a cloud of probabilities around the nucleus.
2. **1s State:** In this ground state, the electron has the highest probability of being found near the nucleus.
3. **Probabilistic Nature:** The wave function provides probabilities, not certainties, about the electron's location.
4. **Modern Atomic Model:** This probabilistic approach to understanding the atom contrasts with earlier deterministic models like the Bohr model.

This concept of the electron cloud revolutionized our understanding of atomic structure and laid the foundation for modern quantum chemistry.

Exploring P States and Higher Orbitals in the Atom

Understanding P States

Beyond the simplest atomic state, the 1s state, we encounter more complex electron distributions associated with higher angular momentum. These are described as P states:

- In P states, the angular momentum quantum number ($l=1$) indicates the presence of angular momentum.

- There are three distinct forms of P states, corresponding to the magnetic quantum number (m_l), which can take values of -1 , 0 , and $+1$.
- These states represent different orientations of the electron's probability cloud in space, resulting in characteristic "dumbbell-shaped" orbital structures.

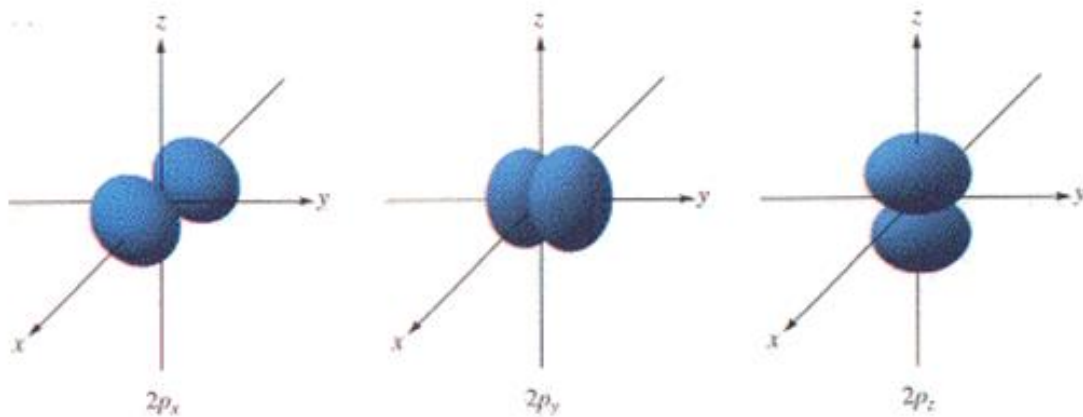


Fig 42.12: The figure shows the shapes and orientations of the three 2p atomic orbitals ($2p_x$, $2p_y$, and $2p_z$), each aligned along a different axis (x, y, and z) in three-dimensional space.

D Orbitals and Beyond

Electrons can also occupy d orbitals, which correspond to an angular momentum quantum number of $l=2$:

- These orbitals exhibit even more intricate shapes, such as "cloverleaf" patterns, reflecting their complex angular momentum distributions.
- As the value of l increases, orbitals become progressively more complex, corresponding to f orbitals ($l=3$) and higher states.

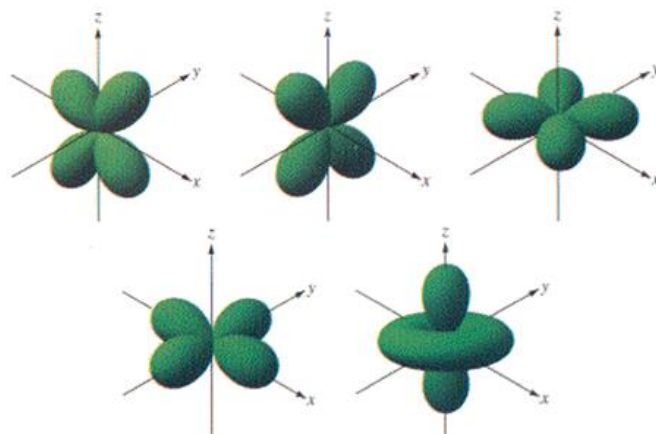


Fig 42.13: Shapes of d-orbitals: Different orientations of d-orbitals (d_{xy} , d_{xz} , d_{yz} , $d_{x^2-y^2}$, and d_{z^2}) in three-dimensional space, showing their unique symmetrical shapes around the nucleus.

The Role of the Schrödinger Equation

The Schrödinger equation serves as the foundation for understanding these orbital shapes and their associated energies. By solving this equation for the hydrogen atom, physicists determine:

1. **Energy Levels:** Each orbital has a specific energy, which depends on the principal quantum number (n) and angular momentum quantum number (l).
2. **Wave Functions:** The mathematical solutions to the Schrödinger equation (the wave functions) describe the probability distributions of electrons in various orbitals.
3. **Orbital Shapes:** These wave functions, when visualized, reveal the characteristic shapes of s, p, d, and higher orbitals.

The Complex Distribution of Electrons

Electrons do not merely "revolve" around the nucleus in simple circular paths, as suggested by earlier models like Bohr's. Instead:

- They occupy regions of space with distinct probability distributions.
- Each orbital corresponds to a unique wave function, which determines the likelihood of finding an electron in a particular region.
- These distributions become increasingly complex with higher angular momentum.

Some Insights

1. P States: Electrons in P orbitals exhibit angular momentum and occupy dumbbell-shaped probability regions.
2. Higher Orbitals: D and f orbitals display even more intricate structures due to their higher angular momentum.
3. Schrödinger Equation: This mathematical framework explains the energy and shapes of orbitals, forming the foundation of quantum mechanics.
4. Modern View of Electrons: Unlike earlier models, quantum mechanics reveals electrons as existing within complex, three-dimensional probability clouds rather than simple orbits.

This modern understanding of atomic orbitals provides profound insights into chemical bonding, molecular shapes, and the fundamental principles governing atomic interactions. For deeper exploration, studying the mathematical solutions of the Schrödinger equation in quantum mechanics texts is highly recommended.

Identical particles are indistinguishable.

If particles A and B are interchanged then

$$|\Psi(1,2)|^2 = |\Psi(2,1)|^2 \Rightarrow$$

either $\Psi(1,2) = +\Psi(2,1)$ BOSONS

or $\Psi(1,2) = -\Psi(2,1)$ FERMIONS

Note: $\Psi(1,1) = -\Psi(1,1) \therefore \Psi(1,1) = 0$

Lecture 43

Introduction to Atomic Physics

Introduction

Thousands of years ago, people were curious about the smallest possible unit of matter. It was only with the advent of modern science, specifically with the discovery of the principles of quantum mechanics about 100 years ago, that we gained a complete understanding of the atom. By combining predictions and observations, we can determine the accuracy of our current understanding of the atom.

To put this in historical context, around 2,500 years ago, a Greek philosopher named Democritus pondered the following question: If I have an apple, and I cut it in half, and keep dividing each half into smaller and smaller parts, will there ever be a point where I can't divide it any further, and will there be a smallest possible unit left? Democritus called this smallest unit an "Atomos." As shown in Fig 43.1.

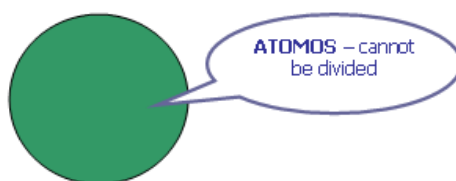


Fig 43. 1: An illustration depicting an atom, the smallest particle in the universe.

Ikhwan al Safa once raised a question about atoms, wondering what happens between them. They named the unknown space between atoms as empty space. After much debate, it was decided that matter comprises of atoms and empty space between the atoms. This led to the concept of the indescribable existence of atoms.

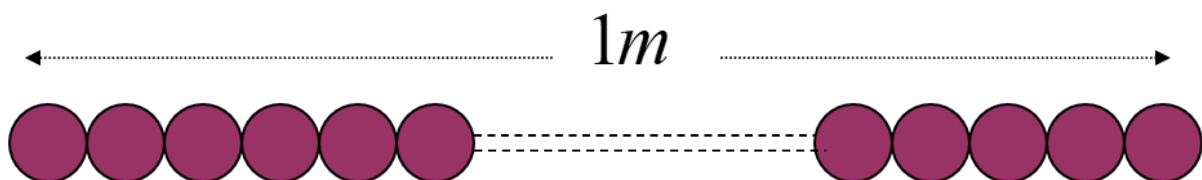
Lavoisier, a French chemist, noted that in all chemical reactions, the total mass of reactants before and after the reaction remains the same. It was concluded that whenever you burn something, the weight of the total material does not change. This is known as the law of conservation of matter.

John Dalton made a significant contribution to the understanding of atoms and matter. In 1803, he laid down some basic principles related to atoms and matter. Dalton stated that every element is made of similar atoms, and you can differentiate between one element, and another based on the weight of their atoms. He also stated that when atoms of different elements combine to form compounds, their ratio is either one and two, one and three, or two and five - simple proportions. When molecules are formed, the properties of that pure matter will neither be related to one element nor to another. For instance, when hydrogen and oxygen gases combine, water is formed, which is liquid and in a molten state - H_2O .

Avogadro's Hypothesis, propounded by Avogadro, is a significant principle in chemistry. It states that if two gases have the same number of molecules, the same pressure, and temperature, then the number of molecules in each gas is equal. For example, if there are equal numbers of hydrogen and oxygen molecules in two gases, the pressure and temperature being the same, then the number of molecules in each gas will be equal.

Assuming that the system has the same number of molecules, if we take 1 kilogram of hydrogen gas, there will be Avogadro's number of molecules present in it, which is $N_0 = 6.023 \times 10^{26}$ molecules. Hydrogen gas consists of hydrogen atoms, whereas hydrogen molecules consist of two hydrogen atoms. Thus, the mass of hydrogen molecules will be twice that of hydrogen atoms. Similarly, if oxygen is present, the mass will be 16 times greater, as oxygen is 16 times heavier than hydrogen. This principle is fundamental, and we can use it to determine the size of atoms and

their mass. If we imagine a solid, we can picture the atoms in it as small spheres or tablets. If the diameter of each sphere is $2r$, then the radius is r , and the number of atoms we can fit in one meter is $1/2r$. We can also calculate the volume of each atom based on the density and atomic weight, which is A/ρ .



In a cube of $1m$ side we then have $(1/2r)^3$ atoms.

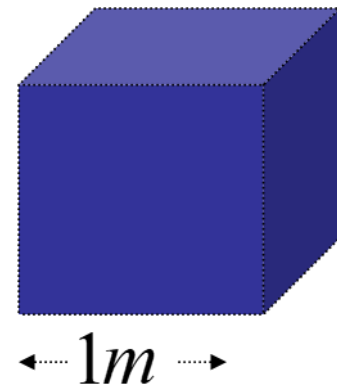
In $1kg$ atom we have 6×10^{26} atoms and each atom occupies a volume $(A/\rho) m^3$.

(where A = atomic weight and ρ = density).

$$\therefore N_0 = (1/2r)^3 \times A/\rho$$

$$\Rightarrow r = \frac{1}{2} \left(\frac{A}{\rho N_0} \right)^{1/3} \quad r_{Ag} \approx r_{Be} \approx 10^{-10} m$$

ATOMS ARE THEREFORE MOSTLY THE SAME SIZE !!



In 1897, J.J. Thomson proposed a model of an atom called the "Plum Pudding" model.

According to this model, the atom is made up of a positively charged material with negatively charged electrons scattered throughout it as shown in Fig 43.2. Thomson's analogy likened the atom to a pudding, with the positive charges spread evenly throughout, and the negatively charged electrons scattered like plums throughout the pudding. This model assumed that if the atom were a sphere, the positive and negative charges would be equally distributed. However, this model was proven incorrect by Ernest Rutherford.

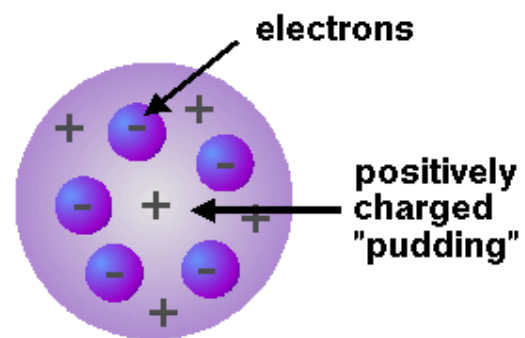


Fig 43. 2: The atom is made up of a positively charged material with negatively charged electrons scattered throughout it, called Plum Pudding Model.

Rutherford's α Scattering Experiment

In 1911, Rutherford conducted an important experiment. In this experiment, alpha particles were released into a chamber, and they collided with a plate of gold as shown in Fig 43.3. Some alpha particles went straight, some went towards the left, some towards the right, but some of them returned after the collision. As alpha particles are heavy and contain two protons and two neutrons, they are small but heavy particle. When they collided with the gold foil, it was observed that some particles either flew out without any change, while others came out at a very low angle as shown in Fig 43.4. Their deflection was very small, but Rutherford was surprised by this fact. He was

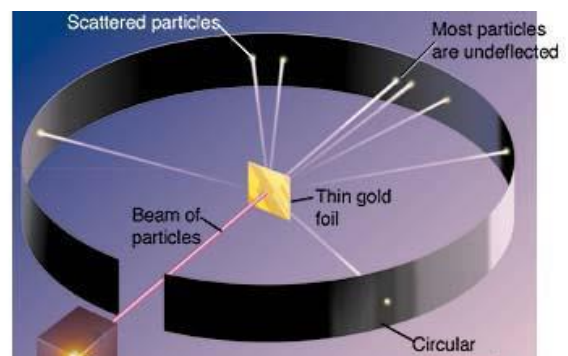


Fig 43. 3: The experiment showed that while most particles passed through the foil, some were deflected at large angles, leading to the groundbreaking discovery of the atomic nucleus. This result suggested that atoms consist of a dense, positively charged core surrounded by a cloud of electrons.

surprised because when a heavy body collides with a light body, the heavy body never goes back, and the light body moves forward. However, if a heavy body collides with another heavy body, it is possible that it will go back after the collision.

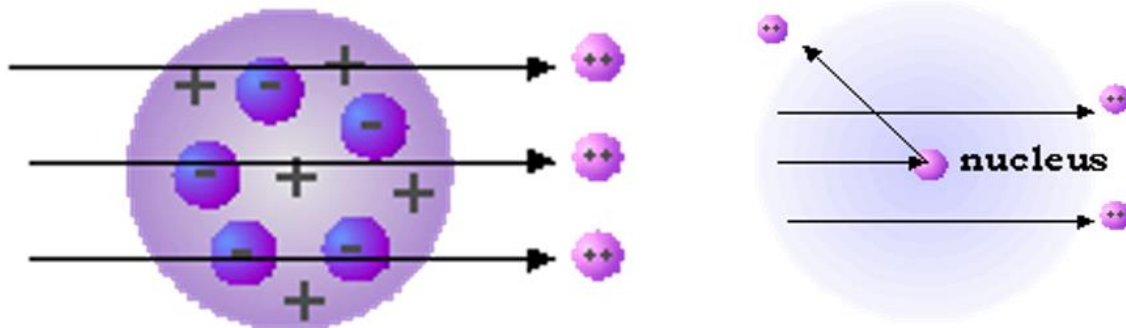


Fig 43. 4: Rutherford's α Scattering Experiment, demonstrating the deflection of alpha particles by a thin gold foil, leading to the discovery of the atomic nucleus.

This experiment proved that there is a small but heavy body somewhere in the atom. Is there such a nucleus, a heavy nucleus, with which this alpha particle would collide and go back? Thus, we now know that the plum pudding model is incorrect. It cannot be that the protons are spread everywhere in the atom along with the electrons. In fact, the shape of the atom is that there is a center in the middle, and the electrons revolve around it. This center is very small, and the electrons are far away from it.

Let me give you an idea of the size of an atom. If we imagine the center of an atom as big as a cricket ball, the nearest electron would be about 8 to 10 kilometers away from it. Therefore, there is a considerable amount of empty space inside an atom, making it mostly empty. Rutherford presented an empty cloud picture of the atom, but there is a significant flaw in this picture that no one has been able to remove. If there is any point around which the electron rotates, it accelerates, which means its velocity (the sum of its speed) keeps changing, even if

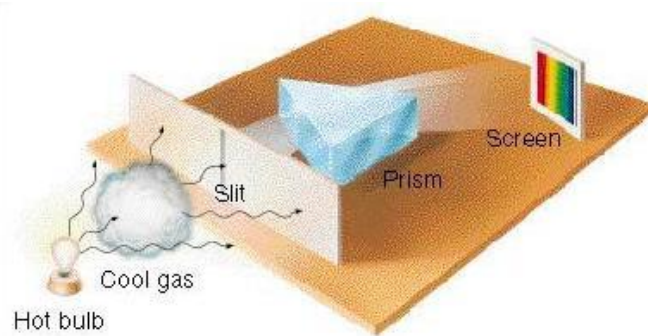


Fig 43. 5: Accelerating electrons emit electromagnetic radiation, leading to a decrease in the atom's radius and potential atomic collapse. This was resolved by the discovery that electrons emit radiation at specific frequencies as observed in absorption spectrum.

its speed remains constant. This acceleration causes electromagnetic radiation to come out from it, as Maxwell had proven a long time ago that whenever a charge accelerates, it emits electromagnetic radiation as shown in Fig 43.5.

When an atom accelerates, it produces radiation, which is related to its speed. As a result, the atom's radius decreases, and its electron collides with the nucleus, leading to the atom's destruction. This problem had no solution until it was discovered that if the electron rotates, the light it emits can have any frequency. However, the radiation coming out of an atom is only of a few specific colors, as seen in an absorbed spectrum experiment. This means that an atom can only absorb frequencies of a certain wavelength as shown in Fig 43.6. When you heat a gas, such as hydrogen or oxygen, the colors and frequencies that emerge are also limited, creating an emission spectrum as shown in Fig 43.7.

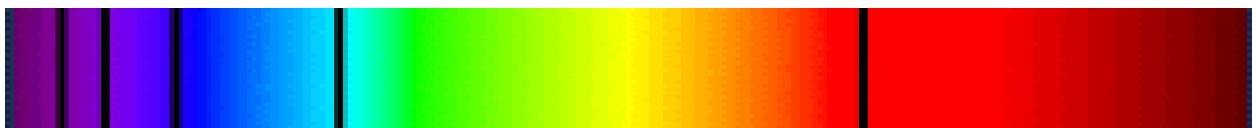


Fig 43. 6: The emission spectrum displays distinct wavelengths of light emitted by an element, while the absorption spectrum shows the specific wavelengths absorbed by the element, leaving dark lines in the continuous spectrum.

At the dawn of the 20th century, the Rutherford model of the atom faced a major challenge. The model failed to account for both absorption and emission, and the light emitted from it was strong enough to destroy the atom. This problem had no solution at the time.

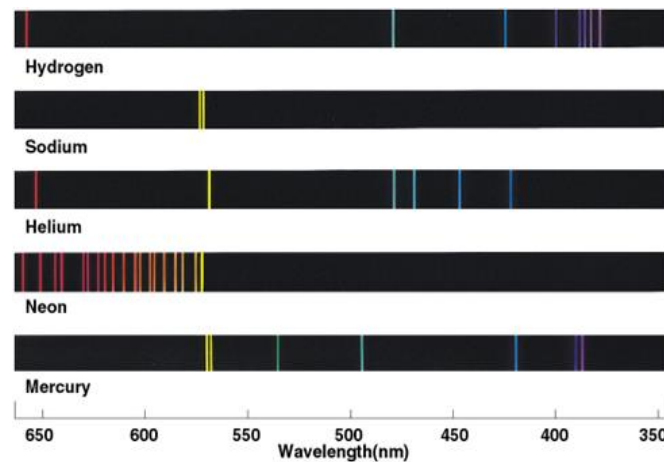


Fig 43. 7: Emission and absorption spectra of hydrogen, sodium, helium, neon, and mercury. The figure compares the characteristic spectral lines emitted and absorbed by each element, highlighting their unique spectral fingerprints and demonstrating how each element's emission and absorption spectra differ.

Bohr's Theory

Niels Bohr proposed a theory about atoms, which states that the energies of atoms are quantized and that transitions between states only occur when they move from one state to another without stopping at any intermediate state as shown in Fig 43.8.

According to Bohr's theory, when an atom absorbs light or energy, it is because a photon collides with it, causing it to move from a lower state to a higher state. The energy of the absorbed photon is exactly equal to the difference between the two states. Similarly, when light is emitted from an atom, it is because the atom moves from a higher state to a lower state.

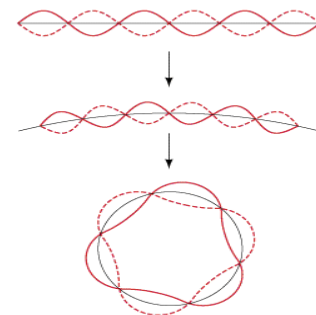


Fig 43. 8: Here is how a standing wave can be bent into a circle. The radius of the circle is just one of the Bohr orbits

Furthermore, the electron states are responsible for the movement between higher and lower states. If the wavelength of the light is not precisely correct, the wave interference will cause the wave to disappear. Therefore, only integral numbers of wavelengths can interfere constructively as shown in Fig 43.9.

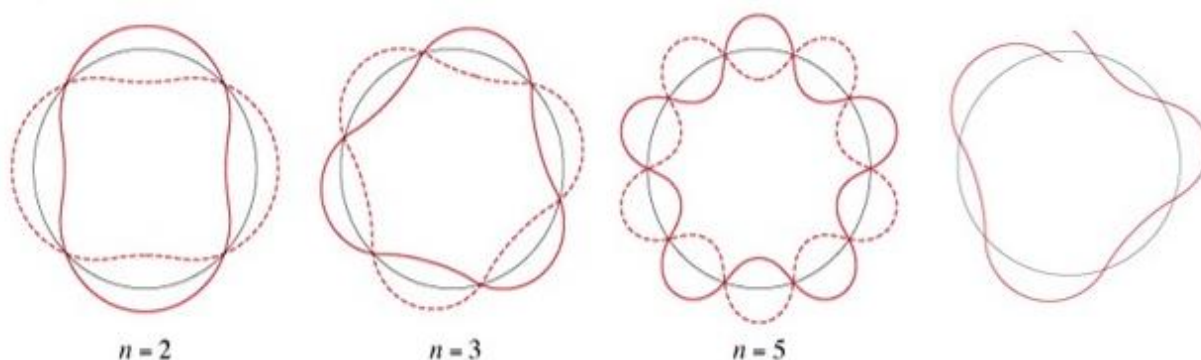


Fig 43. 9: The electron states are responsible for the movement between higher and lower states. If the wavelength is not exactly correct, wave interference makes the wave disappear. So only integral numbers of wavelengths can interfere constructively.

Bohr's Hypothesis

- Electrons can only take discrete energies (energy is related to radius of the orbit)
- Electrons can jump between different orbits due to the absorption or emission of photons.
- Dark lines in the absorption spectra are due to photons being absorbed.
- Bright lines in the emission spectra are due to photons being emitted.
- The electron has a wavelength and forms standing waves in its orbit around the nucleus.
- An integral number of electron wavelengths must fit into the circumference of the circular orbit.

$$n\lambda = 2\pi r$$

$$p = mv = \frac{h}{\lambda}$$

$$= \frac{h}{(2\pi r / n)} = \frac{n\hbar}{r} \quad n = 1, 2, 3, \dots$$

$$L = rmv = n\hbar$$

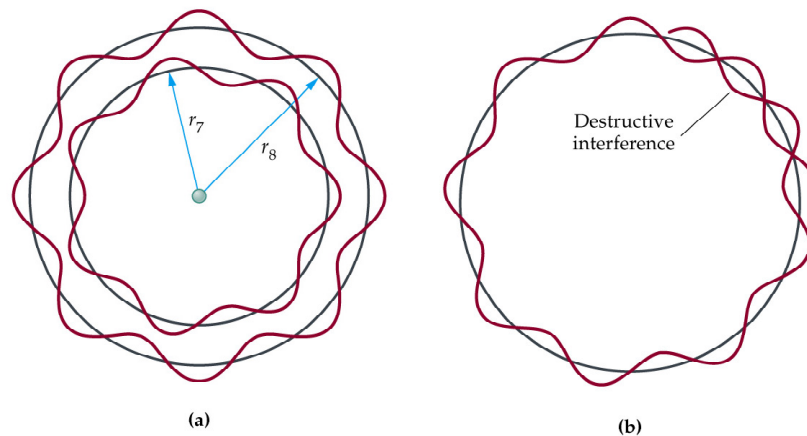


Fig 43. 10: The electron has a wavelength and forms standing waves in its orbit around the nucleus (a). If the wavelength is not exactly correct, wave interference makes the wave disappear (b).

Bohr Orbits

There are two forces that act on an electron: the electrostatic force, which pulls it closer to the proton, and the centrifugal force, which pushes it outwards as shown in Fig 43.11. If the electron is in a state of rest, meaning it's not falling towards the nucleus or running away from it, then these two forces should be equal.

We also know that angular momentum is quantized, and the value is directly proportional to n^2 . Here, n can be 1, 2, 3, and so on, and n^2 can be 1, 4, 9, 16, and so on, which shows that the electron can be one unit away from the nucleus or four, nine, sixteen units away and so on.

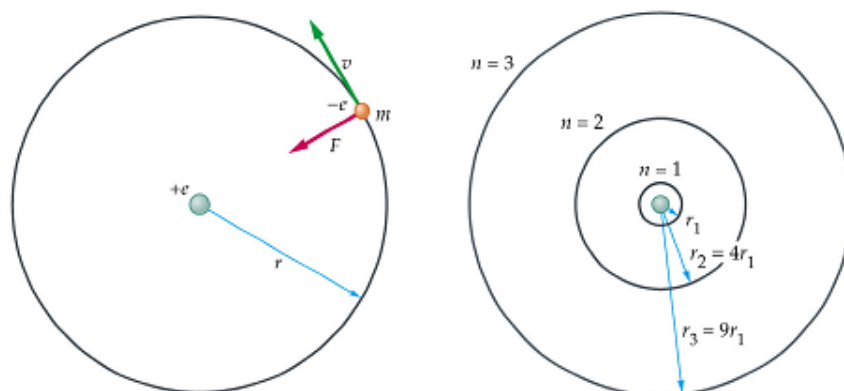


Fig 43. 11: Bohr's orbit illustrating the two forces acting on an electron: the electrostatic force pulling it towards the proton and the centrifugal force pushing it outward. In a stable orbit, these forces are balanced. The figure also shows that angular momentum is quantized, with permissible values proportional to n^2 , where n is a positive integer. This quantization indicates that the electron can occupy discrete orbits at distances corresponding to n^2 units from the nucleus.

$$\frac{mv_n^2}{r} = k \frac{e^2}{r^2}$$

$$L_n = n\hbar$$

$$v_n = \frac{n\hbar}{mr_n}$$

$$r_n = \left(\frac{\hbar^2}{mke^2} \right) n^2 \quad n = 1, 2, 3, \dots$$

The radius of the electron in different states is directly proportional to n^2 , which is called the Bohr Orbit. The energy of these different orbits will also be different.

- For $n = 1$ the electron orbit is closest to the nucleus, $r_n = a_0 \equiv 0.53 \times 10^{-8} \text{ cm}$ (the Bohr radius).
- For higher n , $r_n = a_0 n^2$.

Note that the electron is smaller in orbits farther from the nucleus.

$$v_n = \frac{ke^2}{n\hbar}$$

If we calculate the energy, we can see that the electron orbit and the one closest to the nucleus (where $n=1$) will have the smallest radius orbit, which is approximately half an angstrom. If the electron is in $n=2$, then the radius will be four times bigger, and if it is in $n=3$, then the radius will be nine times bigger, and so on.

Energy of a Bohr Orbit

As the distance between an electron and the nucleus increases, its speed decreases. This change in distance results in two forms of energy - potential and kinetic. The negative sign in the potential energy equation represents the fact that energy is needed to separate positively and negatively charged particles.

$$E = K + U = \frac{1}{2}mv^2 + U$$

$$= \frac{1}{2} \left(\frac{ke^2}{r} \right) - \frac{ke^2}{r} = -\frac{ke^2}{2r}$$

$$\therefore E_n = - \left(\frac{ke^2}{2} \right) \left(\frac{mke^2}{\hbar^2} \right) \frac{1}{n^2} =$$

$$- \left(\frac{mk^2e^4}{2\hbar^2} \right) \frac{1}{n^2} = -(13.6 \text{ eV}) \frac{1}{n^2}$$

In the case of removing an electron from a hydrogen atom, 13.6 electron volts are required when the electron is in the $n = 1$ state, and 4 times as much energy is needed when the electron is in the $n = 2$ state. The kinetic energy is lower than the potential energy, leading to a negative total energy.

The Spectrum of Hydrogen

In this picture, you can see the energy plotted against the principal quantum number, n . The lowest energy level occurs for $n = -R_H$, followed by $-1/4 R_H$ and so on as shown in Fig 43.12. These different levels represent the various states of the hydrogen atom. According to Bohr, transitions can occur between these different levels. If a lot of energy is given to the electron of a hydrogen atom, it can transition from the $n = 1$ state to the $n = 2$ state. The amount of energy required can be calculated by the difference in energy between the two states. For instance, the energy for $n = 1$ is -13.6 and for $n = 2$, it is $-13.6/4$. If a photon with the exact amount of energy required for the transition comes into contact with the electron, the photon is absorbed, and the atom moves from one state to another. Collisions between two hydrogen atoms can also cause excitations. However, the most common way is through light absorption and emission. Many different transitions are possible, resulting in various frequencies being emitted from the hydrogen atom. Each atom has its own unique frequency or fingerprint. Bohr's model has been successful in understanding the spectral lines of hydrogen. It is based on the hypothesis that the delay should be complete, and the number of waves can be one, two, three, four, etc. Quantum mechanics has since been developed, and it has confirmed the same principles that Bohr predicted. When the Schrödinger equation is solved for the wave function, we can get all the energy levels of the hydrogen atom and other atoms, which are the same as predicted by Bohr's model.

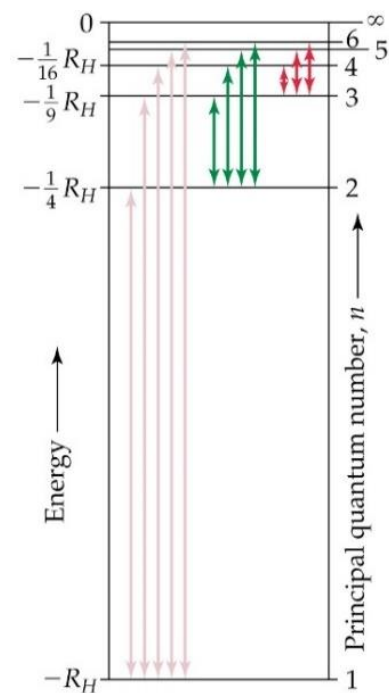


Fig 43. 12: The spectrum of hydrogen energy levels represents the various states of the hydrogen atom.

- The principal quantum number $n = 1, 2, 3 \dots$
- The orbital angular momentum quantum number l , where $L = \sqrt{l(l+1)}\hbar$ and $l = 0, 1, 2, 3 \dots, n-1$.
- The magnetic quantum number m_l where $m_l = -l, \dots, -2, -1, 0, 1, 2, \dots, l$ ($L_z = m_l \hbar$)
- The electron spin quantum number m_s where $m_s = -\frac{1}{2}, \frac{1}{2}$

I would like to find a solution to a problem without solving the Schrödinger equation. This equation yields a function with two parts: one part is called the radial wave function, which is related only to the distance from the center as shown in Fig 43.13. In this graph, you can see a picture of the square of the radial wave function, which shows the probability of finding an electron at distance R . The highest probability is at a_0 .

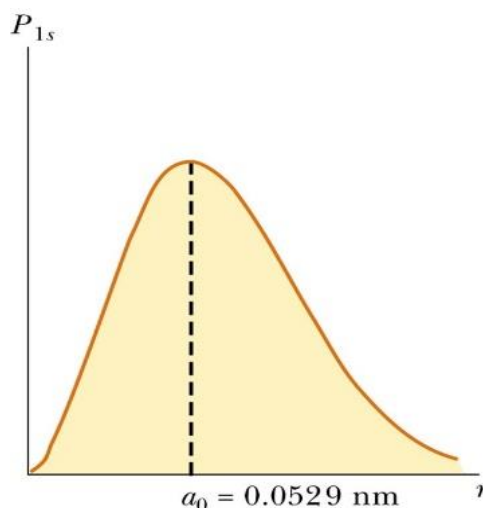


Fig 43. 13: The probability per unit length of finding the electron versus distance from the nucleus for the hydrogen atom in the 1s (ground) state. Note that the graph has its maximum value when r equals the first Bohr radius, a_0 .

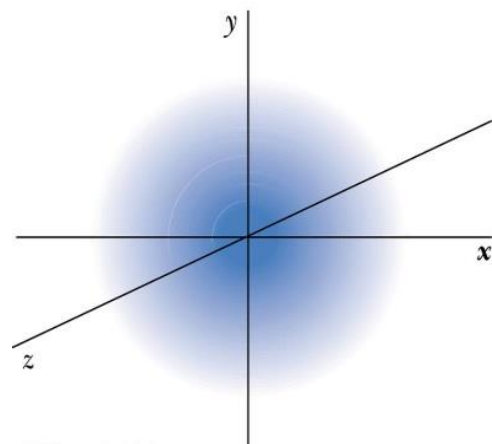


Fig 43. 14: The spherical electron cloud for the hydrogen atom in its 1s state.

When the electron is in the symmetric state, which is equal to 0 or in the s state, it is spread around the nucleus or proton with spherical symmetry. The highest density of the electron cloud is at a_0 . As you move away from the atom, the probability of finding an electron decreases as shown in Fig 43.14. This is different from classical physics where an electron rotates only in a radius of one Bohr radius. We can compare this to smoke - it is spread everywhere, but where it is denser, there is a higher probability of finding an electron. This information is specifically

about the s-state. Similarly, we have $l = 1$, $l = 2$, and $l = 3$ states, with the $l = 1$ state also known as the P state.

- There are three p -orbitals p_x , p_y , and p_z .
- The three p -orbitals lie along the x -, y -, and z -axes of a Cartesian system as shown in Fig 43.15.
- The letters correspond to allowed values of m_l of -1, 0, and +1.
- The orbitals are dumbbell shaped.
- As n increases, the p -orbitals get larger.
- All p -orbitals have a node at the nucleus.

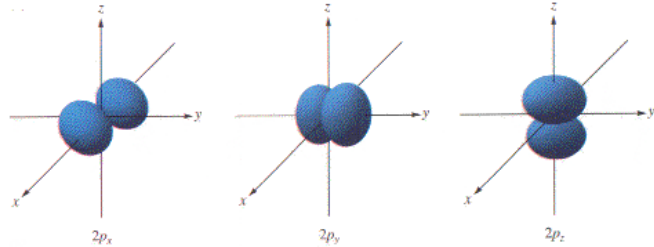


Fig 43. 15: Illustration of the three p -orbitals (p_x , p_y , p_z), aligned along the Cartesian axes. These dumbbell-shaped orbitals correspond to m_l values of -1, 0, and +1, get larger with increasing n , and have a node at the nucleus.

It has maxima on three different axes, and D state is more complicated than that. All these things were for the hydrogen atom.

Electron Spin and the Pauli Exclusion Principle

The hydrogen atom is the simplest atom, consisting of only one electron rotating around the nucleus. However, when an atom has more than one electron, a new phenomenon occurs. As mentioned in my last lecture, electrons are fermions, meaning that if two electrons are exchanged, the function changes its sign. This property of electrons leads to the Pauli exclusion principle, which states that two electrons cannot be in the same quantum state and therefore cannot occupy the same point in space. However, an exception to this rule is allowed for electrons with opposite spins, allowing them to be identified as distinct particles as shown in Fig 43.16. This principle dictates

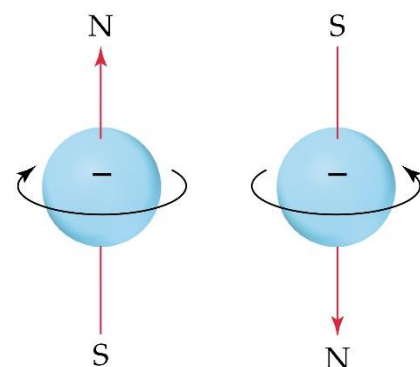


Fig 43. 16: As an electron moves in its orbit about the nucleus, its spin can be either (a) up or (b) down.

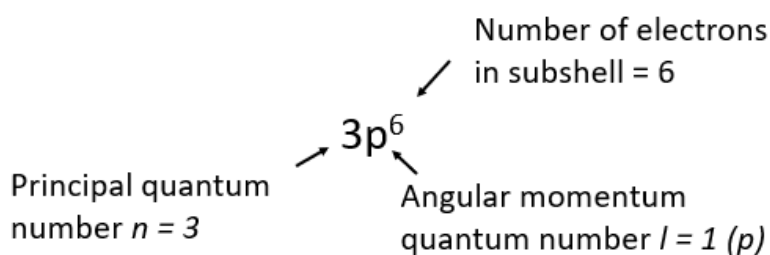
that only one electron with a specific spin can occupy an orbital at a given time. If another electron is added to the same orbital, it must have a different spin. This is how the Pauli exclusion principle works, and it helps to explain the behavior of different atoms.

n	Possible Values of l	Subshell Designation	Possible Values of m_l	Number of Orbitals in Subshell	Total Number of Orbitals in Shell
1	0	1s	0	1	1
2	0	2s	0	1	4
	1	2p	1, 0, -1	3	
3	0	3s	0	1	
	1	3p	1, 0, -1	3	9
	2	3d	2, 1, 0, -1, -2	5	
4	0	4s	0	1	
	1	4p	1, 0, -1	3	16
	2	4d	2, 1, 0, -1, -2	5	
	3	4f	3, 2, 1, 0, -1, -2, -3	7	

Table 43. 1: Table illustrating the relationship between orbitals and quantum numbers. It details how the principal quantum number (n), angular momentum quantum number (l), magnetic quantum number (m_l), define the unique states of electrons within an atom's orbitals.

Helium has two protons in its nucleus and one or two neutrons depending on its isotope. It also has two electrons, each with opposite spin. For Helium, $n = 1$, $l = 0$,

and $m_l = 0$, completing one shell. To make Lithium neutral, one more electron is needed, which goes into the $n = 2$ state where $l = 0$ or 1 , starting a second shell. By adding more electrons, more complex atoms are formed. The process continues with Hydrogen, Helium, Lithium, Beryllium, Boron, Carbon, and so on. When the second shell is filled, eight electrons can come in. There are two possible values of l for $n = 2$, either s-state or P-state. For $l = 0$, two electrons can come in, one with an up spin and one with a down spin. For $l = 1$, there are three possible m_l values ($-1, 0, 1$), and for each one, two electrons can come in, one up and one down. This means that 2 electrons come from the s-state and 6 come from the P-state. This process can be repeated for $n = 3, 4$, and so on, to make electronic configurations for all elements in the periodic table.



Electronic configuration refers to the arrangement of electrons in the shells of an atom. The configuration is determined by the principal quantum number, which is the energy level of the electron. For instance, if the principal quantum number is 3 and the angular momentum

quantum number is 1 (P state), then the subshell can hold up to 6 electrons. Using this method, we can create the electronic structures of various atoms.

For example, oxygen has 8 protons, 8 neutrons, and 8 electrons, while phosphorus has 15 protons, 16 neutrons, and 15 electrons, and zinc has 30 protons, 35 neutrons, and 30 electrons. The structure of an atom consists of a nucleus surrounded by S and P electrons, followed by outer shells.

The Pauli exclusion principle ensures that the electrons occupy the lowest available energy levels, and each shell has a maximum number of electrons it can hold. The $n = 1$ shell is completely filled with a maximum of two electrons, and no electron can enter it from outside. If the Pauli exclusion principle is eliminated, all atoms except hydrogen and helium will come to an $n = 1$ state, causing a significant change in the world.

Subatomic particles in some atoms

16	31	65
O	P	Zn
8	15	30
8 p⁺	15 p⁺	30 p⁺
8 n	16 n	35 n
8 e⁻	15 e⁻	30 e⁻

Lecture 44

Introduction to Nuclear Physics

The Misunderstanding of Nuclear Physics

Many people think that nuclear physics is only about making bombs. This idea comes from the fact that about 60 years ago, nuclear bombs were dropped on Hiroshima and Nagasaki, causing the deaths of around 200,000 innocent people. However, nuclear physics has many other applications. In this chapter, we will explore the true nature of nuclear physics. We aim to understand the centre of the atom and the wonders it holds. This chapter is the beginning of our journey into nuclear physics.

Discovery of the Atomic Nucleus

In 1911, Ernest Rutherford discovered that the atom has a centre, known as the nucleus. Before this, it was believed that an atom contained both positive and negative charges. The negative charges, called electrons, were already known. When we rub certain materials, they become charged because of the electrons. To balance these negative charges, there must be positive charges, which we call protons.

Earlier, the idea was that these positive and negative charges were mixed together, like a pudding. This was known as the plum pudding model. However, when Rutherford conducted his famous gold foil experiment, he found that some alpha particles, which are very heavy, bounced back after hitting the foil. This proved that atoms have a dense centre, or nucleus, with electrons far away from it.

Structure of the Atom

The new understanding is that an atom has a nucleus at its centre, with electrons orbiting around it as shown in Fig 44.1. This is similar to how planets orbit the Sun. However, according to quantum mechanics, this is not entirely accurate. Instead, electrons are like clouds of probability, surrounding the nucleus. Inside the nucleus, we find protons and neutrons.

Protons and neutrons are not elementary particles; they are made of even smaller particles, which we will discuss later. The two basic particles in nuclear physics are the proton and

neutron. The mass of a proton is approximately 1.67×10^{-27} kg, and a neutron is slightly heavier, with a mass of 1.67×10^{-27} kg. Protons are positively charged, while neutrons are neutral.

Units of Measurement in Nuclear Physics

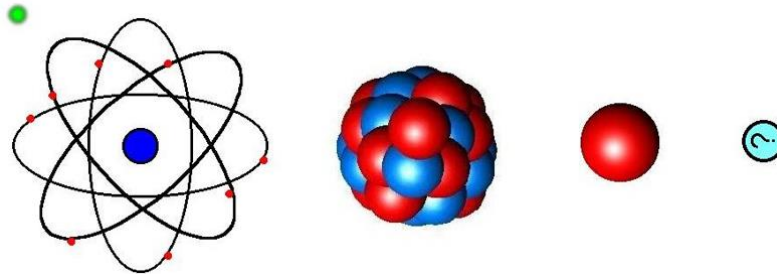


Fig 44. 1: Modern understanding of the atom, with a dense nucleus containing protons and neutrons, surrounded by a cloud of electrons. This model replaces the earlier plum pudding model.

Using standard units like kilograms can be inconvenient in nuclear physics. Instead, we use Einstein's famous equation, $E = mc^2$, to convert mass into energy units. For example, the mass of a proton is $938 \text{ MeV}/c^2$, and a neutron is $940 \text{ MeV}/c^2$. Electrons are much lighter, with a mass of about $0.5 \text{ MeV}/c^2$.

Another unit used in nuclear physics is the fermi, which is equal to 10^{-15} meters. This unit helps measure the size of nuclei, which is around 1.2 fermis.

Identifying Different Nuclei

Nuclei are identified by the number of protons (denoted by Z) and the total number of protons and neutrons (denoted by A). The atomic number (Z) represents the number of protons, while the mass number (A) is the sum of protons and neutrons. Each nucleus has a chemical symbol based on its proton count.

Isotopes

Isotopes are versions of an element with the same number of protons but varying numbers of neutrons. For example, hydrogen has one proton and no neutrons, deuterium has one proton and one neutron, and tritium has one proton and two neutrons as shown in Fig 44.2. These are all isotopes of hydrogen, meaning they have the same number of protons but different numbers of neutrons.

ISOTOPES OF HYDROGEN

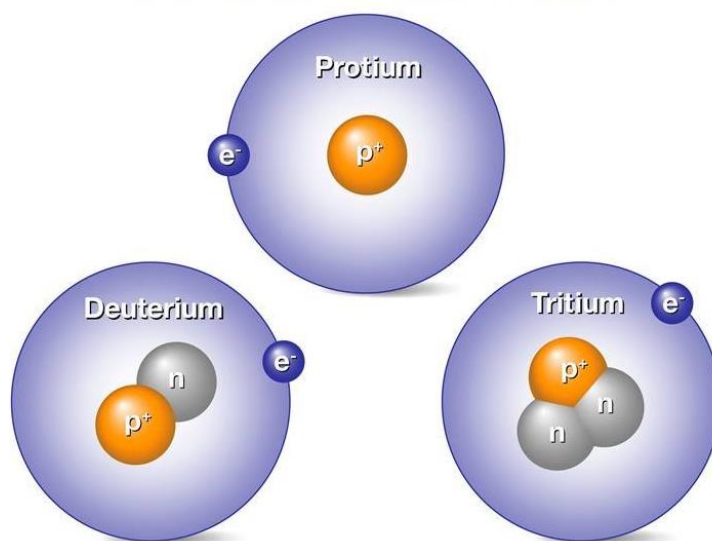


Fig 44. 2: Illustration of hydrogen's three isotopes: protium (one proton), deuterium (one proton, one neutron), and tritium (one proton, two neutrons), highlighting their differences in atomic mass and composition.

Despite their structural differences, all these isotopes of hydrogen exhibit identical chemical properties. This uniformity arises because the number of protons determines an element's chemical behaviour, such as its ability to form water (H_2O), regardless of the number of neutrons. Similarly, carbon exists in several isotopic forms: Carbon-12, Carbon-13, and Carbon-14. Carbon-12, the most abundant form, has six protons and six neutrons. Carbon-13 has an additional neutron, while Carbon-14 contains two extra neutrons. This variability in neutron count among isotopes contributes to different physical properties and behaviours, such as radioactive decay rates.

Imaging the Nucleus

The nucleus, though minute, holds significant mysteries. Unlike atoms, whose structure is observable with light microscopy, the nucleus poses a challenge due to the wavelength limitations of light. Light, being a wave, cannot resolve objects smaller than its wavelength, rendering nuclei invisible to optical microscopes.

To overcome this, scientists employ electron accelerators. Despite electrons being particles, they exhibit wave-like properties described by a wavelength (λ) inversely proportional to their momentum (p). By accelerating electrons to high speeds, their wavelengths shorten enough to interact with nuclei.

Probing the Nucleus

In an electron accelerator, electrons collide with nuclei. These collisions scatter electrons in detectable patterns, revealing crucial details about nuclear structure and composition. The size of a nucleus, approximated by the formula $r = r_0 A^{1/3}$, depends on the number of protons (Z) and neutrons (N). Here, $r_0 = 1.2 \text{ fm}$ represents the nuclear radius constant, reflecting the compact yet dense nature of nuclei.

$$r \approx r_0 A^{1/3} \quad \text{where } r_0 = 1.2 \text{ fm}$$

$$\text{note: } V = \frac{4}{3} \pi r^3 = \frac{4}{3} \pi r_0^3 A$$

$$\therefore \frac{A}{V} = \frac{3}{4\pi r_0^3}$$

This is the number of nucleons per cubic fermi.

Understanding Nuclear Density

Unlike everyday objects with defined boundaries, nuclei exhibit a nearly uniform density throughout. This means that regardless of the element—whether lead, gold, or platinum—the volume occupied per nucleus remains relatively consistent. This phenomenon highlights the unique density distribution within nuclei, where high-density regions blend seamlessly into surrounding space, rather than abruptly ending as in macroscopic objects.

The Nature of Nuclear Force

Inside the nucleus of an atom, there are protons and neutrons. Protons are positively charged, and like charges repel each other. This means that protons naturally push each other away. However, the nucleus remains stable because of a force that is much stronger than the repulsive force between protons. This force is known as the strong nuclear force.

The strong nuclear force is incredibly powerful but has a very short range. If it weren't for this

force, the nucleus would fall apart because the repulsion between protons would be too strong to keep them together. To understand how strong this force is, consider that it takes about 8 million electron volts (MeV) to remove a nucleon (a proton or neutron) from the nucleus, whether the nucleus is small or large. This uniformity indicates the strong nuclear force is very powerful but only acts over very short distances.

Range of the Nuclear Force

The range of the nuclear force is critical in understanding nuclear stability. If the force had a long range, it would pull all nuclei towards each other, making it extremely difficult to remove a nucleon from a larger nucleus. However, since it takes the same amount of energy to remove a nucleon from both small and large nuclei (about 8 MeV), it indicates that the force has a short range, typically between one and two fermis (1 fermi = 10^{-15} meters). This range is much smaller than the size of even the largest nuclei, which are around 10 fermis in diameter.

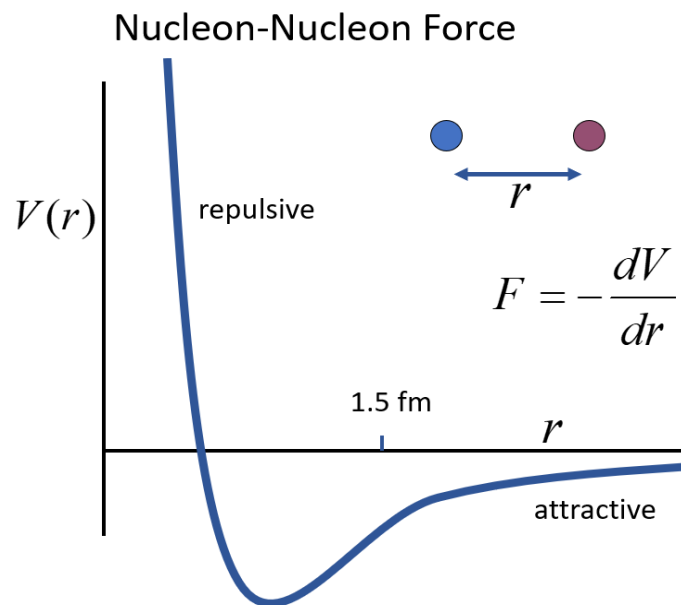


Fig 44. 3: The strong nuclear force between nucleons is attractive but short range; the Coulomb force between protons is long range, but repulsive.

The nuclear force has two main components: an attractive force that pulls nucleons together when they are within a certain range, and a repulsive force that pushes them apart if they get too close as shown in Fig 44.3. This dual nature ensures that the nucleus remains stable, with nucleons packed closely but not too tightly together.

The Yukawa Theory

In 1935, Japanese scientist Hideki Yukawa proposed a theory to explain the nuclear force. He suggested that the force between nucleons (protons and neutrons) is mediated by an exchange of particles he called mesons. According to Yukawa, a proton or neutron sends a meson to another nucleon, and this exchange creates an attractive force, similar to two people playing catch with a ball.

Yukawa's theory was revolutionary because it provided a way to quantify the range of the nuclear force using principles of quantum mechanics, specifically Heisenberg's uncertainty principle. The uncertainty principle states that the product of the uncertainty in energy (ΔE) and the uncertainty in time (Δt) is approximately equal to Planck's constant (h). This means that for a very short time, the conservation of energy can be violated, allowing for the temporary existence of particles like mesons.

Calculating the Range of the Force

Using Heisenberg's uncertainty principle, we can estimate the range of the nuclear force. The strong nuclear force is transmitted between a proton and neutron by the creation and exchange of a pion as shown in Fig 44.4. If a meson with mass m is exchanged between nucleons, the uncertainty in energy ΔE can be approximated by mc^2 , where c is the speed of light. The uncertainty in time Δt is then

$\Delta t \approx \frac{\hbar}{2mc^2}$. The distance d that the meson can

travel in this time is given by $c \times \Delta t$, which leads to $d = \hbar/(mc)$.

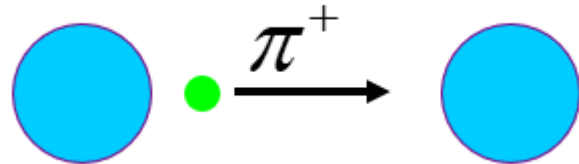


Fig 44. 4: The strong nuclear force is transmitted between a proton and neutron by the creation and exchange of a pion. The pion is created through a temporary violation of conservation of mass-energy and travels from the proton to the neutron and is recaptured. It is not directly observable and is called a virtual particle.

$$\Delta E \Delta t \approx \hbar / 2$$

$$\Delta E = \text{Minimum energy of pion} \approx mc^2$$

$$\therefore \Delta t \approx \frac{\hbar}{2mc^2}$$

$$\text{distance travelled by pion} \approx c \Delta t \approx \frac{\hbar}{2mc}$$

$$\text{Put } \frac{\hbar}{2mc} \approx 1.2 \text{ fm (range of nuclear force)}$$

$$\Rightarrow mc^2 \approx 124 \text{ MeV} / c^2$$

If we substitute typical values, we find that the range of the nuclear force is about 1.2 fermis. Yukawa predicted that the mass of the meson would be around $124 \text{ MeV}/c^2$, and later experiments confirmed the existence of mesons with masses in this range, validating his theory.

Nuclear Fission and Energy Release

One of the most important applications of nuclear physics is the process of nuclear fission, where a large nucleus splits into two or more smaller nuclei as shown in Fig 44.5. When this happens, the total mass of the resulting nuclei is less than the mass of the original nucleus. This mass difference, when multiplied by c^2 (Einstein's famous equation $E = mc^2$), gives the amount of energy released in the fission process.

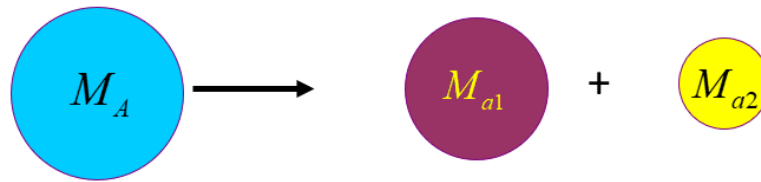


Fig 44. 5: A large atomic nucleus splits into two smaller nuclei upon neutron impact with releasing energy.

$$Q = (M_A - M_{a1} - M_{a2})c^2 = \text{energy released}$$

For example, when a uranium nucleus undergoes fission, it splits into smaller nuclei such as barium and krypton, along with a few free neutrons. The combined mass of these products is less than the original uranium nucleus, and the missing mass is converted into a significant amount of energy. This energy release is the principle behind nuclear power plants and atomic bombs.

Introduction to Binding Energy

When we talk about the nucleus of an atom, one important concept to understand is binding energy. Binding energy is the energy that holds the nucleus together. If we look at different nuclei, we can measure how much binding energy they have. Iron has one of the highest binding energies, around 8.6 or 8.7 MeV (million electron volts) per nucleon as shown in Fig 44.6. In comparison, uranium has a binding energy about 7% lower. Some nuclei are stable, meaning they will stay the same for millions of years, while others are unstable and will eventually decay.

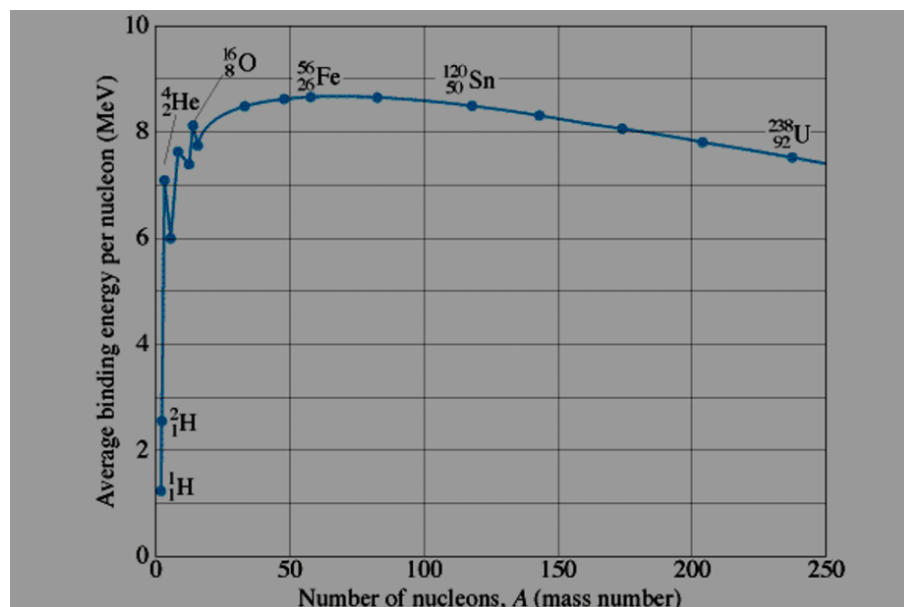


Fig 44. 6: The curve of binding energy, showing how fusion and fission can result in the release of nuclear energy.

Nuclear Decay and Stability

The stability of a nucleus depends on its protons and neutrons. When we talk about unstable nuclei, they decay over time. We cannot predict exactly when a particular nucleus will decay, but we can talk about the decay of many nuclei in statistical terms. If we have a large number of unstable nuclei, the number those decays over a certain time period is proportional to the number of remaining nuclei.

This can be written as:

$$\Delta N \propto N$$

$$\Delta N \propto \Delta t$$

Here, ΔN is the number of decaying nuclei, N is the number of nuclei present, and Δt is the time interval. Combining these, we get:

$$\Delta N = -\lambda N \Delta t$$

The negative sign indicates that the number of nuclei is decreasing over time. For very small-time intervals, this equation can be written as:

$$\frac{dN}{dt} = -\lambda N$$

The solution to this differential equation is:

$$N(t) = N_0 e^{-\lambda t}$$

Here, N_0 is the initial number of nuclei, and λ is the decay constant. The decay constant λ determines how quickly the number of nuclei decreases. The greater λ , the more radioactive the sample.

Half-Life

The concept of half-life is used to describe how long it takes for half of a sample of radioactive material to decay. The half-life ($T_{1/2}$) is the time required for half of the nuclei in a sample to decay as shown in Fig 44.7. This can be derived from the decay equation:

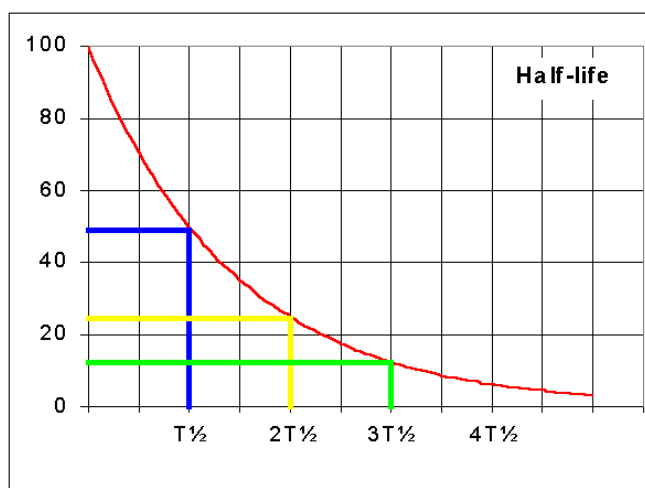


Fig 44. 7: Plot of the exponential decay law for radioactive nuclei. The vertical axis represents the number of radioactive nuclei present at any time t , and the horizontal axis is time. The parameter $T_{1/2}$ is the half-life of the sample.

$$N(t) = \frac{N_0}{2}$$

Substituting this into the decay equation, we get:

$$\frac{N_0}{2} = N_0 e^{-\lambda T_{1/2}}$$

Solving for $T_{1/2}$, we get:

$$T_{1/2} = \frac{\ln 2}{\lambda} \approx \frac{0.693}{\lambda}$$

Stability of Nuclei

The stability of a nucleus depends on the number of protons (Z) and neutrons (N) it has. For small nuclei, a roughly equal number of protons and neutrons is ideal. For example, the most stable isotope of oxygen, oxygen-16, has 8 protons and 8 neutrons. Adding more neutrons makes the nucleus less stable. Heavy elements like lead (Z = 82) have more neutrons than protons. Lead-208, for instance, has 82 protons and 126 neutrons. If we add one more neutron to lead-208, it becomes unstable. This shows that the balance between protons and neutrons is crucial for stability.

No stable nuclei exist naturally with more than 82 protons. When the proton number exceeds 92, no stable nuclei are found at all. This is because the repulsive force between protons becomes too great to be balanced by the strong nuclear force.

Examples of Half-Life

Different substances have very different half-lives. For example, polonium has a half-life of just 1.64×10^{-4} seconds, meaning it decays almost instantly. Krypton has a half-life of 3.16 minutes, while strontium's is 28.5 years. Radium has a half-life of 1600 years, carbon-14's is 5730 years, and uranium-238 has a half-life of 4.5 billion years. The amount of uranium-238 on Earth has decreased by half since the planet formed about 4.5 billion years ago.

Practical Application: Radon Decay

Radon gas decaying to polonium. Radon-222 decays to polonium-218 by emitting an alpha particle. The half-life of radon-222 is 3.8 days. If we start with 20,000 radon atoms, after 3.8 days, 10,000 radon atoms will be left, and 10,000 polonium atoms will have been produced. After another 3.8 days, 5,000 radon atoms will remain, and the number of polonium atoms will have increased.

Problem Example: Silicon Decay

In a lab there is 2 micrograms of pure silicon-31, which has a half-life of 2.5 hours. How many silicon-31 nuclei we have initially and how many are left after 4 hours.

Solution:

First, we need to calculate the number of nuclei in 2 micrograms of silicon-31. Silicon-31 has an atomic mass of 31 grams per mole, and Avogadro's number (the number of atoms in a mole) is 6.02×10^{23} . Therefore, in 31 grams of silicon-31, there are 6.02×10^{23} atoms. In 2 micrograms (2×10^{-6} grams), the number of atoms is:

$$N_0 = \left(\frac{2 \times 10^{-6} \text{ g}}{31 \text{ g/mol}} \right) \times 6.02 \times 10^{23} \text{ atoms/mol} = 3.89 \times 10^{16} \text{ nuclei}$$

Next, we calculate the decay constant λ using the half-life formula: Half-life = 2.5 h = 9000 s

$$\lambda = \frac{0.693}{T_{1/2}} = \frac{0.693}{9000} = 7.7 \times 10^{-5} \text{ s}^{-1}$$

Now, we use the decay equation to find the number of nuclei remaining after 4 hours:

$$\begin{aligned} N(t) &= N_0 e^{-\lambda t} \\ t &= 4h = 14400 \text{ s} \\ N(t) &= 3.89 \times 10^{16} e^{(-7.7 \times 10^{-5} \times 14400)} \\ N(t) &= 3.89 \times 10^{16} \times 0.348 = 1.35 \times 10^{16} \text{ nuclei} \end{aligned}$$

So, after 4 hours, there are approximately 1.35×10^{16} silicon-31 nuclei remaining from the initial 3.89×10^{16} nuclei.

Introduction to Radioactive Decay

Radioactive decay is a fundamental concept in nuclear physics. It helps us understand how old certain objects are, such as ancient artifacts, and even how long-ago animals or humans died. This understanding is based on the Law of Radioactive Decay. This law allows us to determine the age of an object or remains by analysing the radioactive elements present in them.

The Role of Carbon Dating

One of the key applications of radioactive decay is carbon dating. This process is essential for dating ancient biological materials. Here's how it works:

Carbon-12 and Carbon-14

Living organisms absorb carbon from their environment, which includes two types of carbon atoms: carbon-12 and carbon-14. Carbon-12 is stable and does not change over time, while carbon-14 is radioactive and decays at a known rate.

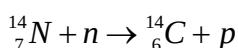
How Carbon-14 Decays

When an organism dies, it stops absorbing carbon. The amount of carbon-12 in its remains stays the same, but the amount of carbon-14 decreases over time. By measuring the ratio of carbon-14 to carbon-12 in a sample, scientists can estimate how long it has been since the organism died. The half-life of carbon-14 is about 5730 years, meaning that half of the carbon-14 in a sample decays over this period.

When a living organism dies, CO_2 no longer absorbed. Thus ratio of carbon 14:12 decreases by half every 5730 years.

Measure activity, $N = N_0 e^{-\lambda t}$ or
activity $A = A_0 e^{-\lambda t}$ with $A_0 = 0.23 \text{ Bq/g}$

Amount of isotopes in atmosphere approximately constant, despite a half-life of 5730 y.



Example of Carbon Dating

A dead body was found in the mountains. Scientists analysed a sample from the body and found that the ratio of carbon-14 to carbon-12 had decreased to half of what it would be in a living organism. This indicated that the person had died approximately 5300 years ago. This calculation uses the decay constant of carbon-14, which is $1.21 \times 10^{-4} \text{ Bq/g}$.

$$\lambda = \frac{0.693}{T_{1/2}} = \frac{0.693}{5730} = 1.21 \times 10^{-4} \text{ y}^{-1}$$

$$0.121 = 0.23 e^{-1.21 \times 10^{-4} \times t}$$

$$\ln \frac{0.121}{0.23} = -1.21 \times 10^{-4} \times t$$

$$t = 5300 \text{ years}$$

Continuous Production of Carbon-14

You might wonder why carbon-14 doesn't keep decreasing universally. The answer is that carbon-14 is continuously produced in the atmosphere when cosmic rays collide with nitrogen atoms. This ongoing production keeps the amount of carbon-14 in the environment relatively constant.

Dating Older Objects with Uranium-238

For dating objects older than what carbon dating can handle, scientists use other radioactive elements like uranium-238. Uranium-238 has a much longer half-life of 4.5 billion years, making it suitable for dating very ancient geological formations and artifacts.

Types of Radioactive Decay

There are three primary types of radioactive decay: alpha decay, beta decay, and gamma decay as shown in Fig 44.8. Each type involves different particles being emitted from the nucleus.

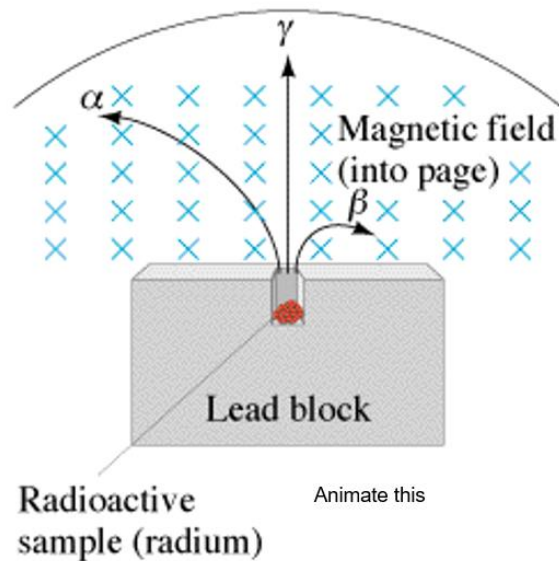


Fig 44.8: The radiation from a radioactive source, such as radium, can be separated into three components using a magnetic field to deflect the charged particles. The detector array at the right records the events. The gamma ray isn't deflected by the magnetic field.

Alpha Decay

Alpha particles consist of two protons and two neutrons. In alpha decay, a large nucleus releases an alpha particle as shown in Fig 44.9. Alpha particles are relatively large and positively charged. They can be ejected from the nucleus when the nucleus is unstable.

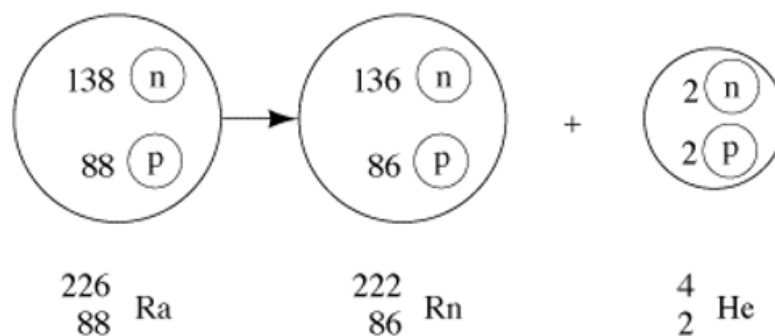


Fig 44. 9: An alpha particle, composed of two protons and two neutrons, is emitted during alpha decay from a large, unstable nucleus. These relatively large, positively charged particles can be expelled from the nucleus due to instability.

$${}_Z^AX \rightarrow {}_{Z-2}^{A-4}X' + {}_2^4He$$

$$Q = (M_{parent} - M_{daughter} - M_{\alpha})c^2$$

- The unstable nuclei simply cannot overcome the proton repulsion indefinitely.
- We saw from the binding energy graph that helium-4 nuclei are very stable.
- Inside big nuclei, two protons and two neutrons seem to form a little gang that hangs together.

Beta Decay

Beta particles are electrons or positrons (the antimatter counterpart of electrons). In beta decay, a neutron in the nucleus transforms into a proton, emitting an electron or a positron. Beta particles are much smaller than alpha particles and can penetrate materials more deeply.

$$n \rightarrow p + e^- + \bar{\nu}$$

$${}_Z^AX \rightarrow {}_{Z+1}^AX' + e^- + \bar{\nu}$$

$${}_Z^AX \rightarrow {}_{Z-1}^AX' + e^+ + \nu$$

- Involves the **weak nuclear force**
- Neutron turns into a proton plus electron plus anti-neutrino
- Proton turns into a neutron plus positron plus neutrino

Gamma Decay

Gamma rays are high-energy photons. They are emitted when a nucleus transitions from a higher energy state to a lower one, releasing energy in the form of gamma radiation. Gamma rays have no mass and no charge, making them highly penetrating.

Gamma decay involves the emission of gamma photons when the nucleus changes energy levels. This is similar to how electrons in an atom emit photons when they transition between energy levels, but the energy involved in gamma decay is much higher, typically in the range of millions of electron volts (MeV).

In-Depth Look at Beta Decay

In beta decay, the emitted beta particles are actually electrons or positrons, which were not initially understood to be such. Today, we know that these particles are fundamental components of the process.

Beta decay is particularly important because it is involved in many natural processes, including the heat and light we receive from the sun.

How Beta Decay Works

In beta decay, a neutron transforms into a proton, an electron, and a third particle called a neutrino. Wolfgang Pauli, a German scientist, theorized the existence of the neutrino to account for the missing momentum and energy during beta decay. Neutrinos are neutral and interact very little with matter, making them difficult to detect.

The Role of the Weak Nuclear Force

Beta decay is governed by the weak nuclear force. This force is weaker and slower than the strong nuclear force but is crucial for many natural processes, including the reactions that power the sun. If the weak force and beta decay did not occur, there would be no heat or light from the sun, and life as we know it would not exist.

Example of Beta Decay

If a neutron is placed in a magnetic field, it can transform into a proton and an electron. The electron will move in one direction and the proton in the opposite direction. However, this transformation does not create a balanced momentum, indicating the presence of a third particle, the neutrino. This process helps maintain the balance and transformation of particles within stars, leading to the production of energy.

Origin of Nuclei

The nuclei of elements found on Earth were not formed at the beginning of the universe. Initially, only hydrogen atoms existed. Over time, through processes occurring within stars, heavier elements like iron, lead, and aluminium were formed.

Formation of Elements in Stars

Inside stars, hydrogen atoms collide and fuse to form heavier elements. When stars explode in supernovae, they scatter these elements across the universe. Gravity then causes these elements to coalesce, forming planets and other celestial bodies. Our Earth and everything on it, including us, are composed of elements formed inside stars.

Summary

The lecture emphasizes that nuclear physics extends beyond weaponry, focusing on unravelling the atom's fundamental nature and natural phenomena. It begins with Ernest Rutherford's 1911 discovery of the atomic nucleus, overturning the previous "plum pudding" model and revealing protons and neutrons at the atom's core. Electrons were described as existing in probability clouds surrounding the nucleus. The study of isotopes, distinguished by neutron numbers, illustrates nuclear structure's complexity, probed using high-energy electron accelerators. The lecture delves into nuclear forces: the strong force binding protons and neutrons via pion exchange, and the short range dictated by quantum principles. It explores decay processes, crucial for dating organic remains like carbon-14 dating, which revealed the age of a 203-year-old specimen as 5300 years. Beta decay, involving electron or positron emission, and gamma decay, emitting high-energy photons, underscore quantum mechanics' role in nuclear processes. Finally, heavy element formation in stars underscores nuclear physics' cosmic implications, enriching our understanding of the universe's elemental composition and evolution.

Lecture 45

Physics of the Sun

Exploring the Physics of Our Sun

Today, we will delve into a fascinating and vital topic: the physics of our Sun. Our Sun, like other stars, is a colossal sphere of gas, but it holds a special place in our universe due to its proximity and crucial role in sustaining life on Earth. Understanding the Sun provides us insights not only about our own star but also about the billions of other stars scattered across the universe.

Basic Facts About the Sun

1. **Mass:** The Sun's mass is approximately 2×10^{30} kg, which is 333,000 times the mass of Earth. We determine the Sun's mass using Newton's Law of Gravitation. When we observe the gravitational interactions between the Sun, Earth, and other celestial bodies like the Moon or other planets, we can calculate the Sun's mass.
2. **Diameter:** The Sun's diameter is about 109 times that of Earth, approximately 1,3920,000 km.
3. **Age:** The Sun is around 4.6 billion years old. We determine this age through various scientific methods, including the study of solar and planetary formation.
4. **Rotation:** The Sun's rotation period varies between its equator and poles which is 25 days at equator, and 36 at poles (at surface). This difference is because the Sun is not a solid body but a sphere of gas. As a result, different parts of the Sun rotate at different speeds.
5. **Temperature:** At its core, the Sun reaches temperatures of about 15 million degrees Celsius. Its surface temperature, however, is much cooler at around 5770 degrees Celsius. We measure these temperatures using black body radiation principles, where the colour and energy spectrum of the Sun's light provide clues about its temperature.
6. **Density:** The Sun's density is about eight times that of gold in its core but averages to about 1.5 times that of water overall. This average density is calculated by dividing the Sun's mass by its volume.
7. **Composition:** The Sun is composed mainly of hydrogen (72%) and helium (25%), with the remaining 3% consisting of heavier elements like iron and carbon. We determine

the Sun's composition by analysing the spectrum of sunlight, which shows absorption lines corresponding to various elements.

8. **Luminosity:** The luminosity of the sun is 3.83×10^{27} joules per second. The Sun emits approximately 4×10^{24} kilowatts of power. To put this in perspective, it would take billions of power plants like the Tarbela Dam (which generates about 5000 MW) to match the Sun's output.

Source of the Sun's Energy

For a long time, scientists speculated about the source of the Sun's immense energy. Early theories suggested chemical reactions or gravitational contraction as possible sources. However, calculations showed that these processes could only sustain the Sun for a few million years, which contradicts the observed age of the Sun. The breakthrough came with Einstein's famous equation, $E=mc^2$, which revealed that mass could be converted into energy. This principle underlies the process of nuclear fusion occurring in the Sun's core, where hydrogen nuclei combine to form helium, releasing vast amounts of energy in the process.

Nuclear Fusion

The Sun generates energy through a process called nuclear fusion. This process involves the fusion of hydrogen nuclei (protons) to form helium nuclei. However, it's essential to understand that the mass of four protons is slightly more than the mass of a helium nucleus, which consists of two protons and two neutrons. This difference in mass results from the nuclear binding energy that holds the nucleus together.

When four protons fuse to form a helium nucleus, energy is released in the form of photons, which we see as light. This process of fusion is what powers the Sun and enables it to emit vast amounts of energy.

Steps of the Fusion Process

1. **Initial Collision:** Two protons collide to form a deuterium nucleus (one proton and one neutron), a positron, and a neutrino as shown in Fig 45.1. The positron quickly encounters an electron, and they annihilate each other, producing gamma rays.
2. **Formation of Helium-3:** The deuterium nucleus collides with another proton, forming helium-3 (two protons and one neutron) and emitting a gamma ray.

3. **Creation of Helium-4:** Two helium-3 nuclei collide to form a helium-4 nucleus (two protons and two neutrons) and release two protons.

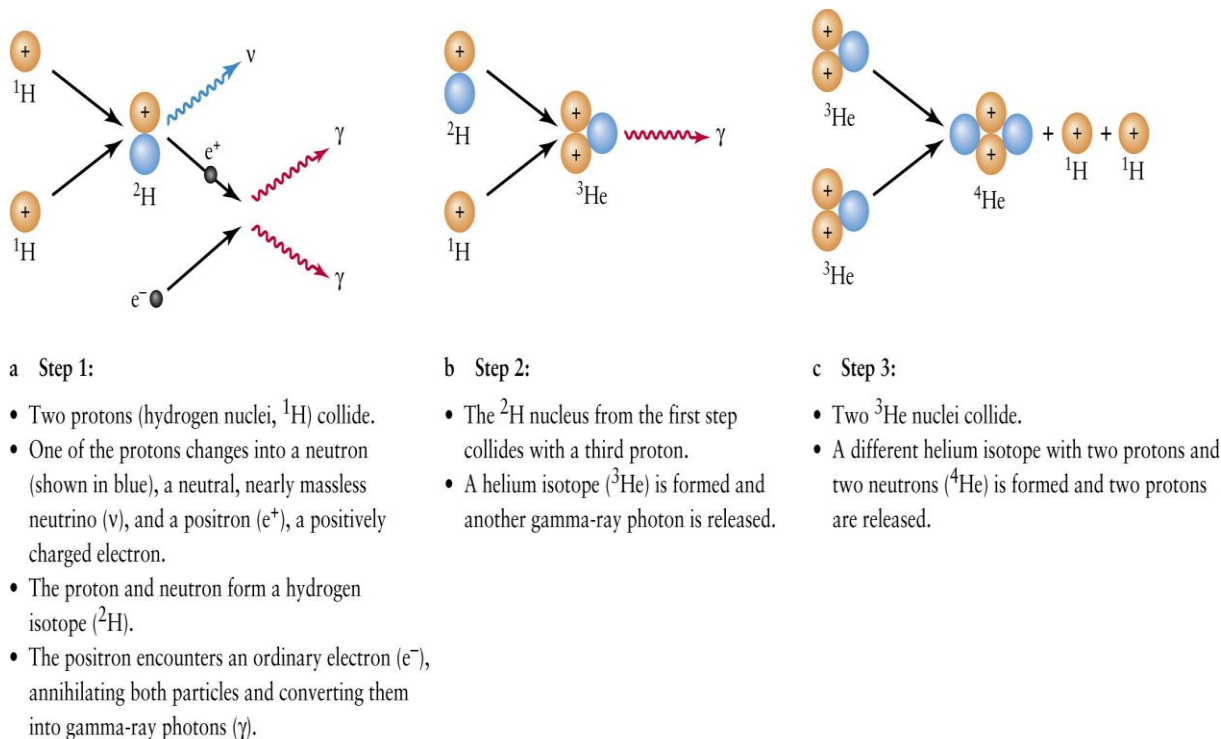


Fig 45. 1: Steps of the fusion process: initial proton collision forming deuterium, formation of helium-3, and creation of helium-4, with gamma rays and protons released during the reactions.

This entire sequence releases energy because the mass of the resulting particles is less than the mass of the initial four protons. The difference in mass is converted into energy according to Einstein's equation, $E=mc^2$.

Nuclear Fusion in the Sun

- Stellar core temperatures are greater than the ~ 8 million K required for hydrogen fusion
- Amount of repulsion increases with more positive charge, so fusion of larger nuclei requires increasingly higher temperatures & densities (100 million K for helium fusion)
- Thus, stars fuse hydrogen first.
- Reaction efficiency ~ 0.008 . Sun can last ~ 10 billion years on core hydrogen fusion

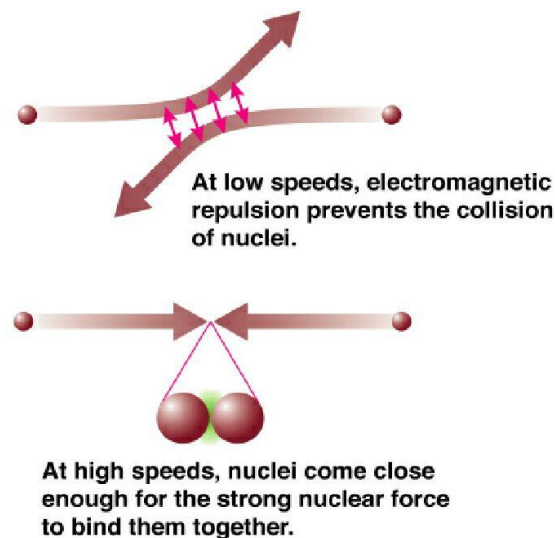


Fig 45. 2: The immense pressure and temperature at the Sun's core provide the necessary conditions for overcoming the electrostatic repulsion between nuclei, enabling fusion reactions.

Why Can't We Replicate Fusion on Earth?

Achieving nuclear fusion on Earth is challenging because it requires extremely high temperatures and pressures to overcome the repulsive force between protons. In the Sun, temperatures reach about 15 million degrees Celsius at the core, allowing fusion to occur. On Earth, creating and maintaining such conditions is currently beyond our technological capabilities.

Structure of the Sun

The Sun can be divided into three main regions:

1. **Core:** The core is the innermost part of the Sun as shown in Fig 45.3, where nuclear fusion occurs. The immense pressure and temperature in the core allow hydrogen nuclei to overcome their repulsive forces and fuse into helium, releasing energy.

2. **Radiation Zone:** Surrounding the core, the radiation zone is where energy from the core is transported outward by photons. However, this energy transfer is slow because photons constantly collide with particles, changing direction and taking a long time to reach the surface. It can take up to 170,000 years for a photon to travel from the core to the surface of the Sun.

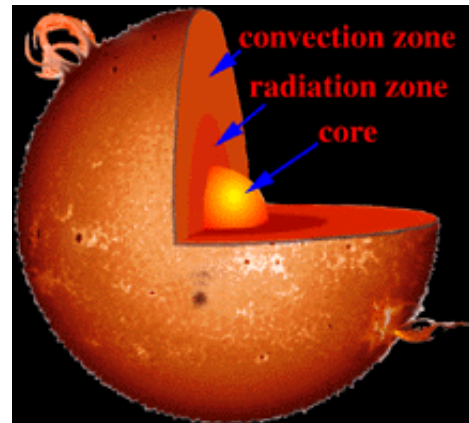


Fig 45. 3: Structure of the Sun's Interior.

3. **Convection Zone:** The outermost layer of the Sun's interior is the convection zone. Here, hot plasma rises to the surface, cools, and then sinks back down in a cyclic process called convection. This process helps transport energy from the inner layers to the Sun's surface.

Solar Granulation

Solar granulation refers to the pattern of small, bright cells, called granules, visible on the Sun's surface. These granules are caused by convection currents of hot plasma rising from the interior. Each granule is typically about 1,000 kilometres in diameter and lasts for several minutes before dissipating.

Surface and Atmosphere

The Sun's surface, known as the photosphere, as shown in Fig 45.4, has a temperature of about 5770 degrees Celsius. Above the photosphere is the corona, a region with much higher temperatures, reaching up to a million degrees Celsius. The corona is visible during a solar eclipse and is studied using spectroscopic analysis to understand its properties.



Fig 45. 4: Outermost layer of Sun – a low-density cloud of plasma; visible during solar eclipses (as above). Hotter than most of the inner layers. Average temperature 1 million K, but can reach 3 million K

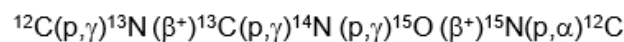
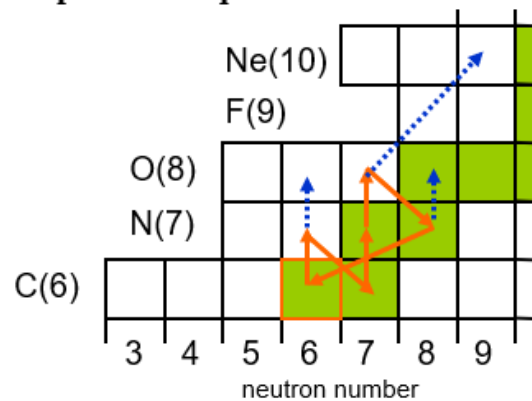
The Sun's Energy Output

The Sun emits a tremendous amount of energy, converting about four million tons of hydrogen into helium every second. This fusion process releases 25 million electron volts of energy and produces neutrinos, which are particles that rarely interact with matter and escape the Sun quickly.

The CNO Cycle

In addition to the proton-proton chain reaction described above, the Sun also uses the CNO cycle (Carbon-Nitrogen-Oxygen cycle) to fuse hydrogen into helium. This cycle involves carbon, nitrogen, and oxygen as catalysts to speed up the fusion process. The presence of these elements is crucial for the CNO cycle to occur.

Example for a sequence of reactions in stars: the CN cycle



Net effect: $4p \rightarrow \alpha + 2e^+ + 2\nu_e$

But requires C or N as catalysts (second and later generation star)

Need reaction rates to:

- establish existence of cycle
- determine total rate of cycle

Fig 45. 5: A sequence of nuclear reactions occurring in stars, known as the Carbon-Nitrogen (CN) cycle. This cycle involves the fusion of hydrogen into helium with carbon, nitrogen, and oxygen acting as catalysts, playing a crucial role in energy production in stars more massive than the Sun.

The Sun generates energy through nuclear fusion, where hydrogen nuclei (protons) combine to form helium. This process releases a tremendous amount of energy in the form of light and heat. Each second, the sun turns 4 million tons of hydrogen into energy. Energy generation is from fusion is



Mass Comparison and Nuclear Binding

If you compare the mass of four protons with the mass of a helium nucleus, which consists of two protons and two neutrons, you'll find that four protons weigh more. This may seem strange because protons are lighter than neutrons. However, the mass of the helium nucleus is less due to the nuclear binding energy that holds it together. This binding energy is what's released during fusion.

Energy Output and the Sun's Lifespan

The Sun emits vast amounts of energy, converting approximately four million tons of hydrogen into helium every second. This fusion process releases 25 million electron volts of energy per reaction. Given the current rate of hydrogen consumption, the Sun can continue this process for about 10 billion years in total, with approximately 5 billion years remaining.

Reaction Efficiency

The reaction efficiency of the fusion process in the Sun is about 8%. This means that not all the hydrogen gets converted into energy; some mass is lost as energy in the form of photons.

Hydrostatic Equilibrium in the Sun

The Sun is in a state of equilibrium due to the balance between two opposing forces: gravity pulling inward and pressure pushing outward as shown in Fig 45.6. This balance prevents the Sun from collapsing under its gravity or exploding due to excessive internal pressure.

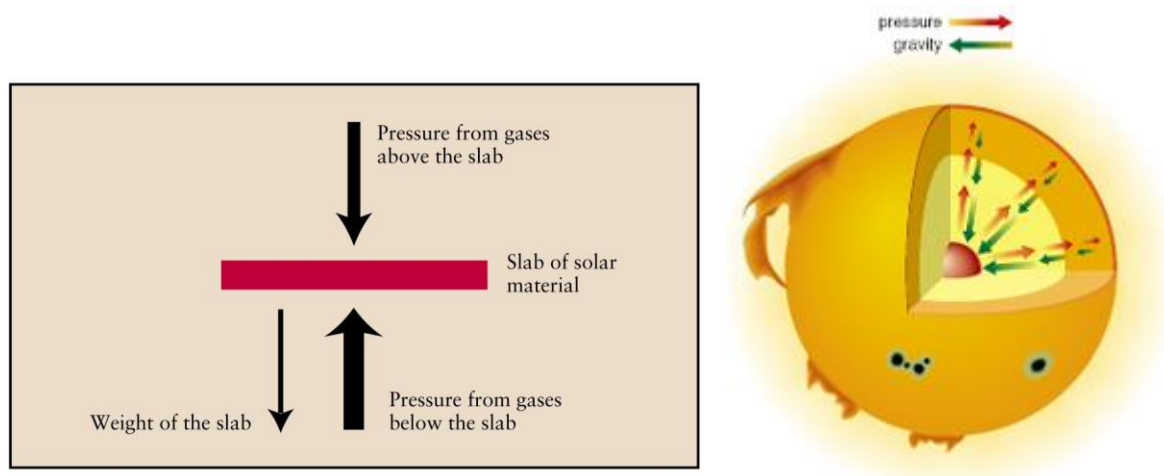


Fig 45. 6: The Sun maintains equilibrium through the balance of two opposing forces: gravity pulling inward and pressure pushing outward. This balance prevents the Sun from collapsing under its own gravity or exploding due to excessive internal pressure.

Calculating the Forces

To understand this equilibrium, imagine a thin shell within the Sun. The mass of this shell experiences gravitational pull inward and pressure from the gas pushing outward as shown in Fig 45.7. By calculating the forces acting on this shell, we can understand how the Sun maintains its structure.

1. **Gravitational Force:** The gravitational force acting on the shell is determined by the mass inside the shell and its distance from the center.
2. **Pressure Force:** The pressure from the gas pushes the shell outward, counteracting gravity.

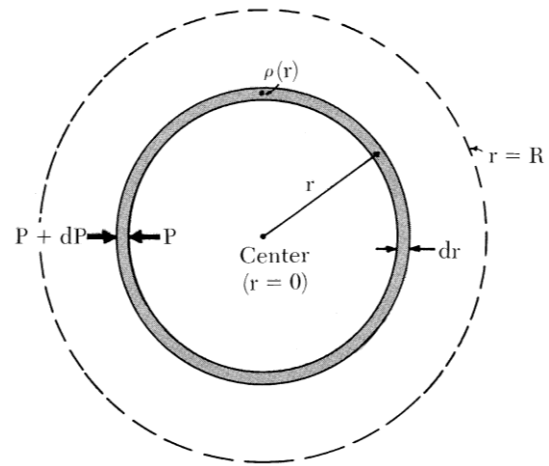


Fig 45. 7: A thin shell within the Sun experiences inward gravitational pull and outward gas pressure. Balancing these forces explains the Sun's stability, preventing expansion or collapse.

When these forces are balanced, the Sun remains stable. If the pressure were too high, the Sun would expand, and if gravity were too strong, it would collapse.

$$dF = -\frac{GM(r) \times \rho(r) 4\pi r^2 dr}{r^2}$$

$$dF = dP \times 4\pi r^2 dr$$

$$\therefore dP = -\frac{GM(r)\rho(r)dr}{r^2}$$

$$\text{or, } \frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2}$$

Similarly,

$$M(r) = \int_0^r \rho(r') 4\pi r'^2 dr'$$

Knowing $\rho(r)$ gives the mass distribution and pressure.

$$\text{Model: assume } \rho \approx \frac{M}{(4\pi/3)R^3} \approx 1.4 \text{ gm cm}^{-3}$$

$$P_{\text{centre}} \approx \frac{GM\rho}{R} \approx 3 \times 10^9 \text{ atmospheres}$$

The Sun's Core and Neutrinos

Photons generated in the core of the Sun take about 170,000 years to reach the surface due to constant collisions with particles. However, neutrinos, which are also produced in the fusion process, can escape from the core to the surface in just a few seconds. These neutrinos provide valuable information about the Sun's core.

Detecting Neutrinos

Scientists have developed methods to detect neutrinos and measure their energy, which helps us understand the conditions inside the Sun. Neutrinos interact very weakly with matter, making them difficult to detect, but advancements in technology have made it possible.

Helioseismology: The Study of Solar Waves

Helioseismology is the science of studying waves on the Sun. This field emerged in 1965 when scientists noticed that different parts of the Sun's surface move at different speeds. By 1975, the first concepts of helioseismology were developed, and today, it's a well-established science. Another way to study the Sun's interior is through helioseismology, the science of studying sound waves traveling through the Sun. The Sun's surface moves up and down, similar to how the Earth's surface moves during an earthquake. By analysing these movements, scientists can infer details about the Sun's internal structure.

The Bouncing Sun

It might surprise you to learn that the surface of the Sun bounces up and down, much like the surface of the Earth during a flood. Obviously, we can't place a meter on the Sun to measure this movement directly. Instead, we use a technique called the Doppler effect to study these changes.

The Doppler Effect

The Doppler effect helps us observe how the Sun's surface moves. When the Sun's surface moves upwards, the frequency of the light (or photons) it emits increases. When the surface moves downwards, the frequency decreases. By analysing these frequency changes, we can map the heat waves on the Sun's surface and understand their intensity.

The Importance of Solar Energy

The Sun emits both heat and light, which travel a long distance to reach us on Earth. Without this energy, life on Earth would be impossible. The Sun acts as a black body, which means it absorbs and emits radiation efficiently.

Black Body Radiation

Black body radiation occurs when the atoms and molecules inside a heated body move, creating electromagnetic waves. These waves are what we perceive as radiation. This process can be described by the Stefan-Boltzmann Law, which states that the power radiated by a black body is proportional to the fourth power of its temperature.

Stefan-Boltzmann Law

The Stefan-Boltzmann Law helps us calculate the radiation emitted by an object. For example, a sphere with a diameter of 10 centimetres at 20 degrees Celsius (or 293 Kelvin) would radiate 418 watts per square meter. If you include the area, the output is around 13 watts. Similarly, your body radiates about 100 watts of energy due to radiation.

Thermal Radiation from a Solid

Calculate the power radiated by a 10-cm-diameter sphere of metal at room temperature (20°C).

$$\begin{aligned}
 P &= \sigma T^4 && \text{(power radiated per area)} \\
 &= (5.670 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4}) (293\text{K})^4 \\
 &= 418 \text{ W/m}^2 \\
 \text{Area: } A &= 4\pi r^2 = 4\pi (5 \times 10^{-2} \text{ m})^2 = 3.14 \times 10^{-2} \text{ m}^2 \\
 \text{Power radiated} &= P \times A = (418 \text{ W/m}^2) (3.14 \times 10^{-2} \text{ m}^2) \\
 &= 13 \text{ Watts}
 \end{aligned}$$

Radiation from the Sun and the Earth

Just like the Sun, the Earth also radiates energy. We receive energy from the Sun, and the Earth emits this energy back into space. The balance between these incoming and outgoing energies determines the Earth's temperature.

Surface Temperature of the Earth

To understand the balance of energy, we define three quantities:

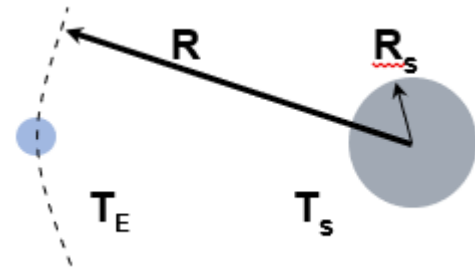
1. **Solar Flux at the Sun's Surface (P_s):** This is the energy emitted by the Sun.
2. **Solar Flux at the Earth (P_R):** This is the energy received by the Earth from the Sun.

3. **Ground Flux:** This is the energy emitted by the Earth.

$$P_s = \text{Sun's flux at its surface} = \sigma T_s^4$$

$$P_R = \text{Sun's flux at earth} = \sigma T^4 \times \left(\frac{R_s}{R}\right)^2$$

$$J_E = \text{Earth's flux at its surface} = \sigma T_E^4$$



Power absorbed from sun = Power radiated by earth

$$P_R \times \pi R_E^2 = P_E \times 4\pi R_E^2$$

$$P_R = 4P_E$$

$$\sigma T_s^4 \times \left(\frac{R_s}{R}\right)^2 = 4\sigma T_E^4$$

$$\therefore R_s = 7 \times 10^8 \text{ metres}$$

$$R = 1.5 \times 10^{11} \text{ metres}$$

$$T_s = 5800^\circ \text{ K}$$

$$T_E = \left(\frac{R_s}{2R}\right)^{1/2} T_s = 280 \text{ K}$$

Fig 45. 8: the star's radius (R_s), the distance from the star to the planet (R), and the effective temperatures of the star (T_s) and the planet (T_E). These relationships help in understanding the equilibrium temperature of a planet based on its distance from the star and the star's properties.

Using these quantities, we can calculate the Earth's equilibrium temperature. The energy received from the Sun must equal the energy emitted by the Earth for the temperature to remain stable. If there is a difference, the Earth's temperature will change.

The Average Temperature of Earth

By calculating the solar flux and the ground flux, we find that the Earth's equilibrium temperature is around 280 degrees Kelvin (or 7 degrees Celsius). However, due to the reflection of sunlight and other factors, the actual average surface temperature of the Earth is around 290 degrees Kelvin (or 13 degrees Celsius). About 30% of the sun's radiation reflects off our atmosphere.

The Greenhouse Effect and Global Warming

The temperature of the Earth is crucial because it relates to global warming, a major issue today. When the Sun's heat reaches the Earth, some of it is trapped by greenhouse gases, like carbon dioxide, in the atmosphere. This trapping of heat raises the Earth's temperature.

Greenhouse Gases and Their Impact

Carbon dioxide is a significant greenhouse gas. It has always been present in the atmosphere, but its concentration has increased due to human activities like burning fossil fuels. These increased levels of carbon dioxide make it harder for heat to escape from the Earth's surface, leading to a rise in temperature.

Molecular Vibrations and Quantum Mechanics

Each molecule, including carbon dioxide, has its own vibrations. In quantum mechanics, these vibrations correspond to specific energy levels. A molecule can only exist in these discrete energy levels as shown in Fig 45.9, much like how you can only sit or stand and not hover halfway in between. These vibrational levels are crucial in understanding how greenhouse gases interact with radiation.

Energy levels of an oscillator:

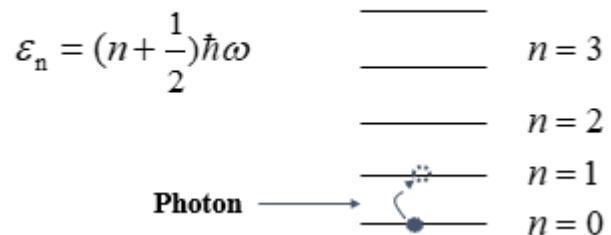


Fig 45. 9: Each molecule, including carbon dioxide, has specific vibrations corresponding to discrete energy levels in quantum mechanics. These levels are essential for understanding how greenhouse gases interact with radiation.

The Greenhouse Effect

When radiation from the Sun reaches the Earth, it tries to escape back into space. However, gases like carbon dioxide and methane in the atmosphere absorb this radiation. This absorption causes these molecules to vibrate, trapping heat and warming the Earth. These gases are called greenhouse gases because their effect is similar to the way a greenhouse traps heat.

- The main greenhouse gases are CO₂, CH₄, H₂O. (N₂, O₂ and Ar are transparent to solar and earth radiation.)

- CO₂ is now 360 ppm of the atmosphere, up from 227 ppm in 1750, before the industrial revolution. Plants turn H₂O + CO₂ into O₂ plus organics.
- Some estimates predict a 3 to 10 °C rise in the earth's surface temperature over the next 100 years due to the increased CO₂ greenhouse effect.

Methane and Carbon Dioxide

Methane (CH₄) is another significant greenhouse gas. It, like carbon dioxide, has molecular vibrations that fall within the same frequency range as the radiation trying to escape Earth. This means that methane also traps heat, contributing to global warming.

The Impact of Greenhouse Gases

The trapped heat causes the Earth's temperature to rise. Estimates suggest that the Earth's temperature could increase by 5 to 10 degrees Celsius in the next 50-100 years if we do not take action to reduce greenhouse gas emissions. This temperature rise could have severe consequences, such as melting ice caps and rising sea levels. To protect our planet, we must control emissions of carbon dioxide and methane.

The Future of the Sun

The Sun will not last forever. It consumes hydrogen through nuclear fusion to produce energy. Currently, this fusion process occurs in the Sun's core, where temperatures are about 8 million degrees Kelvin. However, in about five billion years, the Sun will have exhausted most of its hydrogen.

The Helium Phase

As hydrogen runs out, the Sun will begin to fuse helium. This process requires even higher temperatures and pressures. The fusion of helium will give the Sun a new lease on life, but it will also cause significant changes. The Sun will expand into a Red Giant, becoming 100 times its current size. This expansion will likely engulf nearby planets, including Earth.

The Red Giant Phase

In the Red Giant phase, the Sun's core will be primarily helium, and the outer layers will expand dramatically. This phase will last until the helium in the core is exhausted. As the core contracts and heats up, it will eventually ignite helium fusion, producing carbon and oxygen.

The Supergiant Phase and White Dwarf

After the helium is depleted, the Sun will enter the Supergiant phase. It will expand even more, up to 300 times its current size. Eventually, the Sun will shed its outer layers, leaving behind a hot core called a white dwarf. This white dwarf will be incredibly dense; a small amount of its material would weigh as much as the Earth. Over time, the white dwarf will cool and fade.

The Ultimate Fate of Stars

Not all stars end as white dwarfs. Massive stars, much heavier than our Sun, can end in more dramatic ways. When their fuel is exhausted, they collapse under their own gravity, creating a black hole. A black hole has such strong gravitational pull that not even light can escape from it.

Understanding the Sun with Physics

Studying the Sun requires knowledge from various branches of physics:

- **Classical Mechanics:** Used to understand the balance of forces within the Sun.
- **Thermodynamics:** Helps us understand the heat transfer and energy production in the Sun.
- **Statistical Mechanics:** Describes the behavior of particles in the Sun's hot interior.
- **Nuclear Physics and Quantum Mechanics:** Explain the fusion reactions that power the Sun.

Applying Physics to the Sun

By applying these branches of physics, we can understand the Sun's structure, energy production, and eventual fate. For example, Newton's laws explain the gravitational forces, while thermodynamics and statistical mechanics describe the movement of particles and energy distribution.