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COURSE HANDOUTS

Dynamics

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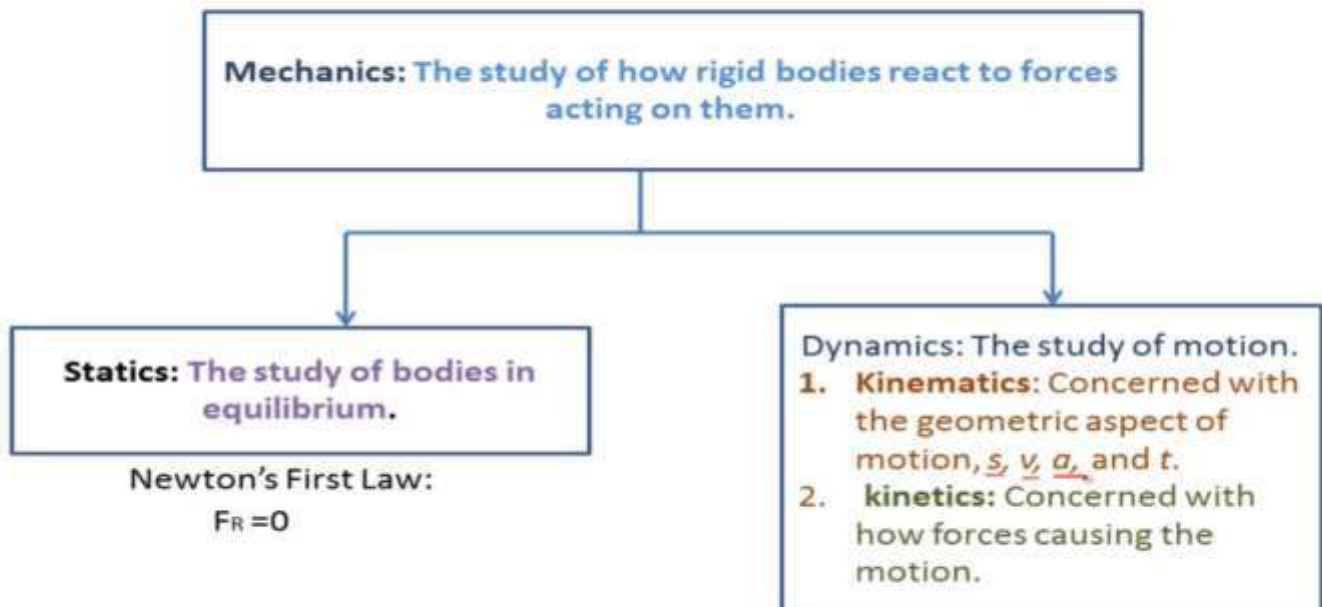
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Introduction

Mechanics-Dynamics



Kinematics: Motion.

1 - how far you went (displacement)

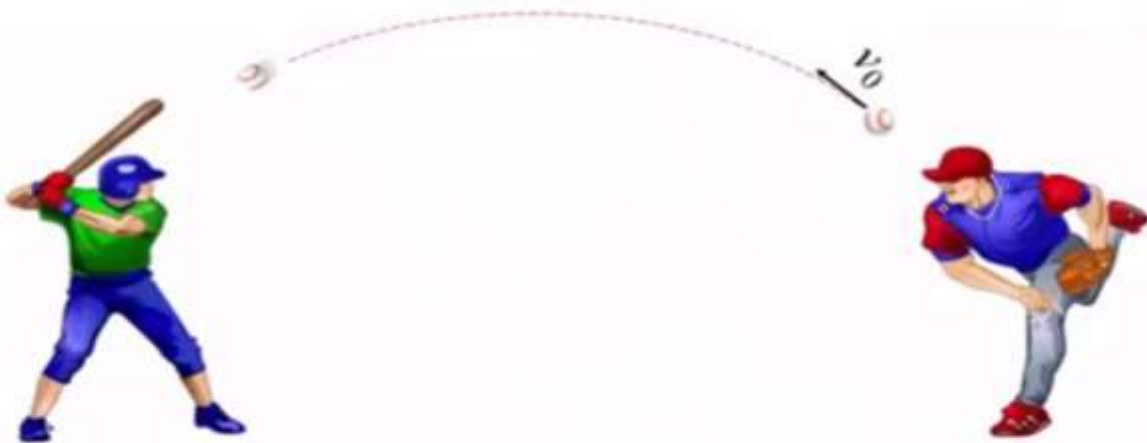
2 - how fast you went.

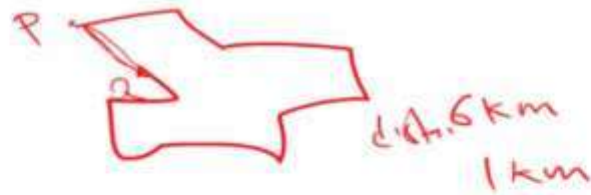
6 KM, 30 mins

Speed = 12 km/hour

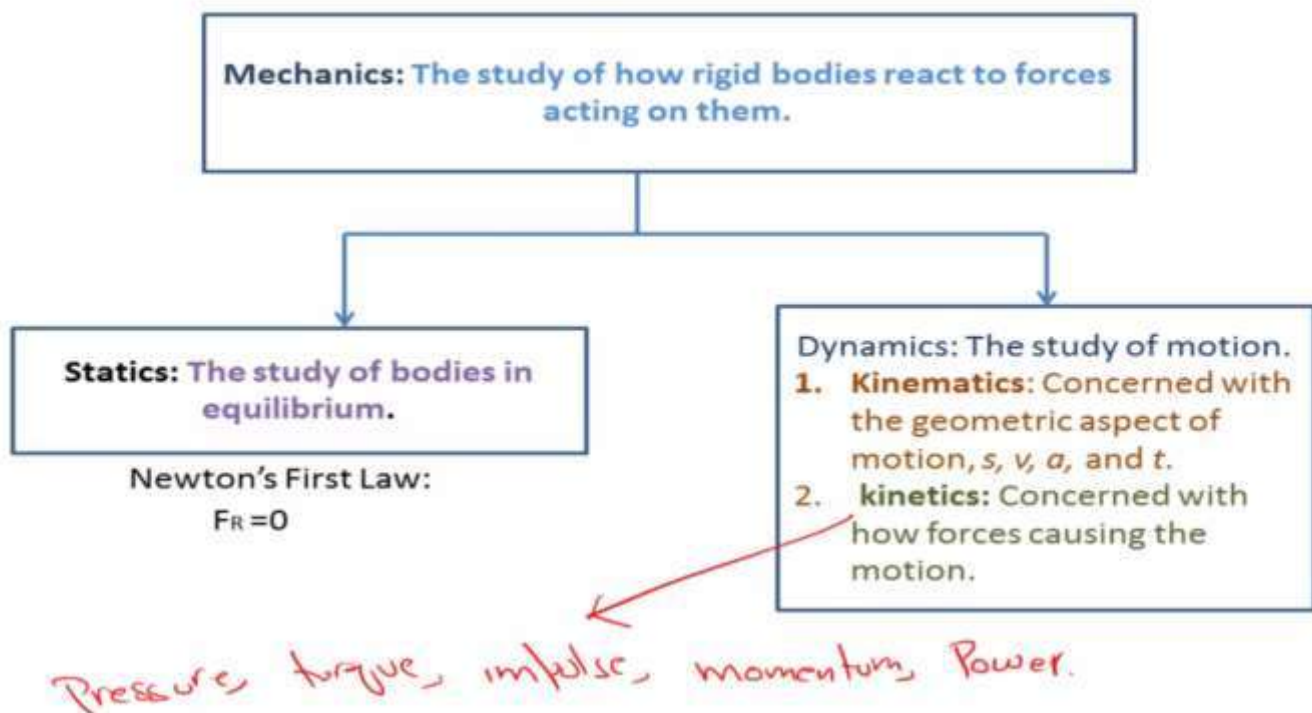
3 - How quickly you speed up or down
acceleration

Example





- **Scalar Quantity:** A quantity that can be specified by its magnitude and has no direction. *time, distance speed*
- **Vector Quantity:** A quantity that has both magnitude and direction. *displacement, velocity, Acceleration*



Velocity and Acceleration

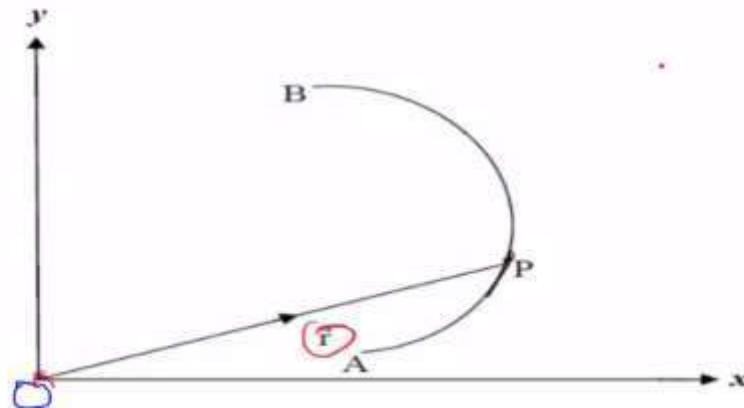
4

Kinematics

4

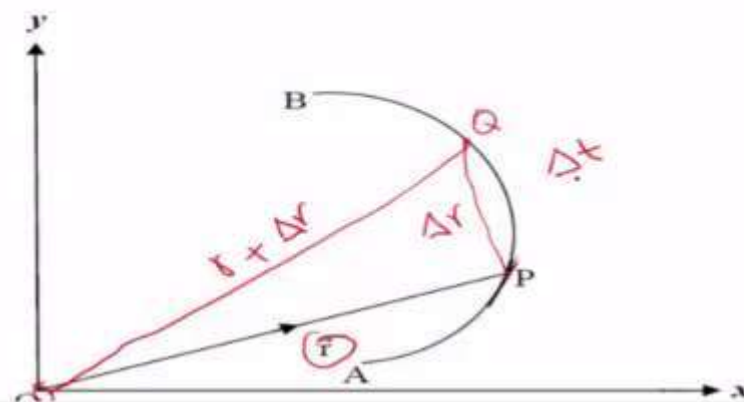
Position Vector

The position of particle can be specified by a vector $\mathbf{r}$ whose initial point is at the origin of some fixed coordinate system and the terminal point is at the particle. This vector is called position vector. If the particle is moving, the vector $\mathbf{r}$ changes with time, i.e. it is a function of time.



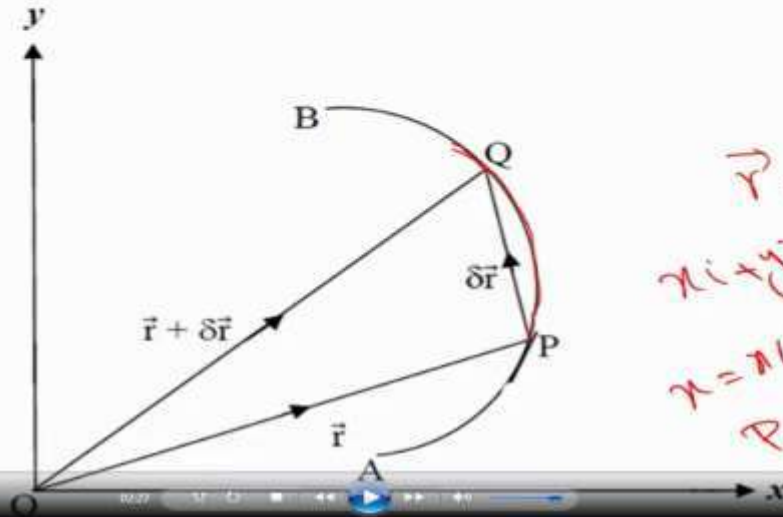
Position Vector

The position of particle can be specified by a vector $\mathbf{r}$ whose initial point is at the origin of some fixed coordinate system and the terminal point is at the particle. This vector is called position vector. If the particle is moving, the vector $\mathbf{r}$ changes with time, i.e. it is a function of time.



Trajectory

The curve traced by moving particle is called the trajectory or path of particle.



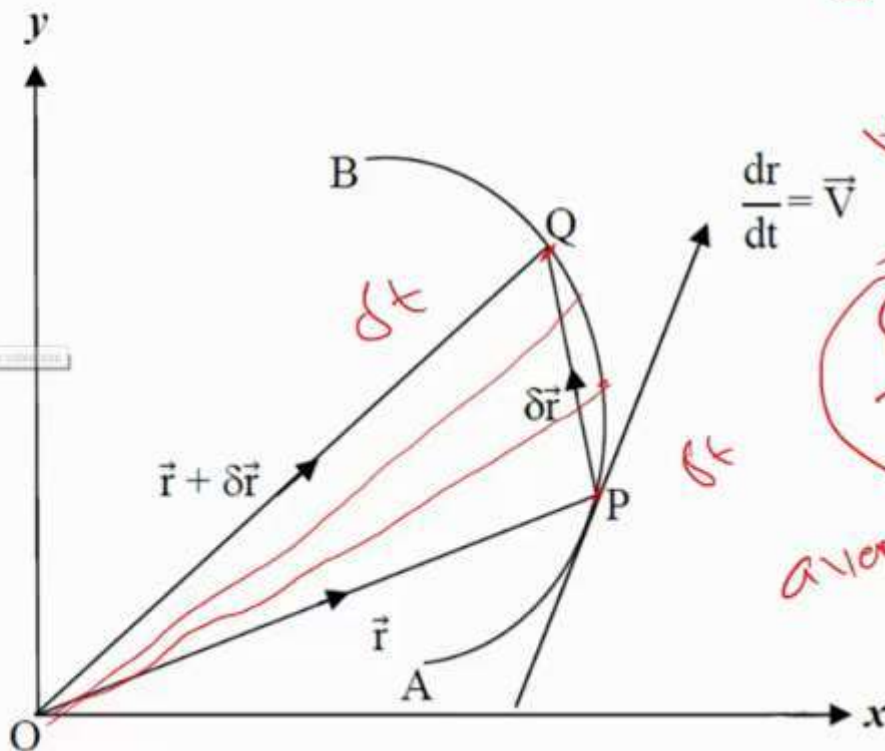
$$\vec{r} = \vec{r}(t)$$

$$x\hat{i} + y\hat{j} + z\hat{k} = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$$

$$x = x(t), y = y(t), z = z(t)$$

Parametric

Velocity



$$\frac{d\vec{r}}{dt} = \vec{V}$$

$$\delta t \rightarrow 0$$

$$V = \lim_{\delta t \rightarrow 0} \frac{\delta \vec{r}}{\delta t} = \frac{d\vec{r}}{dt}$$

$$\left(\frac{\delta \vec{r}}{\delta t} \right)$$

average rate of change of displacement.

Acceleration

instantaneous rate of change of velocity.

$$\vec{a} = \lim_{\delta t \rightarrow 0} \frac{\delta \vec{v}}{\delta t} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(\vec{v})$$

$$= \frac{d}{dt} \left(\frac{d\vec{r}}{dt} \right) = \frac{d^2 \vec{r}}{dt^2}$$

$$\vec{a} = \frac{d^2 \vec{r}}{dt^2}$$

Cartesian coordinate system

$$\vec{r} = x\hat{i} + y\hat{j}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j}$$

$$\vec{a} = \frac{d^2 x}{dt^2} \hat{i} + \frac{d^2 y}{dt^2} \hat{j}$$

Different ways to change $\vec{v}$



Car speeds up



Car slows up

If a car goes from 0 to 100km/hr in only 3 seconds then what is the acceleration.

$$\begin{aligned} \vec{a} &= \frac{d\vec{v}}{dt} = \frac{\text{change in velocity}}{\text{change in time}} \\ &= \frac{100 \text{ km/h}}{3} = 33 \text{ km/h s} \\ &= \frac{33 \times 1000}{3600 \text{ s}} \quad (1 \text{ hr} = 3600 \text{ sec}) \\ \vec{a} &= \frac{33 \times 10^3}{2.6 \times 10^3 \text{ s}^2} = 9.1 \text{ m/s}^2 \end{aligned}$$

Ex 1

A particle is moving in such a way that its position at any time t is specified by

$$r = (t^3 + t^2)i + (\cos t + \sin^2 t)j + (e^t + \log t)k$$

Find its velocity and acceleration.

$$\begin{aligned} v &= \frac{dr}{dt} = (3t^2 + 2t)i + (-\sin t + 2\sin t \cos t)j + (e^t + \frac{1}{t})k \\ &= (3t^2 + 2t)i + (\sin 2t - \sin t)j + (e^t + \frac{1}{t})k \\ a &= \frac{d^2r}{dt^2} = (6t + 2)i + (2\cos 2t - \cos t)j + (e^t - \frac{1}{t^2})k \end{aligned}$$

0 at $t=0$.

$$\vec{r} = at^2\hat{i} + 4at\hat{j}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = 2at\hat{i} + 4a\hat{j}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = 2a\hat{i} + 0 = 2a\hat{i}$$

$$\vec{r} = a\cos t\hat{i} + b\sin t\hat{j} \quad a > b$$

Find the Position of the Particle where velocity has maximum and minimum magnitude.

$$\vec{v} = \frac{d\vec{r}}{dt} = -a\sin t\hat{i} + b\cos t\hat{j}$$

$$v = x^2 + y^2$$

$$|v| = \sqrt{x^2 + y^2}$$

$$|v| = \sqrt{a^2\sin^2 t + b^2\cos^2 t} = \sqrt{a^2\sin^2 t + b(1-\sin^2 t)}$$

$$|v| = \sqrt{a^2\sin^2 t + b^2\cos^2 t} = \sqrt{a^2\sin^2 t + b(1-\sin^2 t)}$$

$$= \sqrt{a^2\sin^2 t + b^2 - b^2\sin^2 t}$$

$$= \sqrt{\sin^2 t(a^2 - b^2) + b^2}$$

$= 90$

$$= a\cos 90\hat{i} + b\sin 90\hat{j}$$

$$= b\hat{j}$$

$$t = 270$$

$$\vec{r} = a\cos 270\hat{i} + b\sin 270\hat{j}$$

$$= -b\hat{j}$$

$$\sin^2 t = 1$$

$$\pm b\hat{j}$$

$$270^\circ \quad 90^\circ$$

$$\sin t = -1 \quad \sin t = 1$$

$$\sin^2 t = 1$$

$$\sin^2 t = 0$$

v is minimum

$$\sin t = 0$$

$$t = 0, 180$$

$$r = a\cos 180\hat{i} + b\sin 180\hat{j} = -a\hat{i}$$

$$\pm a\hat{i}$$

$$t = 0$$

$$r = a\cos 0\hat{i} + b\sin 0\hat{j}$$

$$r = a\hat{i}$$

Question:

Find The Radial and Transverse component of velocity of a particle moving along the curve

$ax^2 + by^2 = 1$ at any time t if the polar angle is $\theta = ct$

$$\theta = ct^2$$

$$\frac{d\theta}{dt} = 2ct$$

$$\frac{d^2\theta}{dt^2} = 2c$$

$$ax^2 + by^2 = 1$$

$$x = r \cos \theta, \quad y = r \sin \theta$$

$$a r^2 \cos^2 \theta + b r^2 \sin^2 \theta = 1$$

$$r^2 (a \cos^2 \theta + b \sin^2 \theta) = 1 \Rightarrow r = \frac{1}{\sqrt{a \cos^2 \theta + b \sin^2 \theta}}$$

$$r = \frac{1}{\sqrt{a \cos^2 \theta + b \sin^2 \theta}}$$

$$\begin{aligned} V_r = \frac{dr}{dt} &= \frac{d}{dt} (a \cos^2 \theta + b \sin^2 \theta)^{-1/2} \\ &= -\frac{1}{2} (a \cos^2 \theta + b \sin^2 \theta)^{-3/2} (a \cdot 2 \cos \theta (-\sin \theta) \cdot \frac{d\theta}{dt} + b \cdot 2 \sin \theta \cos \theta \cdot \frac{d\theta}{dt}) \\ &= -\frac{1}{2} (a \cos^2 \theta + b \sin^2 \theta)^{-3/2} (a - b) 2 \sin \theta \cos \theta \frac{d\theta}{dt} \\ &= -\frac{1}{2} (a \cos^2 \theta + b \sin^2 \theta)^{-3/2} (a - b) \sin 2\theta \frac{d\theta}{dt} \\ &= \frac{(a - b) \sin 2\theta \cdot 2ct}{2 (a \cos^2 \theta + b \sin^2 \theta)^{3/2}} \end{aligned}$$

$$\begin{aligned} &= -\frac{1}{2} (a \cos^2 \theta + b \sin^2 \theta)^{-3/2} (a \cdot 2 \cos \theta (-\sin \theta) \cdot \frac{d\theta}{dt} + b \cdot 2 \sin \theta \cos \theta \cdot \frac{d\theta}{dt}) \\ &= -\frac{1}{2} (a \cos^2 \theta + b \sin^2 \theta)^{-3/2} (a - b) 2 \sin \theta \cos \theta \frac{d\theta}{dt} \\ &= -\frac{1}{2} (a \cos^2 \theta + b \sin^2 \theta)^{-3/2} (a - b) \sin 2\theta \frac{d\theta}{dt} \\ &= \frac{(a - b) \sin 2\theta \cdot 2ct}{2 (a \cos^2 \theta + b \sin^2 \theta)^{3/2}} \end{aligned}$$

$$V_r = \frac{dr}{dt} = \frac{ct (a - b) \sin 2\theta}{(a \cos^2 \theta + b \sin^2 \theta)^{3/2}}$$

$$V_\theta = r \frac{d\theta}{dt} = \frac{1}{\sqrt{a \cos^2 \theta + b \sin^2 \theta}} \cdot 2ct$$

Radial & Transverse Component of Velocity & Acceleration

$$x = r \cos \theta$$

$$y = r \sin \theta$$

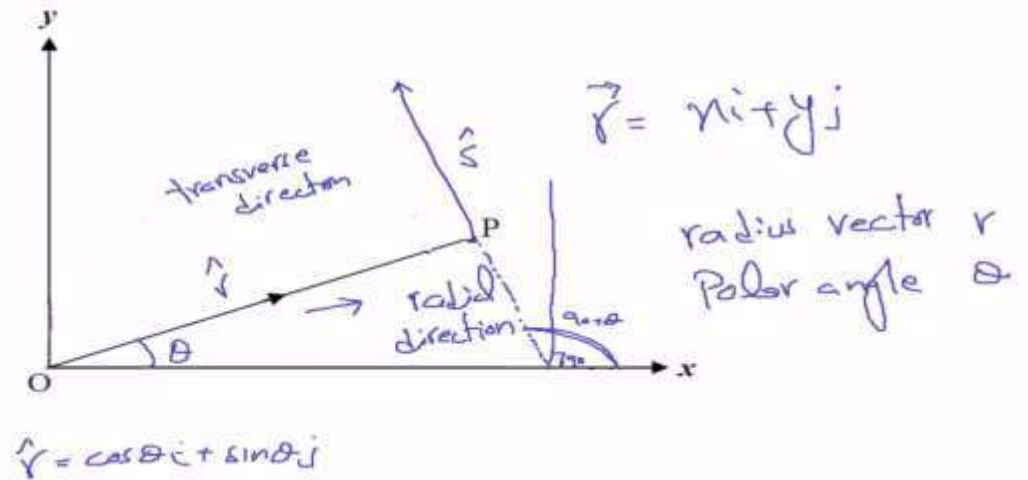
$$\vec{r} = x\mathbf{i} + y\mathbf{j}$$

$$= r \cos \theta \mathbf{i} + r \sin \theta \mathbf{j}$$

$$\vec{r} = r (\cos \theta \mathbf{i} + \sin \theta \mathbf{j})$$

$$\frac{\vec{r}}{r} = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}$$

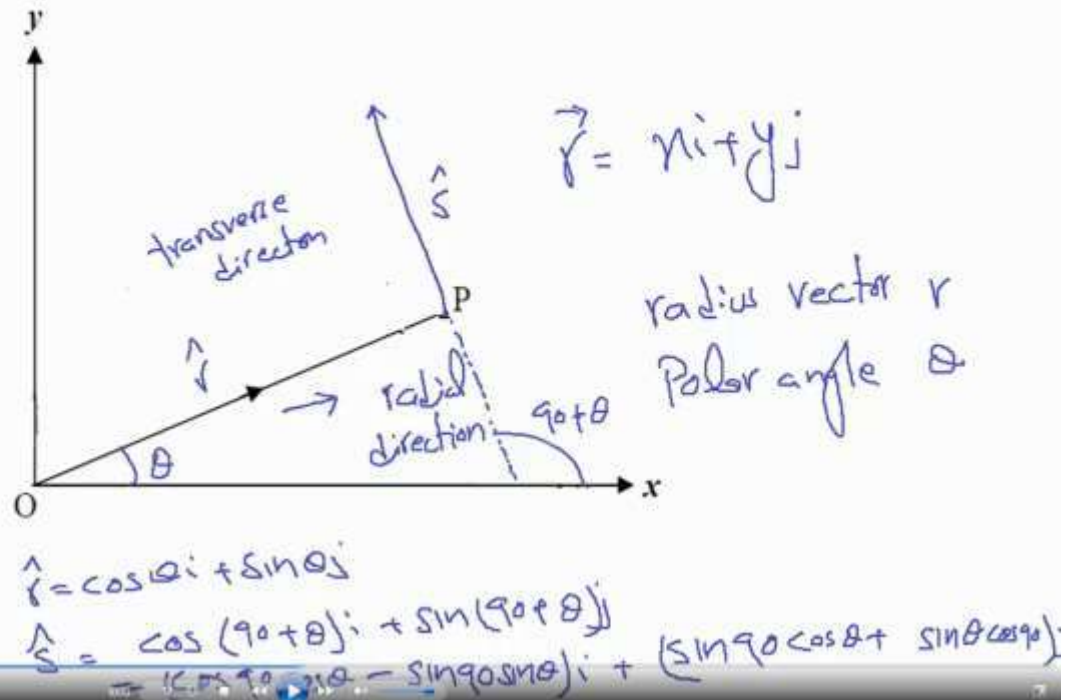
$$\hat{r} = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}$$



Radial & Transverse Component of Velocity & Acceleration

$$x = r \cos \theta$$

$$y = r \sin \theta$$



Polar coordinate system

Coord
system

$$\begin{aligned}\hat{s} &= \cos(\theta + 90^\circ)\mathbf{i} + \sin(\theta + 90^\circ)\mathbf{j} \\ &= (\cos 90^\circ \cos \theta - \sin 90^\circ \sin \theta)\mathbf{i} + (\sin 90^\circ \cos \theta + \cos 90^\circ \sin \theta)\mathbf{j} \\ &= (0 - \sin \theta)\mathbf{i} + (\cos \theta + 0)\mathbf{j} \\ &= -\sin \theta \mathbf{i} + \cos \theta \mathbf{j}\end{aligned}$$

$$\begin{aligned}\hat{r} &= \cos \theta \mathbf{i} + \sin \theta \mathbf{j} \\ \frac{d\hat{r}}{dt} &= -\sin \theta \mathbf{i} \cdot \frac{d\theta}{dt} + \cos \theta \mathbf{j} \cdot \frac{d\theta}{dt} \\ &= \frac{d\theta}{dt} (-\sin \theta \mathbf{i} + \cos \theta \mathbf{j}) \\ &= \frac{d\theta}{dt} \cdot \hat{s}\end{aligned}$$

$$\begin{aligned}\hat{s} &= -\sin \theta \mathbf{i} + \cos \theta \mathbf{j} \\ \frac{d\hat{s}}{dt} &= -\cos \theta \mathbf{i} \cdot \frac{d\theta}{dt} - \sin \theta \mathbf{j} \cdot \frac{d\theta}{dt} \\ &= -\frac{d\theta}{dt} (\cos \theta \mathbf{i} + \sin \theta \mathbf{j}) \\ &= -\frac{d\theta}{dt} \cdot \hat{r}\end{aligned}$$

$$\begin{aligned}\hat{r} &= \frac{\vec{r}}{r} & \vec{r} &= r \cdot \hat{r} \\ \vec{v} &= \frac{d\vec{r}}{dt} = \frac{d}{dt}(r \cdot \hat{r}) \\ &= \frac{dr}{dt} \cdot \hat{r} + r \cdot \frac{d\hat{r}}{dt} \\ &= \left(\frac{dr}{dt}\right) \cdot \hat{r} + r \cdot \left(\frac{d\theta}{dt}\right) \cdot \hat{s}\end{aligned}$$

v_r = Radial component of velocity = $\frac{dr}{dt}$
 v_θ = Transverse " " = $r \frac{d\theta}{dt}$

$V_\theta = \text{Transverse} = \frac{d\theta}{dt} r$

$$\vec{v} = \frac{dr}{dt} \hat{r} + r \cdot \frac{d\theta}{dt} \hat{s}$$

$$\begin{aligned} a = \frac{d\vec{v}}{dt} &= \frac{d}{dt} \left(\frac{dr}{dt} \hat{r} + r \cdot \frac{d\theta}{dt} \hat{s} \right) \\ &= \frac{d^2r}{dt^2} \hat{r} + \frac{dr}{dt} \cdot \frac{d\hat{r}}{dt} + \frac{dr}{dt} \cdot \frac{d\theta}{dt} \hat{s} + r \cdot \frac{d}{dt} \cdot \frac{d\theta}{dt} \hat{s} + r \cdot \frac{d\theta}{dt} \cdot \frac{d\hat{s}}{dt} \end{aligned}$$

$$+ r \cdot \frac{d}{dt} \cdot \frac{d\theta}{dt} \hat{s} + r \cdot \frac{d\theta}{dt} \cdot \frac{d\hat{s}}{dt}$$

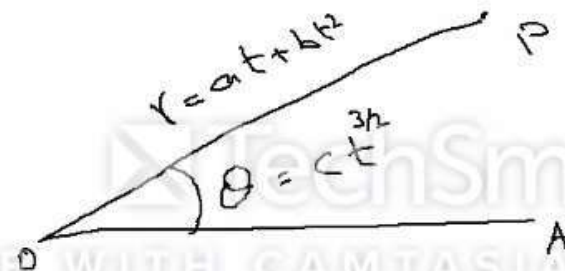
$$\begin{aligned} &= \frac{d^2r}{dt^2} \hat{r} + \frac{dr}{dt} \cdot \frac{d\theta}{dt} \hat{s} + \frac{dr}{dt} \cdot \frac{d\theta}{dt} \hat{s} + \\ &\quad r \cdot \frac{d^2\theta}{dt^2} \hat{s} + r \cdot \frac{d\theta}{dt} \cdot \left(-\frac{d\theta}{dt} \hat{r} \right) \end{aligned}$$

$$\begin{aligned}
 &= \frac{d^2 r}{dt^2} \hat{r} + 2 \frac{dr}{dt} \cdot \frac{d\theta}{dt} \hat{s} + r \cdot \frac{d^2 \theta}{dt^2} \hat{s} - \frac{d^2 \theta}{dt^2} \cdot r \hat{r} \\
 &= \frac{d^2 r}{dt^2} \hat{r} - \frac{d^2 \theta}{dt^2} \cdot r \hat{r} + 2 \frac{dr}{dt} \cdot \frac{d\theta}{dt} \hat{s} + r \cdot \frac{d^2 \theta}{dt^2} \hat{s} \\
 &= \left[\frac{d^2 r}{dt^2} - r \frac{d^2 \theta}{dt^2} \right] \hat{r} + \left[2 \frac{dr}{dt} \cdot \frac{d\theta}{dt} + r \frac{d^2 \theta}{dt^2} \right] \hat{s}
 \end{aligned}$$

$a_r = \text{radial component of acceleration} = \frac{d^2 r}{dt^2} - r \frac{d^2 \theta}{dt^2}$
 $a_\theta = \text{Transverse} = 2 \frac{dr}{dt} \cdot \frac{d\theta}{dt} + r \frac{d^2 \theta}{dt^2}$

Question:

A particle P moves in a plane in such away that at any time t , its distance from point O is $r = at + bt^2$ and the line connecting O and P makes an angle $\theta = ct^{3/2}$ with a fixed line OA. Find the radial and transverse components of velocity and acceleration of particle at $t = 1$



$$\begin{aligned}
 v_r &= \frac{dr}{dt} & v_\theta &= r \frac{d\theta}{dt} \\
 a_r &= \frac{d^2 r}{dt^2} - r \left(\frac{d\theta}{dt} \right)^2 & a_\theta &= 2 \frac{dr}{dt} \cdot \frac{d\theta}{dt} + r \frac{d^2 \theta}{dt^2}
 \end{aligned}$$

$$a_r = \frac{d^2 r}{dt^2} - r \left(\frac{d\theta}{dt} \right)^2$$

$$a_\theta = 2 \frac{dr}{dt} \cdot \frac{d\theta}{dt} + r \frac{d^2 \theta}{dt^2}$$

$$r = at + bt^2 \quad \theta = ct^{3/2}$$

$$\frac{dr}{dt} = a + 2bt$$

$$\frac{d\theta}{dt} = \frac{3}{2} ct^{1/2}$$

$$\frac{dr}{dt} \Big|_{t=1} = a + 2b$$

$$\frac{d\theta}{dt} \Big|_{t=1} = \frac{3}{2} c$$

$$\frac{d^2 r}{dt^2} = 2b$$

$$\frac{d^2 r}{dt^2} = 2b$$

$$\frac{d^2 \theta}{dt^2} = \frac{3}{2} \cdot \frac{1}{2} ct^{-1/2} = \frac{3}{4} ct^{-1/2}$$

$$\frac{d^2 \theta}{dt^2} \Big|_{t=1} = \frac{3}{4} c$$

at

$$\frac{dr}{dt} \Big|_{t=1} = a + 2b$$

$$\frac{d\theta}{dt} \Big|_{t=1} = \frac{3}{2} c$$

$$\frac{d^2 r}{dt^2} = 2b$$

$$\frac{d^2 r}{dt^2} = 2b$$

$$\frac{d^2 \theta}{dt^2} = \frac{3}{2} \cdot \frac{1}{2} ct^{-1/2} = \frac{3}{4} ct^{-1/2}$$

$$\frac{d^2 \theta}{dt^2} \Big|_{t=1} = \frac{3}{4} c$$

$$v_r = \frac{dr}{dt} = a + 2b$$

$$v_\theta = r \frac{d\theta}{dt} = (a+b) \frac{3}{2} c$$

$$a_r = \frac{d^2 r}{dt^2} - r \left(\frac{d\theta}{dt} \right)^2 = 2b - (a+b) \left(\frac{3}{2} c \right)^2 = 2b - \frac{9}{4} c^2 (a+b)$$

$$a_\theta = 2 \frac{dr}{dt} \left(\frac{d\theta}{dt} \right) + r \frac{d^2 \theta}{dt^2} = 2(a+2b) \left(\frac{3}{2} c \right) + (a+b) \frac{3}{4} c = \frac{3}{4} c (5a+5b)$$

Question:

Find the Radial and Transverse component of acceleration of a particle moving along the circle $x^2 + y^2 = a^2$ with constant velocity c .

$$\frac{d\theta}{dt} = c$$

$$\frac{d^2\theta}{dt^2} = 0$$

$$x^2 + y^2 = a^2$$

$$r^2 \cos^2 \theta + r^2 \sin^2 \theta = a^2$$

$$r^2 (\cos^2 \theta + \sin^2 \theta) = a^2$$

$$r^2 = a^2 \Rightarrow r = a$$

$$\frac{dr}{dt} = 0 \Rightarrow \frac{d^2r}{dt^2} = 0$$

$$r^2 \cos^2 \theta + r^2 \sin^2 \theta = a^2$$

$$r^2 (\cos^2 \theta + \sin^2 \theta) = a^2$$

$$r^2 = a^2 \Rightarrow r = a$$

$$\frac{dr}{dt} = 0 \Rightarrow \frac{d^2r}{dt^2} = 0$$

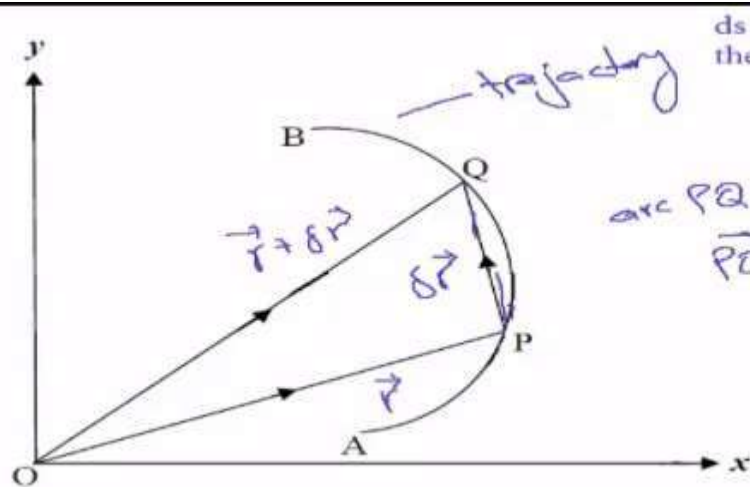
$$a_r = \frac{d^2r}{dt^2} - r \left(\frac{d\theta}{dt} \right)^2$$

$$= 0 - ac^2 = -ac^2$$

$$a_\theta = 2 \frac{dr}{dt} \left(\frac{d\theta}{dt} \right) + r \frac{d^2\theta}{dt^2}$$

$$= 0 + 0 = 0$$

Lecture: 5



ds is the distance on the curve at any time the fixed point.

$$\text{arc } PQ = \delta s$$

$$\vec{PQ} = \delta \vec{r}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds} \left(\frac{ds}{dt} \right) v$$

$$\vec{v} = \frac{d\vec{r}}{ds} v$$

$$\frac{ds}{dt}$$

$$\frac{d\vec{r}}{dt} = \lim_{\delta t \rightarrow 0} \frac{\delta \vec{r}}{\delta t}$$

$$\frac{d\vec{r}}{ds} = \lim_{\Delta s \rightarrow 0} \frac{\Delta \vec{r}}{\Delta s}$$

which shows that the direction of $\frac{d\vec{r}}{ds}$ is along the tangent to the path at the point P.

Let $\hat{t}$ be the unit tangent vector along the tangent the curve at P.

Let P be the curve at P.

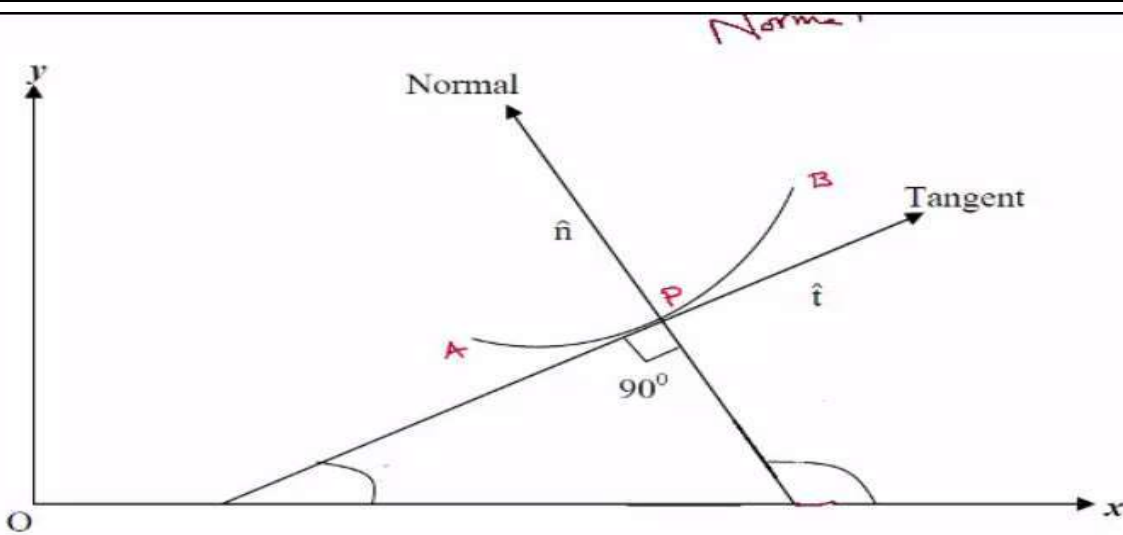
$$\left| \frac{d\vec{r}}{ds} \right| = \lim_{\Delta s \rightarrow 0} \left| \frac{\Delta \vec{r}}{\Delta s} \right| = \lim_{\Delta s \rightarrow 0} \frac{\text{length of chord } PQ}{\text{length of arc } PQ} = 1$$

$$\frac{d\vec{r}}{ds} = 1 \cdot \hat{t} = \hat{t}$$

$$\frac{d\vec{r}}{ds} = 1 \cdot \hat{t} = \hat{t}$$

$$\vec{v} = v\hat{t} + 0\hat{n}$$

tangential component of velocity = v
Normal component = 0



$$\vec{v} = v\hat{t} + 0\hat{n}$$

$$\vec{v} = v\hat{t}$$

$$a = \frac{d\vec{v}}{dt} = \frac{d}{dt}(v\hat{t}) = \frac{dv}{dt} \cdot \hat{t} + \left(\frac{d\hat{t}}{dt} \right) v$$

$$\hat{t} \cdot \hat{t} = 1$$

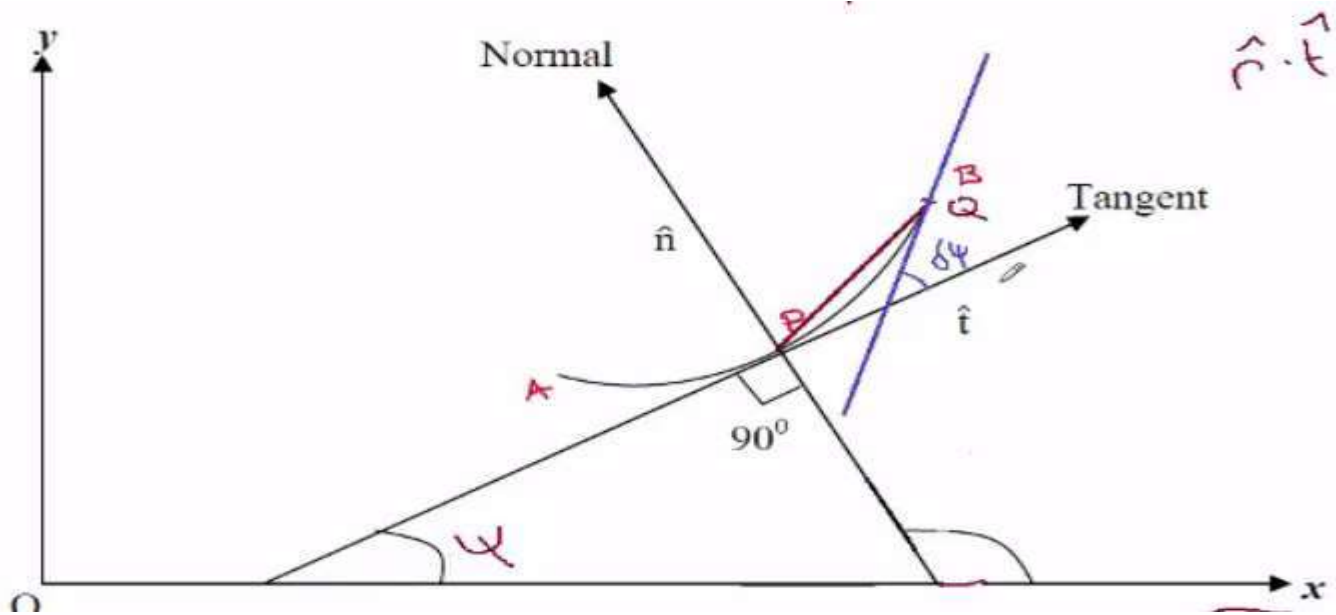
$$\hat{t} \cdot \frac{d\hat{t}}{dt} + \frac{d\hat{t}}{dt} \cdot \hat{t} = 0$$

$$2 \frac{d\hat{t}}{dt} \cdot \hat{t} = 0$$

$$\frac{d\hat{t}}{dt} \cdot \hat{t} = 0$$

$$\hat{n} \cdot \frac{d\hat{t}}{dt} \hat{t} = 0$$

$$\left| \frac{d\hat{t}}{dt} \right| = \lim_{\delta t \rightarrow 0} \left| \frac{\delta \hat{t}}{\delta t} \right|$$



$$a = \frac{d\vec{v}}{dt} = \frac{d}{dt} (v\hat{t}) = \frac{dv}{dt} \cdot \hat{t} + \frac{d\hat{t}}{dt} v$$

$$= \frac{dv}{dt} \cdot \hat{t} + \frac{d\hat{t}}{d\psi} \cdot \frac{d\psi}{ds} \cdot \left(\frac{ds}{dt} \right) = v$$

$$= \frac{dv}{dt} \cdot \hat{t} + \frac{d\hat{t}}{d\psi} \cdot \frac{d\psi}{ds} v$$

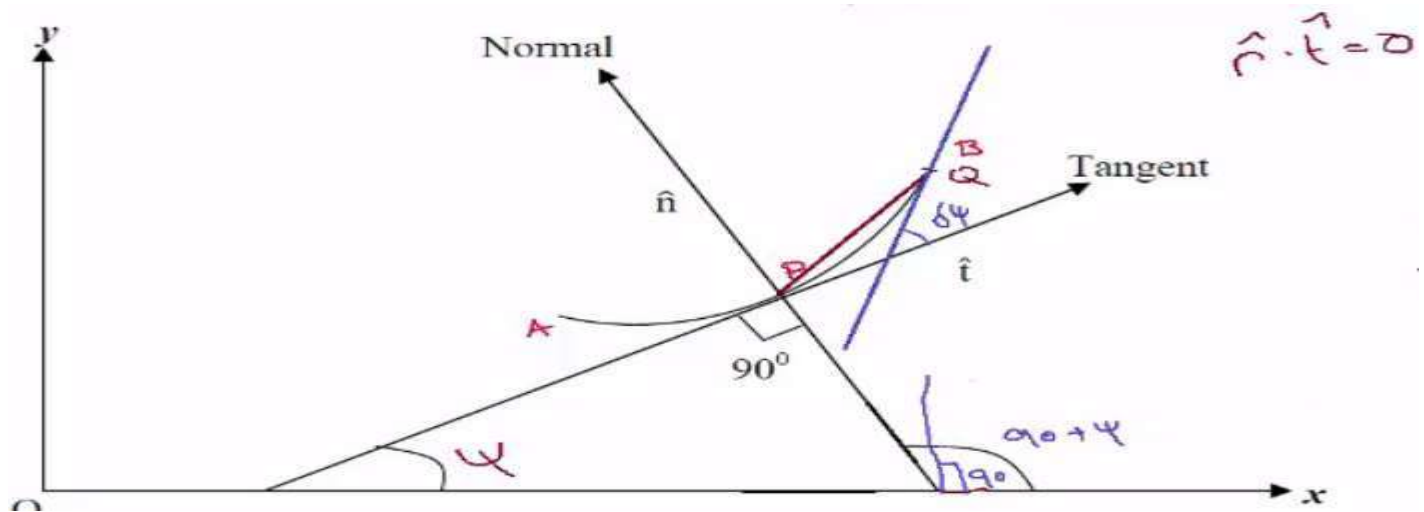
$$= \frac{dv}{dt} \cdot \hat{t} + \frac{d\hat{t}}{d\psi} \cdot \frac{d\psi}{ds} v$$

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

$$\frac{d\psi}{ds} = \text{curvature} = \kappa = \frac{1}{\rho}$$

$$= \frac{dv}{dt} \cdot \hat{t} + \frac{d\hat{t}}{d\psi} \cdot \frac{1}{\rho} v^2 \quad \text{--- } \boxed{1}$$

$$\hat{t} = \cos \psi i + \sin \psi j$$



$$\hat{t} = \cos \psi i + \sin \psi j$$

$$\begin{aligned} \hat{n} &= \cos(90 + \psi)i + \sin(90 + \psi)j \\ &= -\sin \psi i + \cos \psi j \end{aligned}$$

$$\frac{d\hat{t}}{d\psi} = -\sin \psi i + \cos \psi j = \hat{n}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dx}{dt} \hat{i} + \frac{dy}{dt} \hat{j}$$

$$a_t = \frac{dv}{dt}$$

$$a_n = \frac{v^2}{r}$$

Question:

A particle is moving along the parabola $x^2 = 4ay$ with constant speed. Determine tangential and normal components of its acceleration when it reaches the point whose abscissa is $\sqrt{5}a$.

$$x^2 = 4ay$$

$$2x = 4a \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{2x}{4a} = \frac{x}{2a}$$

$$\frac{d^2y}{dx^2} = \frac{1}{2a}$$

$$x = \sqrt{5}a$$

$$\frac{dy}{dx} = \frac{\sqrt{5}a}{2a} = \frac{\sqrt{5}}{2}$$

$$r = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{\left[1 + \left(\frac{\sqrt{5}}{2}\right)^2\right]^{3/2}}{\frac{1}{2a}} = 2a \cdot \left[1 + \frac{5}{4}\right]^{3/2} = 2a \left[\frac{9}{4}\right]^{3/2} = 2a \left[\frac{3}{2}\right]^{3/2} = 2a \left[\frac{27}{8}\right]^{1/2}$$

$$= 2a \times \frac{27}{8} = \frac{27a}{4}$$

$$\frac{dv}{dt} = 0$$

tangential
Normal

$$\text{Component of acceleration} = a_t = \frac{dv}{dt} = 0$$

$$a_n = \frac{v^2}{r} = \frac{v^2}{\frac{27a}{4}} = \frac{4v^2}{27a}$$

Find the tangential and normal component of acceleration of a point describing ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

With uniform speed v when the particle is at $(0, b)$.

$$a_t = \frac{dv}{dt} = 0 \quad a_n = \frac{v^2}{\rho}$$

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{d^2y/dx^2}$$

$$x^2 b^2 + y^2 a^2 = a^2 b^2$$

$$2xb^2 + 2y \frac{dy}{dx} a^2 = 0$$

$(0, b)$

$$\frac{dy}{dx} = \frac{-xb^2}{ya^2} = \frac{-b^2 x}{a^2 y} = 0$$

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{-b^2}{a^2} \left[\frac{y \cdot 1 - x \frac{dy}{dx}}{y^2} \right] = \frac{-b^2}{a^2} \left[\frac{y - x \left(\frac{-b^2 x}{a^2 y} \right)}{y^2} \right] \\ &= \frac{-b^2}{a^2} \left[\frac{y + \frac{b^2 x^2}{a^2 y}}{y^2} \right] = \frac{-b^2}{a^2} \left[\frac{a^2 y^2 + b^2 x^2}{a^2 y^3} \right] \end{aligned}$$

$$\frac{d^2y}{dx^2} \Big|_{(0,b)} = \frac{-b^2}{a^2} \left[\frac{a^2 b^2}{b^3} \right] = -\frac{b}{a^2}$$

$$\frac{d^2y}{dx^2} = -\frac{b}{a^2} \left[\frac{\frac{2x}{\sqrt{1+b^2x^2}}}{\frac{2x}{\sqrt{1+b^2x^2}}} \right] = -\frac{b}{a^2}$$

$$\rho = \frac{[1 + (10)^2]^{3/2}}{|-\frac{b}{a^2}|} = \frac{1}{\frac{b}{a^2}} = \frac{a^2}{b}$$

$$a_n = \frac{v^2}{\rho} = \frac{v^2}{a^2/b} = \frac{bv^2}{a^2}$$

A particle is moving with uniform speed along the curve

$$x^2y = a \left(x^2 + \frac{a^2}{\sqrt{5}} \right) \quad \text{Show that acceleration has maximum value } \frac{10v^2}{9a}$$

$$\frac{dv}{dt} = 0$$

$$a_t = \frac{dv}{dt} = 0$$

$$\rho = \frac{[1 + (\frac{dy}{dx})^2]^{3/2}}{\frac{d^2y}{dx^2}}$$

$$\frac{d\rho}{dx} = 0$$

$x = \text{critical points}$

$$x^2y = a \left(x^2 + \frac{a^2}{\sqrt{5}} \right)$$

$x = \text{center band}$

$$x^2 y = a \left(x^2 + \frac{a^2}{5} \right)$$

$$y = a \left(1 + \frac{a^2}{5x^2} \right)$$

$$y = a \left(1 + \frac{a^2}{5} x^{-2} \right)$$

$$\frac{dy}{dx} = a \left(0 - \frac{2a^2 x^{-3}}{5} \right) = a \left(\frac{-2a^2 x^{-3}}{5} \right)$$

$$\frac{dy}{dx} = a \left(\frac{-2a^2}{5x^3} \right)$$

$$\frac{dy}{dx} = \frac{-2a^3}{5x^3} = \frac{-2a^3 x^{-3}}{5}$$

$$\frac{d^2y}{dx^2} = -2(-3) \frac{a^3 x^{-4}}{5} = \frac{6a^3 x^{-4}}{5} = \frac{6a^3}{5x^4}$$

$$I = \frac{\left[1 + \left(\frac{-2a^3}{5x^3} \right)^2 \right]^{3/2}}{6a^3} = \frac{\sqrt{5} x^4}{6a^3} \left[1 + \frac{4a^6}{5x^6} \right]$$

$$= \frac{\sqrt{5} x^4}{6a^3} \left[\frac{5x^6 + 4a^6}{5x^6} \right]^{3/2}$$

$$\begin{aligned}
 &= \frac{20\pi^4}{6a^3} \left[\frac{5\pi^6}{5\pi^6} \right]^{3/2} \\
 &= \frac{20\pi^4}{6a^3} \left[\frac{5\pi^6 + 4a^6}{5\pi^6} \right]^{3/2} \\
 &= \frac{20\pi^4}{6a^3} \left[\frac{5\pi^6 + 4a^6}{5\pi^6} \right]^{3/2} \cdot \frac{5\pi^6}{5\pi^6} \\
 &= \frac{(5\pi^6 + 4a^6)^{3/2}}{30a^3\pi^5}
 \end{aligned}$$

$$\int = \frac{(5\pi^6 + 4a^6)^{3/2}}{30a^3\pi^5}$$

$$\begin{aligned}
 \frac{dP}{dn} &= \frac{1}{30a^3} \frac{5\pi^6 - \frac{3}{2} [5\pi^6 + 4a^6] \cdot 30\pi^5 - (5\pi^6 + 4a^6)^{3/2} \cdot 5\pi^6}{(\pi^5)^2} \\
 &= \frac{(5\pi^6 + 4a^6)^{1/2} \cdot (4\pi^6 - 4a^6)}{6a^3\pi^5}
 \end{aligned}$$

$$\begin{aligned}
 \frac{dP}{dn} &= \frac{1}{30a^3} \frac{5\pi^6 - \frac{3}{2} [5\pi^6 + 4a^6] \cdot 30\pi^5 - (5\pi^6 + 4a^6)^{3/2} \cdot 5\pi^6}{(\pi^5)^2} \\
 &= \frac{(5\pi^6 + 4a^6)^{1/2} \cdot (4\pi^6 - 4a^6)}{6a^3\pi^5}
 \end{aligned}$$

$$\begin{aligned}
 \frac{dP}{dn} &= 0 \\
 4\pi^6 - 4a^6 &= 0 \Rightarrow \pi^6 = a^6 \\
 \pi &= a \\
 5\pi^6 + 4a^6 &= 0 \\
 \Rightarrow 5\pi^6 &= -4a^6 \\
 \pi &= -a \\
 \frac{dP}{dn} &< 0 & \pi > a & \frac{dP}{dn} > 0
 \end{aligned}$$

Separate step

$$= \frac{\cancel{5}x^4}{5a^2} \left[\cancel{5}\dot{x} + 4a^2 \right] / \cancel{5}\cancel{5} \cdot x^5$$

$$\int = \frac{(5x^6 + 4a^6)^{3/2}}{30a^3 x^5} = \frac{(5a^6 + 4a^6)^{3/2}}{30a^3 a^5} = \frac{1}{30a^8} 27a^9 = \frac{9a}{10}$$

Separate step

$x^2 y = a \left(x^2 + \frac{a^2}{\sqrt{5}} \right)$ Show that acceleration has maximum value $\frac{10v^2}{9a}$

$$\frac{dv}{dt} = 0$$

$$a_x = \frac{dv}{dt} = 0$$

$$\frac{d}{dt} \left(\frac{v^2}{2} \right) = 0$$

$$\frac{d}{dt} \left(\frac{v^2}{2} \right) = 0$$

$$\frac{d}{dt} \left(\frac{v^2}{2} \right) = 0$$

$$\frac{d}{dt} \left(\frac{v^2}{2} \right) = 0$$

$$a_x = \frac{v}{r} = \frac{v^2}{\frac{9a}{10}} = \frac{10v^2}{9a}$$

Lecture 6:

Rectilinear Motion

Rectilinear motion means the motion of a particle along a straight line.



Average velocity = $\frac{8x}{\delta t}$

$$v = \frac{dx}{dt}$$

$$\lim_{\delta t \rightarrow 0} \frac{dx}{\delta t} = \frac{dx}{dt}$$

Average velocity = $\frac{8x}{\delta t}$

$$\lim_{\delta t \rightarrow 0} \frac{dx}{\delta t} = \frac{dx}{dt}$$

$$v = \frac{dx}{dt}$$

$$a = \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = v \frac{dv}{dx}$$

$$a = \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = v \frac{dv}{dx}$$

boundary conditions or initial conditions

Lecture 7:

Motion with Constant Acceleration

$$a = \frac{dv}{dt}$$

$$v = \frac{dx}{dt}$$

$$a = \frac{d}{dt} \left(\frac{dx}{dt} \right) = \frac{d^2x}{dt^2}$$

$$v = \frac{dx}{dt}$$

$$a = \frac{d}{dt} \left(\frac{dx}{dt} \right) = \frac{d^2x}{dt^2}$$

$$a = \frac{d^2x}{dt^2}$$

$$at + A = \frac{dx}{dt}$$

$$at + A = v$$

$$v = at + u$$

$$t=0 \quad v=u$$

$$a \cdot 0 + A = u$$

$$A = u$$

This eq gives the final velocity of the particle at any time t

$$\frac{dn}{dt} = at + u$$

$$dn = at dt + u dt$$

$$n = \frac{at^2}{2} + ut + B \quad B=0$$

$$0 = 0 + 0 + B$$

$$\textcircled{n} = \frac{1}{2}at^2 + ut$$

This equation gives the distance travelled by the particle at any time t .

$$a = \frac{dv}{dt} = \frac{dv}{dn} \cdot \frac{dn}{dt} = v \cdot \frac{dv}{dn}$$

$$a dn = v dv$$

$$an = \frac{v^2}{2} + C$$

$$a dn = v dv$$

$$an = \frac{v^2}{2} + \textcircled{C}$$

$$n=0$$

$$v=U$$

$$0 = \frac{U^2}{2} + C$$

$$C = -\frac{U^2}{2}$$

$$an = \frac{v^2}{2} - \frac{U^2}{2}$$

$$2an = v^2 - U^2$$

$$2ax = v^2 - u^2$$

This equation gives the distance travelled by the particle when initial and final both velocities are given.

$$v = u + at$$

$$v^2 - u^2 = 2ax$$

$$x = ut + \frac{1}{2}at^2$$

Example. A particle moving in a straight line starts from rest and is accelerated uniformly to attain a velocity of 60 miles per hour in 4 seconds. Find the acceleration of motion and the distance travelled by the particle in the last three seconds.



$$v = \frac{60 \times 5280}{60 \times 60} = 88 \text{ ft/sec}$$



$$v = \frac{60 \times 5280}{60 \times 60} = 88 \text{ ft/sec}$$

$$v = u + at$$

$$u = 0$$

$$t = 4$$

$$v = 0 + a(4)$$

$$88 = 0 + a(4)$$

$$a = 88/4 = 22 \text{ ft/sec}$$

$$x = \frac{1}{2}at^2 + ut$$

$$x = \frac{1}{2}(22)(4)^2 + 0 \cdot t$$

$$x = \frac{1}{2} \cdot 22 \cdot 16 = 176 \text{ ft}$$

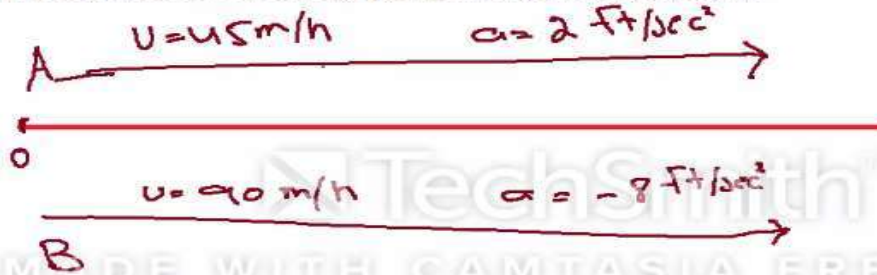
$$x = ut + \frac{1}{2}at^2$$

$$x_2 = 0 \cdot t + \frac{1}{2}(22)(1)^2$$

$$= \frac{1}{2} \cdot 22 = 11 \text{ Ft}$$

$$x_1 - x_2 = 176 - 11 = 165 \text{ Ft}$$

Question: Two particles start simultaneously from a point O and move in a straight line; one with a velocity of 45 miles per hour and an acceleration of 2 ft/sec^2 and the other with a velocity of 90 miles per hour and a retardation (the rate of decrease of velocity) of 8 ft/sec^2 . Find the time after which the velocities of the particles are the same and the distance of O from the point where they meet again.



$$U_A = \frac{45 \times 5280}{3600}$$

$$U_A = \frac{45 \times 5280}{3600} = 66 \text{ Ft/sec}^2$$

$$U_B = \frac{90 \times 5280}{3600} = 132 \text{ Ft/sec}^2$$

$$V_A = U_A + at$$

$$= 66 + 2t$$

$$V_B = U_B + at$$

$$= 132 + (-8)t$$

$$= 132 - 8t$$

$$66 + 2t = 132 - 8t$$

$$10t = 66$$

$$t = 6.6 \text{ sec}$$

$$\begin{aligned} x_A &= u_A t + \frac{1}{2} a_A t^2 \\ &= 66t + \frac{1}{2} \cdot 2 \cdot t^2 \\ &= 66t + t^2 \end{aligned}$$

$$\begin{aligned} x_B &= u_B t + \frac{1}{2} a_B t^2 \\ &= 132t + \frac{1}{2} (-8) t^2 \\ &= 132t - 4t^2 \end{aligned}$$

$$66t + t^2 = 132t - 4t^2$$

$$5t^2 = 66t$$

$$5t = 66$$

$$t = 66/5 \text{ sec}$$

$$x = 66 \cdot \frac{66}{5} + \left(\frac{66}{5} \right)^2 = 1045.4 \text{ Ft}$$

Lecture 8:

$$a = \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = \frac{dv}{dx} \cdot v$$

$$a(v) = v \cdot \frac{dv}{dx}$$

$$dx = v \cdot \frac{dv}{a(v)}$$

$$dx = v \cdot \frac{dv}{a(v)}$$

$$x = \int \frac{v}{a(v)} \cdot dv + C$$

$$\frac{dv}{dt} = a(v)$$

$$x = \int \frac{v}{a(v)} \cdot dv + C$$

$$\frac{dv}{dt} = a(v)$$

$$dt = \frac{dv}{a(v)}$$

$$t = \int \frac{dv}{a(v)} + D$$

Example: A particle moves in a straight line with an acceleration kv^3 . If its initial velocity is u , find the velocity and the time spent when the particle has travelled a distance x .

$$\begin{array}{c} t=0 \longrightarrow x \longleftarrow t=t \\ \text{initial } v=u \quad \text{Find } v=x \\ a = kv^3 \end{array}$$

$$v \frac{dv}{dx} = kv^3$$

$$\frac{v}{v^3} \frac{dv}{dx} = k dx$$

$$\int \frac{1}{v^2} \frac{dv}{dx} = \int k dx$$

$$-\frac{1}{v} = kx + A$$

$$x=0, \quad v=u$$

$$-\frac{1}{u} = 0 + A \Rightarrow A = -\frac{1}{u}$$

$$-\frac{1}{v} = kx - \frac{1}{u} \Rightarrow \frac{1}{v} = -kx + \frac{1}{u}$$

$$x=0, v=U$$

$$-\frac{1}{U} = 0 + A \Rightarrow A = -\frac{1}{U}$$

$$-\frac{1}{v} = kx - \frac{1}{U} = \frac{U - kUx + 1}{U}$$

$$v = \frac{U}{1 - kUx}$$

$$Q = KV^3$$

$$\frac{dv}{dt} = KV^3$$

$$\frac{dv}{v^3} = K dt$$

$$\int \frac{dv}{v^3} = \int K dt$$

$$-\frac{1}{2v^2} = Kt + \textcircled{B}$$

$$t=0, v=U$$

$$-\frac{1}{2U^2} = 0 + B$$

$$-\frac{1}{2v^2} = Kt - \frac{1}{2U^2}$$

$$t = \frac{1}{2K} \left[\frac{1}{U^2} - \frac{1}{v^2} \right]$$

$$-\frac{1}{2v^2} = 0 + B$$

$$-\frac{1}{2v^2} = kt - \frac{1}{2v^2}$$

$$t = \frac{1}{2k} \left[\frac{1}{v^2} - \frac{1}{v^2} \right]$$

$$t = \frac{1}{2k} \left[\frac{1}{v^2} - \left(\frac{1-kv^2}{v^2} \right)^2 \right]$$

$$t = \frac{1}{2kv^2} (2 - kv^2)$$

Motion with Variable Acceleration

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

$$= \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = v \frac{dv}{dx}$$

1) Time dependent Acceleration

$$\frac{d^2x}{dt^2} = a(t)$$

$$\frac{dx}{dt} = \int a(t) dt + A = b(t) + A$$

$$x = \int b(t) + At + B$$

Example c

Find the distance travelled and the velocity attained by a particle moving in a straight line, at any time t if it starts from rest at $t=0$ and is subject to an acceleration $t^2 + \sin t + e^t$

$$a = \frac{dv}{dt} = t^2 + \sin t + e^t$$

$$\frac{dv}{dt} = \frac{t^3}{3} - \cos t + e^t + A$$

$$0 = 0 - 1 + 1 + A$$

$$A = 0$$

$$A = 0$$

$$\frac{dv}{dt} = \frac{t^3}{3} - \cos t + e^t$$

$$v = \frac{t^4}{12} - \sin t + e^t + B$$

$$t=0, \quad v=0$$

$$v = \frac{t^4}{12} - \sin t + e^t + B$$

$$t=0, \quad v=0$$

$$0 = 0 - 0 + 1 + B$$

$$B = -1$$

$$v = \frac{t^4}{12} - \sin t + e^t - 1$$

Distance Dependent Acceleration

$$a(x) = \frac{dv}{dt} = \frac{dv}{dx} \cdot \left(\frac{dx}{dt} \right) = v \cdot \frac{dv}{dx}$$

$$\int a(x) \cdot dx = \int v \cdot dv$$

$$F(x) = \frac{v^2}{2} + C$$

$$F(x) = \frac{v^2}{2} + C$$

$$-2C = 0$$

$$v^2 = 2Fx - 2C$$

$$v^2 = 2Fx + D$$

$$v = \pm \sqrt{2Fx + D}$$

$$v^2 = 2Fx + D$$

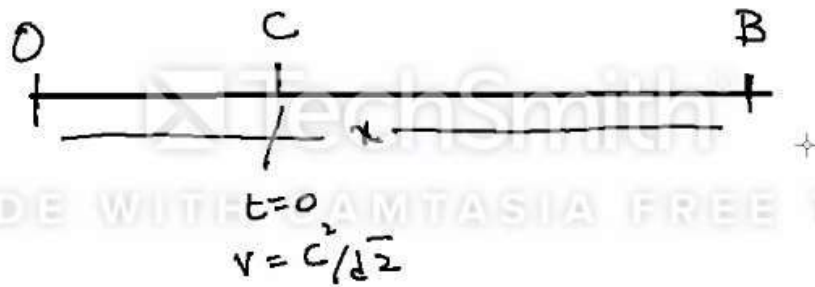
$$v = \pm \sqrt{2Fx + D}$$

$$\frac{dx}{dt} = \pm \sqrt{2Fx + D}$$

$$\int \pm \frac{dx}{\sqrt{2Fx + D}} = \int dt$$

$$t = \pm \int \frac{dx}{\sqrt{2Fx + D}} + E$$

Discuss the motion of a particle moving in a straight line with an acceleration x^3 , where x is the distance of the particle from a fixed point O on the line, if it starts at $t=0$, from a point $x=c$ with velocity $\frac{c^2}{\sqrt{2}}$.



$$v \frac{dv}{dx} = a = x^3$$

$$\int v dv = \int x^3 dx$$

$$\frac{v^2}{2} = \frac{x^4}{4} + D \quad \text{--- (1)}$$

$$\frac{v^2}{2} = \frac{x^4}{4} + D \quad \text{--- (1)}$$

$$x=c, \quad v = \frac{c^2}{\sqrt{2}}$$

$$\frac{1}{2} \left(\frac{c^2}{\sqrt{2}} \right)^2 = \frac{c^4}{4} + D$$

$$\frac{1}{2} \cdot \frac{c^4}{2} = \frac{c^4}{4} + D$$

$$D = 0$$

$$\frac{v^2}{2} = \frac{x^4}{4} + 0$$

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} + 0$$

$$v^2 = \frac{x^4}{2}$$

$$v = \frac{1}{\sqrt{2}} x^2$$

$$\frac{dx}{dt} = \frac{1}{\sqrt{2}} x^2$$

$$\int \sqrt{2} \frac{dx}{x^2} = \int dt$$

$$-\frac{\sqrt{2}}{x} + D = t$$

$$-\frac{\sqrt{2}}{x} + D = t \quad \text{value}$$

$$t=0, \quad x=c$$

$$-\frac{\sqrt{2}}{c} + D = 0$$

$$D = \frac{\sqrt{2}}{c}$$

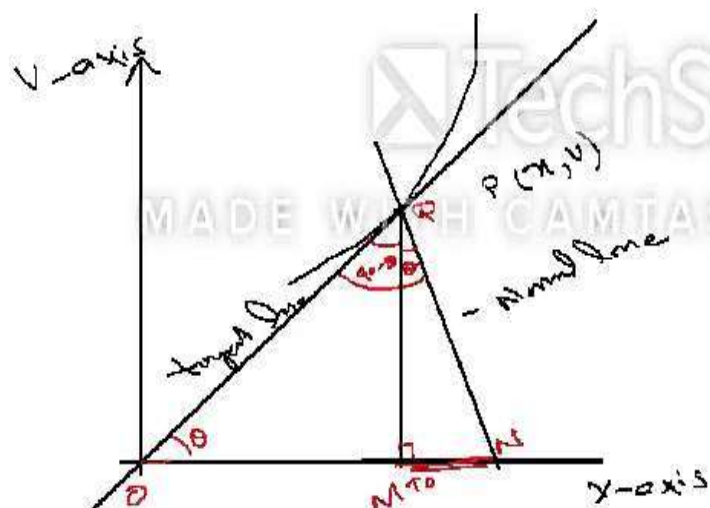
$$t = -\frac{\sqrt{2}}{x} + \frac{\sqrt{2}}{c}$$

$$t = \sqrt{2} \left(\frac{1}{c} - \frac{1}{x} \right)$$

Lecture 10:

Velocity Space Curve

Question: Show that the length of the subnormal at any point of the velocity space curve gives the corresponding acceleration of the particle.



$$v = \frac{dx}{dt}$$

$$a = \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt}$$

$$a = \frac{dv}{dx} \cdot v$$

$$\frac{dv}{dx} = \tan \theta$$

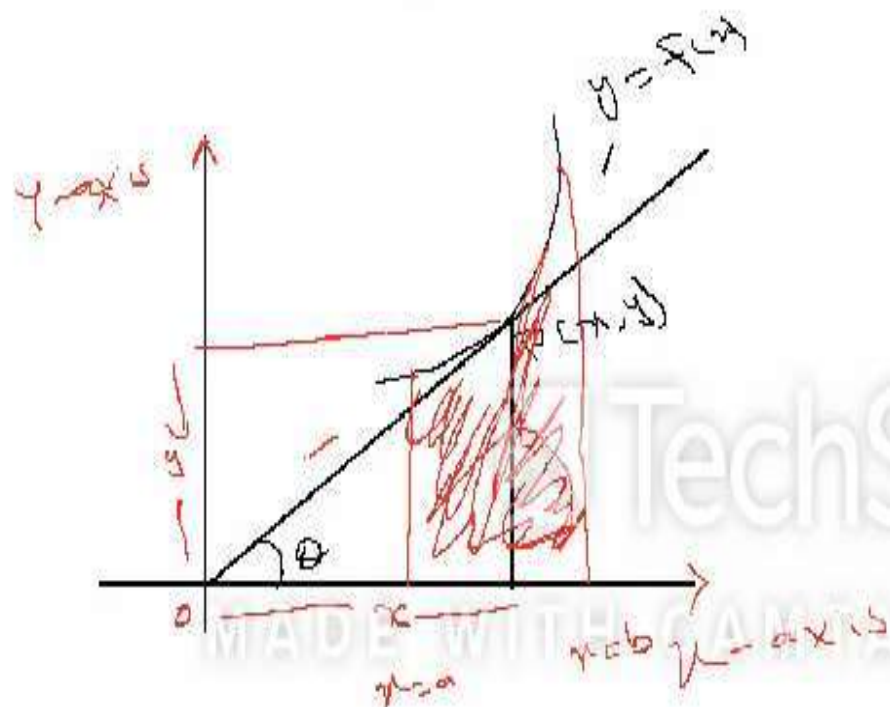
$$\tan \theta = \frac{MN}{PM}$$

$$MN = PM \tan \theta$$

$$MN = v \tan \theta = v \frac{dv}{dx}$$

$$PM = v$$

Graphical Methods

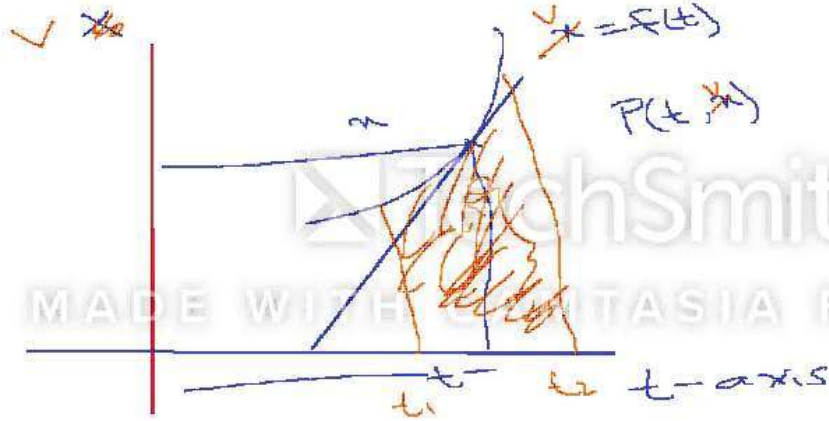


$$A = \int_{x=a}^{x=b} y \, dx$$

pos neg x -axis

velocity

Space Time Curve



$$A = \int v \, dt$$

$$= \int \frac{dx}{dt} \, dt$$

$$= \int_{x_1}^{x_2} dx = x(t_2) - x(t_1)$$

Acceleration = slope of the tangent line = $\frac{dv}{dt}$ = ~~velocity~~ ^{acceleration}

velocity of the particle at any point P = slope of the tangent line at the point P to the velocity time curve

$$\frac{dv}{dt} = a$$

$$\frac{dv}{dt} = a$$

$$dv = a \, dt$$

$$\int dv = \int a \, dt$$

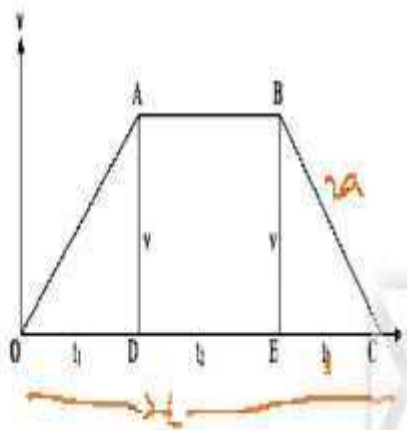
$$\int dv = \int a \, dt$$

$$v = at + B$$

$$x = ct + A$$

Question

A particle starts from rest with a constant acceleration a . When its velocity acquires a certain value v , it moves uniformly and then its velocity starts decreasing with a constant retardation $2a$ till it comes to rest. Find the distance travelled by the particle, if the time taken from rest to rest is t .



$$\frac{dv}{dt} = a$$

$$a = \frac{v}{t_1}$$

$$2a = \frac{v}{t_3}$$

$$t_1 = \frac{v}{a}, \quad t_3 = \frac{v}{2a}$$

$$t = t_1 + t_2 + t_3$$

$$\begin{aligned} t_2 &= t - t_1 - t_3 \\ &= t - \frac{v}{a} - \frac{v}{2a} \\ &= t - \frac{3v}{2a} \end{aligned}$$

$$x = \Delta OAD + \Delta BEED + \Delta BEC$$

$$= \frac{1}{2} t_1 v + t_2 v + \frac{1}{2} t_3 v$$

$$= \frac{1}{2} v \left(t_1 + 2t_2 + t_3 \right) = \frac{1}{2} v \left(\frac{v}{a} + 2 \left(t - \frac{3v}{2a} \right) + \frac{v}{2a} \right) = \frac{1}{2} v \left(2t - \frac{3v}{a} \right)$$

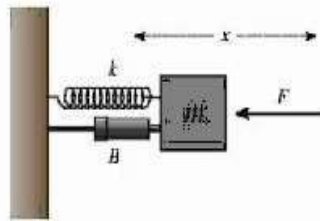
Simple Harmonic Motion

Periodic motion is the motion in which an object repeats its path in equal intervals of time. For example the motion of the hands of a clock is periodic motion, leaves of a tree moving to and fro due to wind breeze, these all are examples of periodic motion.

The particle performs the same set of movement again and again in a periodic motion. One such set of movement is called an Oscillation. An example of oscillatory motion is Simple Harmonic Motion.

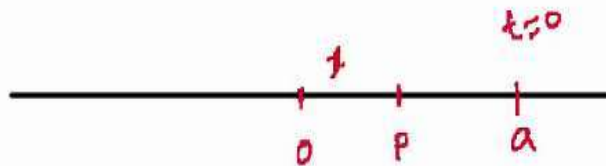
The motion of a particle moving along a straight line with an acceleration whose direction is always towards a fixed point on the line and whose magnitude is proportional to the distance from the fixed point is called simple harmonic motion.

Now, consider there is a spring that is fixed at one end. When there is no force applied to it, it is at its equilibrium position. Now,



If we pull it outwards, there is a force exerted by the string that is directed towards the equilibrium position.
If we push the spring inwards, there is a force exerted by the string towards the equilibrium position.

If we pull it outwards, there is a force exerted by the string that is directed towards the equilibrium position.
If we push the spring inwards, there is a force exerted by the string towards the equilibrium position.



$$\frac{d^2x}{dt^2} = -kx$$

$$\int 2 \frac{dx}{dt} \frac{d^2x}{dt^2} = \int -kx \frac{dx}{dt}$$

$$\left(\frac{dx}{dt} \right)^2 = -kx^2 + A$$

$$\frac{d^2 x}{dt^2} = -kx$$

$$\left(\frac{dx}{dt}\right)^2 = \dots$$

$$t=0 \quad x=a$$

$$\frac{dx}{dt} = 0$$

$$\left(\frac{dx}{dt}\right)^2 = -kx^2 + k a^2$$

$$\left(\frac{dx}{dt}\right)^2 = k(a^2 - x^2)$$

$$\frac{dx}{dt} = \pm \sqrt{k(a^2 - x^2)}$$

$$0 = -kx^2 + A$$

$$A = kx^2 = ka^2$$

$$\frac{dx}{dt} = +\sqrt{k(a^2 - x^2)}$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \int \sqrt{k} dt$$

$$\sin^{-1} \frac{x}{a} = \sqrt{k} t + B \rightarrow \sin^{-1} \frac{x}{a} = \sqrt{k} t + \frac{\pi}{2}$$

$$\frac{x}{a} = \sin\left(\sqrt{k} t + \frac{\pi}{2}\right)$$

$$t=0, \quad x=a$$

$$\sin^{-1} \frac{a}{a} = 0 + B$$

$$B = \sin^{-1} 1 = \pi/2$$

$$x = a \sin\left(\sqrt{k} t + \frac{\pi}{2}\right)$$

$$x = a \cos \sqrt{k} t$$

$$\frac{dx}{dt} = -\sqrt{k(a^2 - x^2)}$$

$$\begin{aligned} t &= 0 \\ x &= a \end{aligned}$$

$$\cos^{-1} \frac{x}{a} = \sqrt{k} t + C$$

$$C = 0$$

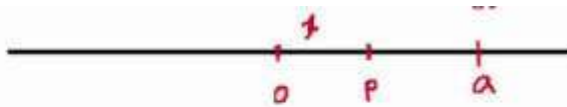
$\Rightarrow$

$$\cos^{-1} \frac{x}{a} = \sqrt{k} t$$

$$\underline{\underline{x = a \cos \sqrt{k} t}}$$

$$x = a \sin \sqrt{k} t$$

Separate part:



$$\frac{d^2 x}{dt^2} = -kx$$

$$m = \pm i k$$

$$x = A e^{i \sqrt{k} t} + B e^{-i \sqrt{k} t}$$

$$= A \cos \sqrt{k} t + B \sin \sqrt{k} t$$

$$t = 0 \quad x = a$$

$$-\frac{dx}{dt} = 0$$

$$x = e^{mt}$$

$$m^2 e^{mt} = -k e^{mt}$$

$$e^{mt} (m^2 + k) = 0$$

$$\Rightarrow m^2 + k = 0$$

$$2 \frac{dx}{dt} \frac{d^2 x}{dt^2} dt$$

$$\left(\frac{dx}{dt} \right)^2 = -kx^2 + A$$

$$0 = -kx^2 + A$$

$$A = kx^2 = ka^2$$

$$\left(\frac{dx}{dt} \right)^2 = -kx^2 + ka^2$$

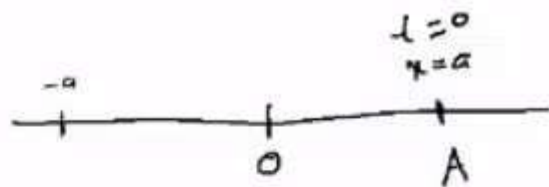
$$\left(\frac{dx}{dt} \right)^2 = k(a^2 - x^2)$$

$$dx = \pm \sqrt{k(a^2 - x^2)}$$

Lecture 12:

Nature of Simple Harmonic Motion

If the particle is at pt. A i.e. $x = a$
& $t = 0$ so SHM is given by
 $x = a \cos \sqrt{\lambda} t$



Amplitude of motion
 $-1, 1$

Now the distance of particle at any time t is given by
 $x = a \cos \sqrt{\lambda} t$
 $= a \cos (\sqrt{\lambda} t + 2\pi)$
 $= a \cos (\sqrt{\lambda} t + 4\pi)$
or $x = a \cos \sqrt{\lambda} (t + \frac{2\pi}{\sqrt{\lambda}})$
 $x = a \cos \sqrt{\lambda} (\frac{2\pi}{\sqrt{\lambda}} + \frac{4\pi}{\sqrt{\lambda}})$

It means particle occupied same position after every $\frac{2\pi}{\sqrt{\lambda}}$

Differentiate w.r.t. t :
 $\frac{dx}{dt} = a(-\sin \sqrt{\lambda} t) \sqrt{\lambda}$
 $\Rightarrow \frac{dx}{dt} = v = -a\sqrt{\lambda} \sin \sqrt{\lambda} t$

It means particle occupied same position after every $\frac{2\pi}{\sqrt{\lambda}}$

Differentiate w.r.t. t :
 $\frac{dv}{dt} = a(-\cos \sqrt{\lambda} t) \sqrt{\lambda}$
 $\Rightarrow \frac{dv}{dt} = -a\sqrt{\lambda} \cos \sqrt{\lambda} t$

$$\sin(\sqrt{\lambda} t) = \sin(\sqrt{\lambda} t + 2\pi)$$

$$T = \frac{2\pi}{\sqrt{\lambda}}$$

time period of oscillation

$$v \cdot T = 1 \Rightarrow v = \frac{1}{T} = \frac{1}{2\pi/\sqrt{\lambda}} = \frac{\sqrt{\lambda}}{2\pi}$$

A particle describe Simple Harmonic Motion with frequency f . If the greatest velocity is V , find the amplitude and maximum value of acceleration of the particle. Also show that the velocity v at a distance x from the centre of motion is given by

$$v = 2\pi f \sqrt{a^2 - x^2} \quad \text{where } a \text{ is the amplitude}$$

Sol

$$v = \sqrt{\lambda(a^2 - x^2)} \quad \text{--- (i)}$$

$$v = \sqrt{\lambda a^2} = \sqrt{\lambda} a$$

$$a = \frac{v}{\sqrt{\lambda}} \quad \text{--- (ii)} \quad T = \frac{2\pi}{\sqrt{\lambda}}$$

$$f = \frac{1}{T} = \frac{1}{2\pi/\sqrt{\lambda}} = \frac{\sqrt{\lambda}}{2\pi}$$

$$\Rightarrow \sqrt{\lambda} = 2\pi f$$

$$a = \frac{v}{2\pi f}$$



$$\text{Acceleration} = |-\lambda x| = \lambda x$$

$$a = a = \lambda a = (\sqrt{\lambda})^2 a = (2\pi f)^2 \frac{v}{2\pi f} = 2\pi v f$$

$$v \cdot \frac{dv}{dx} = -\lambda x$$

$$v \cdot dv = -\lambda x dx$$

$$\int v \cdot dv = -\lambda \int x dx$$

$$v \cdot \frac{dv}{dx}$$

$$v \cdot dv = -\lambda x dx$$

$$\int v \cdot dv = -\lambda \int x dx$$

$$\frac{v^2}{2} = -\lambda \left[\frac{x^2}{2} \right] + B$$

$$x=a, v=0$$

$$0 = -\frac{\lambda a^2}{2} + B \Rightarrow B = \frac{\lambda a^2}{2}$$

$$\frac{v^2}{2} = -\lambda \left[\frac{x^2}{2} \right] + \frac{\lambda a^2}{2}$$

$$\frac{v^2}{2} = -\lambda \left[\frac{x^2}{2} \right] + B$$

$$x=a, v=0$$

$$0 = -\frac{\lambda a^2}{2} + B \Rightarrow B = \frac{\lambda a^2}{2}$$

$$\frac{v^2}{2} = -\lambda \left[\frac{x^2}{2} \right] + \frac{\lambda a^2}{2}$$

$$\frac{v^2}{2} = \frac{\lambda}{2} [a^2 - x^2]$$

$$v = \sqrt{\lambda(a^2 - x^2)}$$

$$= \sqrt{\lambda} \cdot \sqrt{a^2 - x^2}$$

$$= 2\pi f \cdot \sqrt{a^2 - x^2}$$

Lecture: 13

A particle describing simple harmonic motion has velocities 5 ft./sec and 4 ft./sec when its distances from the centre are 12 ft. and 13 ft. respectively. Find the time period of motion.

$$v = \sqrt{\lambda(a^2 - x^2)}$$

$$v^2 = \lambda(a^2 - x^2)$$

$$5^2 = \lambda(a^2 - 12^2)$$

$$25 = \lambda a^2 - 144\lambda$$

— (A)

$$4^2 = \lambda(a^2 - 13^2)$$

$$16 = \lambda(a^2 - 169)$$

$$= \lambda a^2 - 169\lambda$$

— (B)

$$25 = \lambda a^2 - 144\lambda$$

$$16 = \lambda a^2 - 169\lambda$$

$$\begin{array}{r} 25 = \lambda a^2 - 144\lambda \\ 16 = \lambda a^2 - 169\lambda \\ \hline 9 = 25\lambda \end{array}$$

$$\Rightarrow \lambda = 9/25$$

$$v = \sqrt{\lambda(a^2 - x^2)}$$

$$5^2 = \lambda(a^2 - 12^2)$$

$$25 = \lambda a^2 - 144\lambda$$

— (A)

$$4^2 = \lambda(a^2 - 13^2)$$

$$16 = \lambda(a^2 - 169)$$

$$= \lambda a^2 - 169\lambda$$

— (B)

$$25 = \lambda a^2 - 144\lambda$$

$$16 = \lambda a^2 - 169\lambda$$

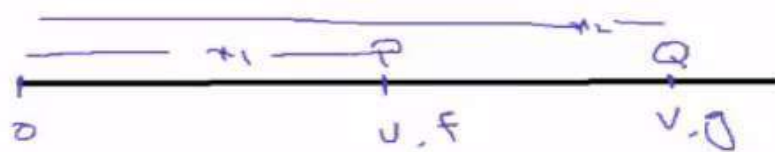
$$\begin{array}{r} 25 = \lambda a^2 - 144\lambda \\ 16 = \lambda a^2 - 169\lambda \\ \hline 9 = 25\lambda \end{array}$$

$$\Rightarrow \lambda = 9/25$$

$$T = \frac{2\pi}{\sqrt{\lambda}} = \frac{2\pi}{\sqrt{9/25}} = \frac{2\pi}{3/5} = \frac{10}{3}\pi$$

Example

A point describes simple harmonic motion in such a way that its velocity and acceleration at a point P are u and f respectively and the corresponding quantities at another point Q are v and g . Find the distance PQ



$$v = \sqrt{\lambda(a^2 - x^2)}$$

$$v^2 = \lambda(a^2 - x^2)$$

$$u^2 = \lambda(a^2 - x_1^2)$$

$$v^2 = \lambda(a^2 - x_2^2)$$

$$\text{acceleration} = -\lambda x$$

$$f = -\lambda x_1$$

$$g = -\lambda x_2$$

$$v^2 = \lambda(a^2 - x^2)$$

$$u^2 = \lambda(a^2 - x_1^2) \quad \text{--- (A)}$$

$$v^2 = \lambda(a^2 - x_2^2) \quad \text{--- (B)}$$

$$\text{acceleration} = -\lambda x$$

$$f = -\lambda x_1$$

$$g = -\lambda x_2$$

$$u^2 = \lambda a^2 - \lambda x_1^2$$

$$v^2 = \lambda a^2 - \lambda x_2^2$$

$$\underline{u^2 - v^2 = \lambda(x_2^2 - x_1^2)}$$

$$u^2 - v^2 = \lambda(x_2 - x_1)(x_2 + x_1)$$

$$u^2 - v^2 = (x_2 - x_1) \lambda(x_2 + x_1)$$

$$= -(x_1 - x_2) \times (f + g)$$

$$= -(x_1 - x_2) \times (f + g)$$

$$\begin{aligned}
 u^2 &= \lambda x^2 - \lambda x_1^2 \\
 v^2 &= \lambda x^2 - \lambda x_2^2 \\
 \hline
 u^2 - v^2 &= \lambda (x_2^2 - x_1^2) \\
 u^2 - v^2 &= \lambda (x_2 - x_1)(x_2 + x_1) \\
 &= (x_2 - x_1) \times \lambda (x_2 + x_1) \\
 &= -(x_2 - x_1) \times (F + G) \\
 &= x_2 - x_1 \\
 -\frac{(u^2 - v^2)}{F + G} & \quad |PQ| = |x_2 - x_1| = \frac{u^2 - v^2}{F + G}
 \end{aligned}$$

$F + G = -\lambda (x_1 + x_2)$

Example

If a point P moves with a velocity v given by $v^2 = n^2 (ax^2 + 2bx + c)$ show that P executes a simple harmonic motion. Find the centre, the amplitude and the time-period of the motion.

$$\begin{aligned}
 2v \frac{dv}{dx} &= n^2 (2ax + 2b) \\
 &= 2an^2 \left(x + \frac{b}{a}\right)
 \end{aligned}$$

$$v \frac{dv}{dx} = an^2 \left(x + \frac{b}{a}\right)$$

$$\text{acceleration} = \lambda x$$

$$x = 0$$

$$x + \frac{b}{a} = 0$$

$$x = -\frac{b}{a}$$

$$an^2 = \lambda$$

$$x + \frac{b}{a} = x$$

$$x = -\frac{b}{a}$$

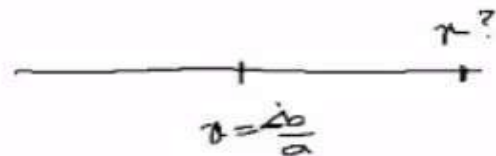
$$a \frac{d^2 x}{dt^2} = \lambda x$$

$$x + \frac{b}{a} = x$$

$$x = 0$$

$$x + \frac{b}{a} = 0$$

$$x = -\frac{b}{a}$$



$$n^2 (ax^2 + 2bx + c) = 0$$

$$\Rightarrow ax^2 + 2bx + c = 0$$

$$x = \frac{-2b \pm \sqrt{4b^2 - 4(ac)}}{2a}$$

$$x = \frac{-2b \pm \sqrt{4b^2 - 4ac}}{2a}$$

$$= \frac{-b \pm \sqrt{b^2 - ac}}{a}$$

$$\frac{-b + \sqrt{b^2 - ac}}{a} - \left(-\frac{b}{a}\right)$$

$$\frac{\sqrt{b^2 - ac}}{a} = \text{amplitude}$$

$$T = \frac{2\pi}{\sqrt{\lambda}}$$

$$\lambda = an^2$$

$$\sqrt{\lambda} = n\sqrt{a}$$

$$T = \frac{2\pi}{n\sqrt{a}}$$

Lecture 14:

Methods of Dynamics

A dynamical system is called two dimensional if the orbit of each particle of the system is parallel to a plane.

Equation of Motion of Particle

$$\begin{aligned}\frac{d}{dt}(mv) &= F \\ m \cdot \frac{d}{dt}v &= F \\ m \cdot \dot{v} &= F \\ \underline{m \cdot a} &= F\end{aligned}$$

$$\begin{aligned}\underline{m \cdot \frac{d}{dt}v} \\ \underline{m \cdot \dot{v}} &= F \\ \underline{m \cdot a} &= F\end{aligned}$$

$$\frac{dr}{dt} = v$$

$$\frac{d^2r}{dt^2} = \frac{dv}{dt}$$

$$\begin{aligned}m \cdot \frac{d^2r}{dt^2} &= F \\ m \cdot \ddot{r} &= F\end{aligned}$$

Central Force

If the force F on a particle always passes through a fixed point O , the force is called a central force and O is called the center of force.

$$\begin{aligned}\frac{d}{dt}(mv) &= 0 \\ \underline{mv} &= C\end{aligned}$$

Lecture 15:

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j}$$

$$\mathbf{F} = X\mathbf{i} + Y\mathbf{j}$$

$$m \frac{d\mathbf{v}}{dt} = \mathbf{F}$$

$$m \cdot \dot{\mathbf{v}} = \mathbf{F}$$

$$m \ddot{\mathbf{r}} = \mathbf{F}$$

$$m(\ddot{x}\mathbf{i} + \ddot{y}\mathbf{j}) = X\mathbf{i} + Y\mathbf{j}$$

$$m \frac{d\mathbf{v}}{dt} = \mathbf{F}$$

$$m \cdot \dot{\mathbf{v}} = \mathbf{F}$$

$$m \ddot{\mathbf{r}} = \mathbf{F}$$

$$m(\ddot{x}\mathbf{i} + \ddot{y}\mathbf{j}) = X\mathbf{i} + Y\mathbf{j}$$

$$m\ddot{x} = X$$

$$m\ddot{y} = Y$$

$$\mathbf{v} = \dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j}$$

$$v^2 = (\dot{x})^2 + (\dot{y})^2$$

Example:

A particle moves in such a way that its position vector at time t is

$$\mathbf{r} = (a \cos nt) \mathbf{i} + (b \sin nt) \mathbf{j}$$

where a, b, n are constants and $a > b > 0$. Show that the path of the particle is an ellipse of semi-major and minor axes a, b respectively, and that the field of force is directed towards the centre of the ellipse. Find also the maximum speed.

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j}$$

$$x = a \cos nt$$

$$y = b \sin nt$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{a^2 \cos^2 nt}{a^2} + \frac{b^2 \sin^2 nt}{b^2} = 1$$

$$v^2 = (\dot{x})^2 + (\dot{y})^2$$

$$\dot{x} = -a \sin nt (n) \\ = -an \sin nt$$

$$\dot{y} = bn \cos nt$$

$$\ddot{x} = -an^2 \cos nt$$

$$\ddot{y} = -bn^2 \sin nt$$

$$v^2 = a^2 n^2 \sin^2 nt + b^2 n^2 \cos^2 nt \\ = a^2 n^2 \sin^2 nt + b^2 n^2 (1 - \sin^2 nt) \\ = n^2 (a^2 - b^2) \sin^2 nt + n^2 b^2$$

$$v_{\max}^2 = n^2 a^2 - n^2 b^2 + n^2 b^2 \\ = n^2 a^2$$

$$v_{\max} = na$$

$$ma = F$$

$$m\dot{v} = F$$

$$m\ddot{r} = F$$

$$\ddot{\mathbf{r}} = \ddot{r}\mathbf{i} + \ddot{\theta}\mathbf{j}$$

$$= -a\dot{n}^2\cos nt\mathbf{i} + (-b\dot{n}^2\sin nt)\mathbf{j}$$

$$= -\dot{n}^2(a\cos nt\mathbf{i} + b\sin nt\mathbf{j})$$

$$= -\dot{n}^2\mathbf{r}$$

$$\mathbf{F} = m\ddot{\mathbf{r}} = m(-\dot{n}^2\mathbf{r}) = -m\dot{n}^2\mathbf{r}$$

If $F < 0$ the force is said to be attractive towards O. If $F > 0$ the force is said to be repulsive from O.

Lecture 16:

Two-Dimensional Polar Form of the Equation of Motion

(r, θ)

$$a_r = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = \ddot{r} - r(\dot{\theta})^2$$

$$a_\theta = 2 \cdot \frac{dr}{dt} \cdot \frac{d\theta}{dt} + r \frac{d^2\theta}{dt^2}$$

$$= 2 \cdot \dot{r} \cdot \dot{\theta} + r\ddot{\theta}$$

$$\hat{\mathbf{r}} \quad \hat{\mathbf{s}}$$

$$\hat{r} \quad \hat{s}$$

$$F = F_r \hat{r} + F_\theta \hat{s}$$

$$a = a_r \hat{r} + a_\theta \hat{s}$$

$$F = ma$$

$$F_r \hat{r} + F_\theta \hat{s} = m(a_r \hat{r} + a_\theta \hat{s})$$

$$F = ma$$

$$F_r \hat{r} + F_\theta \hat{s} = m(a_r \hat{r} + a_\theta \hat{s})$$

$$F_r = ma_r = m(\ddot{r} - r\dot{\theta}^2)$$

$$F_\theta = ma_\theta = m(2\dot{r}\dot{\theta} + r\ddot{\theta})$$

$$r = a, \quad \dot{r} = 0, \quad \ddot{r} = 0$$

$$-m(a\dot{\theta}^2) = F_r$$

$$ma\ddot{\theta} = F_\theta$$

Example:

A particle of mass m moves on xy -plane under the force $F = -\frac{k}{r^3} \mathbf{r}$ where r is its distance from the origin O . If it starts on the positive x -axis at a distance a from O with speed v_0 in a direction making an angle α with the positive x -direction, prove that at time t

$$\ddot{r} = \frac{m a^2 v_0^2 \sin^2 \alpha - k}{m r^3}$$

Sol

$$\mathbf{r} = r \hat{r}$$

$$\mathbf{F} = -\frac{k}{r^3} \mathbf{r}$$

$$= -\frac{k}{r^3} (x \hat{r})$$

$$= -\frac{k}{r^3} \hat{r}$$

$$F_r = m(\ddot{r} - r(\dot{\theta})^2)$$

$$\underline{\underline{F_\theta}} = m(2r\dot{\theta} + r\ddot{\theta})$$

$$m(\ddot{r} - r(\dot{\theta})^2) = -\frac{k}{r^3}$$

$$\ddot{r} - r(\dot{\theta})^2 = \frac{-k}{mr^3} \Rightarrow \ddot{r} = \frac{-k}{mr^3} + r\dot{\theta}^2$$

$$\therefore 0 = m(2r\dot{\theta} + r\ddot{\theta})$$

$$\hookrightarrow 0 = m(2\dot{r}\ddot{\theta} + r\ddot{\theta})$$

$$\Rightarrow 2\dot{r}\ddot{\theta} + r\ddot{\theta} = 0$$

$$\frac{1}{r} \left(\frac{d}{dt} (r^2 \dot{\theta}) \right) = 0$$

$$\frac{d}{dt} (r^2 \dot{\theta}) = 0$$

$$r^2 \dot{\theta} = C$$

$$\ddot{r} = -\frac{k}{mr^3} + r\dot{\theta}^2$$

$$= -\frac{k}{mr^3} + r \left(\frac{C^2}{r^4} \right)$$

$$= -\frac{k}{mr^3} + \frac{C^2}{r^3}$$

$$= \frac{1}{mr^3} (mC^2 - k)$$

$$\mathbf{v} = \dot{r} \hat{\mathbf{r}} + r\dot{\theta} \hat{\mathbf{s}} \quad \text{at } t=0$$

$$r=a \quad v=v_0$$

$$\mathbf{v}_0 = (v_0 \cos \alpha) \hat{\mathbf{r}} + (v_0 \sin \alpha) \hat{\mathbf{s}}$$

$$(\dot{r})_{r=a} = v_0 \cos \alpha$$

$$(r\dot{\theta})_{r=a} = v_0 \sin \alpha$$

$$a\dot{\theta} = v_0 \sin \alpha$$

$$\dot{\theta} = \frac{v_0 \sin \alpha}{a}$$

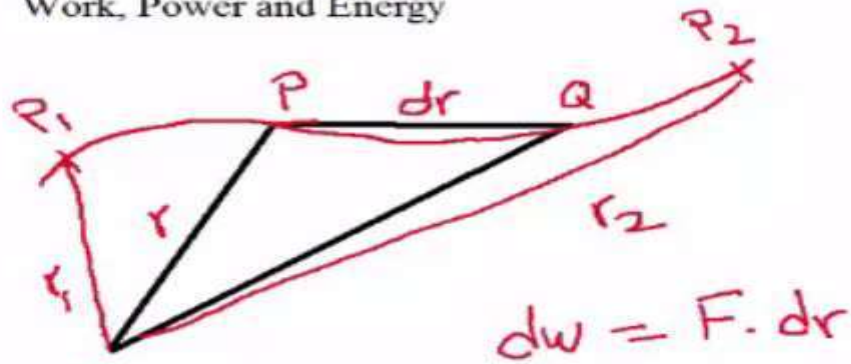
$$C = r^2 \dot{\theta}$$

$$r=a$$

$$C = a^2 \cdot \frac{v_0 \sin \alpha}{a} = a v_0 \sin \alpha$$

Lecture 17:

Work, Power and Energy



$$W = \int_{P_1}^{P_2} F \cdot dr$$
$$= \int_{r_1}^{r_2} F \cdot dr$$

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Diagram showing a force vector F acting on a particle moving along a path from P_1 to P_2 . The work done is given by $W = \int_{P_1}^{P_2} F \cdot dr = \int_{r_1}^{r_2} F \cdot dr$.

$$dw = F \cdot dr$$

$$\frac{dw}{dt} = F \cdot \frac{dr}{dt}$$
$$= F \cdot v$$

$$\underline{T = \frac{1}{2}mv^2}$$

$$\underline{P = \frac{dw}{dt}}$$

Theorem (Principle of Energy)

The total work done on a particle in moving it along a curve C from P_1 to P_2 is equal to the increase $T_2 - T_1$ in the kinetic energy, where T_1, T_2 are respectively the kinetic energies of the particle at times t_1, t_2 corresponding to the positions P_1, P_2 .

Proof

$$W = \int_{r_1}^{r_2} F \cdot dr$$

$$= \int_{r_1}^{r_2} m \frac{dv}{dt} \cdot dr$$

$$= \int_{r_1}^{r_2} m \frac{dr}{dt} \cdot dv$$

$$F = ma = m \frac{dv}{dt}$$

$$= \int_{r_1}^{r_2} m \frac{dr}{dt} \cdot dv$$

$$= \int_{r_1}^{r_2} m \cdot v \cdot dv$$

$$= m \frac{v^2}{2} \Big|_{r_1}^{r_2}$$

$$= \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2$$

$$= T_2 - T_1$$

Cor The rate of increase of kinetic energy of a particle is equal to the power applied to the particle.

$$T = \frac{1}{2} mv^2$$

$$\begin{aligned} \frac{dT}{dt} &= \frac{1}{2} \cdot 2mv \cdot \frac{dv}{dt} \\ &= v \cdot m \frac{dv}{dt} \\ &= v \cdot ma \\ &= v \cdot F \\ &= P \end{aligned}$$

Lecture 18:

Conservative Force and Principle of Conservation of Energy

$$\mathbf{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$\mathbf{F}$

The work done by a conservative force is independent of the path; in other words, the work done by a conservative force is the same for any path connecting two points.

$$\begin{aligned} \mathbf{F} &= -\frac{\partial V}{\partial x} \hat{i} - \frac{\partial V}{\partial y} \hat{j} - \frac{\partial V}{\partial z} \hat{k} \\ &= -\nabla V = \text{gradient of } V \end{aligned}$$

Let $f(x, y, z)$ be a scalar-valued function. Then its **gradient**

$$\nabla f(x, y, z) = \left(\frac{\partial f}{\partial x}(x, y, z), \frac{\partial f}{\partial y}(x, y, z), \frac{\partial f}{\partial z}(x, y, z) \right) \quad \hat{i} \quad \hat{j} \quad \hat{k}$$

is a vector field, which we denote by $\mathbf{F} = \nabla f$. We can easily calculate that the **curl** of $\mathbf{F}$ is zero.

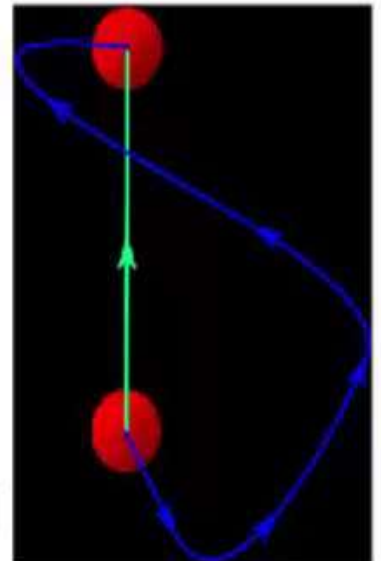
We use the formula for **curl** $\mathbf{F}$ in terms of its components

$$\text{curl } \mathbf{F} = \begin{pmatrix} \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \end{pmatrix}$$

Since each component of $\mathbf{F}$ is a derivative of f , we can rewrite the curl as

$$\text{curl } \nabla f = \begin{pmatrix} \frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}, \frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}, \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \end{pmatrix}$$

If f is twice continuously differentiable, then its second derivatives are independent of the order in which the derivatives are applied. All the terms cancel in the expression for $\text{curl } \nabla f$, and we conclude that $\text{curl } \nabla f = \mathbf{0}$.



If f is twice continuously differentiable, then its second derivatives are independent of the order in which the derivatives are applied. All the terms cancel in the expression for $\text{curl } \nabla f$, and we conclude that $\text{curl } \nabla f = 0$.

$$\text{curl } (\text{grad } V) = 0$$

$$\text{curl } (F) = 0$$

Theorem.

The work done on the particle in moving it from position P_1 to P_2 under a conservative field of force, is the difference between the potential energies of the particle at P_1 and P_2 respectively.

$$W = \int_{P_1}^{P_2} F \cdot dr$$

$$W = \int_{P_1}^{P_2} F \cdot dr$$

$$= \int_{P_1}^{P_2} - \left(\frac{\partial V}{\partial x} i + \frac{\partial V}{\partial y} j + \frac{\partial V}{\partial z} k \right) \cdot (dx i + dy j + dz k)$$

$$= - \int_{P_1}^{P_2} \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz$$

$$= - \int_{P_1}^{P_2} dV = - V \Big|_{P_1}^{P_2}$$

$$= -V_2 - (-V_1)$$

$$= -V_2 + V_1$$

$$= V_1 - V_2$$

$$T \cup V$$

$$E = T + V$$

Theorem. In a conservative field of force the total energy of a particle remains constant throughout the motion.

Proof

$$W = T_2 - T_1$$

$$W = V_1 - V_2$$

$$T_2 - T_1 = V_1 - V_2$$

$$T_2 + V_2 = V_1 + T_1$$

Lecture 19:

Earth Gravitational Field

$\vec{g}$

$$F = ma$$

$$W = mg = -mg\hat{j}$$

$$\vec{g} = -g\hat{j}$$

$$V = mgy$$

0

$$F = ma$$

$$W = mg = -mg\hat{j}$$

$$\vec{g} = -g\hat{j}$$

$$V = mgy$$

y is the height of particle at time t then

$$-\frac{dV}{dt} = -mg = W$$

Conservative

$$F = -\frac{\partial V}{\partial x}\hat{i} - \frac{\partial V}{\partial y}\hat{j} - \frac{\partial V}{\partial z}\hat{k}$$

Example

A particle of mass m falls freely in a vertical plane under gravitational field from a height h with speed v_0 . Find the speed with which it strikes the ground.

$$T = \frac{1}{2}mv^2 \quad V = mgy$$

$$\frac{1}{2}mv^2 + mgy = C \quad \text{--- (1)}$$

$$t=0, \quad y=h, \quad v=v_0$$

$$\frac{1}{2}mv_0^2 + mgh = C$$

$$\frac{1}{2}mv_0^2 + mgh = C$$

$$\frac{1}{2}mv^2 + mgy = \frac{1}{2}mv_0^2 + mgh$$

$$\frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = mg(h-y)$$

$$\frac{1}{2}m(v^2 - v_0^2) = 2mg(h-y)$$

$$v^2 = v_0^2 + 2g(h-y)$$

$$y=0$$

$$v^2 = v_0^2 + 2gh$$

$$v = \sqrt{v_0^2 + 2gh}$$

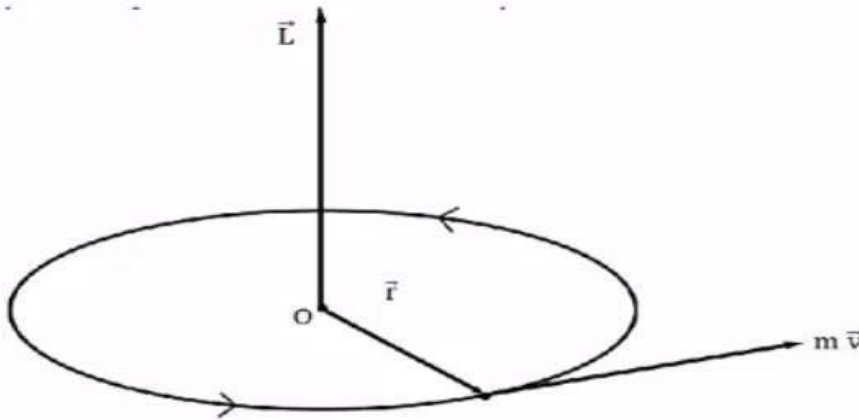
Lecture 20:

Principle of Angular Momentum

The moment of linear momentum of an object is called angular momentum. Angular momentum is represented by L .

Momentum is the motion contained in a body. Quantity of motion possessed by a body depends upon both of its mass and velocity. So the product of mass and velocity is the measure of the momentum.

The angular momentum of an object about an axis of rotation is measured by the product of the linear momentum of the object and the perpendicular distance between the object and the axis of rotation.



Suppose an object of mass m is revolving around a circle of radius r with speed v about an axis passing through center O as shown in the figure.

The linear momentum of the object, $p = mv$.

The angular momentum of an object, $L = \text{linear momentum} \times \text{perpendicular distance from the axis of rotation}$

$$L = p r$$

$$L = m \underline{v} r = m \cdot (v \times r)$$

Since $v = \omega r$, where ω is the angular velocity of the object, the angular momentum can be written as

$$L = m (\omega r) r$$

$$L = m r^2 \omega$$

The equations above are the expressions for the angular momentum of the body. It is a vector quantity.

$$\tau = r \times F$$

Theorem:

The rate of change of angular momentum of a particle about a point O is equal to the torque of the forces acting on the particle.

$$\frac{dL}{dt} = \tau$$

$$L = m (v \times r)$$

$$\frac{dL}{dt} = \frac{d}{dt} (m (v \times r))$$

$$= \frac{d}{dt} (r \times (mv))$$

$$= \frac{dr}{dt} \times (mv) + r \times \frac{d}{dt} (mv)$$

$$= v \times mv + r \times \frac{d}{dt} (mv)$$

$$= \frac{m(v \times v)}{m(v \times v)}$$

$$= 0 + r \times \frac{d}{dt} (mv)$$

$$\vec{F} = \frac{d}{dt} (mv)$$

$$= \frac{dr}{dt} \times (mv) + r \times \frac{d}{dt} (mv)$$

$$= v \times mv + r \times \frac{d}{dt} (mv)$$

$$= \frac{m(v \times v)}{m(v \times v)}$$

$$= 0 + r \times \frac{d}{dt} (mv)$$

$$\vec{F} = \frac{d}{dt} (mv)$$



$$= r \times F$$

$$\frac{dL}{dt} = \tau$$

Lecture 21:

Theorem

The total work done by the forces on a system of particles in moving from the configuration at time t_1 to the configuration at time t_2 is equal to the increase in kinetic energy.

$$W = \int_P^Q \mathbf{F} \cdot d\mathbf{r}$$

$$W = \sum_{i=1}^n \int_{P_i}^{Q_i} \mathbf{F}_i \cdot d\mathbf{r}_i$$

$$W = \sum_{i=1}^n \int_{P_i}^{Q_i} \mathbf{F}_i \cdot d\mathbf{r}_i$$

$$= \sum_{i=1}^n \int_{P_i}^{Q_i} m_i \frac{d\mathbf{v}_i}{dt} \cdot d\mathbf{r}_i$$

$$= \sum \int_{P_i}^{Q_i} m_i \mathbf{v}_i \cdot d\mathbf{v}_i$$

$$= \sum \left(\frac{m_i v_i^2}{2} \right) \Big|_{P_i}^{Q_i}$$


$$= \sum \left(\frac{1}{2} m_i v_i^2 \right) \Big|_{P_i}^{Q_i}$$

$$= T_2 - T_1$$

$$F_i = -\nabla V_i$$

$$= - \left(\frac{\partial V_i}{\partial x} i + \frac{\partial V_i}{\partial y} j + \frac{\partial V_i}{\partial z} k \right)$$

$$F_i \cdot dr_i = (\quad) \times (dx, dy, dz)$$

$$= -dV^{(i)}$$

$$W = - \sum \int_{P_i}^{Q_i} dV^{(i)} =$$

$$F_i \cdot dr_i = (\quad) \times (dx, dy, dz)$$

$$= -dV^{(i)}$$

$$W = - \sum \int_{P_i}^{Q_i} dV^{(i)} = - \sum (V^{(i)})_{P_i}^{Q_i}$$

$$= - (V_2 - V_1)$$

$$= V_1 - V_2$$

$$T_2 - T_1 = V_1 - V_2$$

$$\Rightarrow T_1 + V_1 = T_2 + V_2$$

Motion of the Centre of Mass of a System of Particles

Let O be the origin and $r_1, r_2, \dots, r_n$ the position vectors of a system S of n particles of masses $m_1, m_2, \dots, m_n$ respectively at time t . Then the centre of mass G of the system S is the point whose position vector is

$$\begin{aligned} r_G &= \frac{\sum_{i=1}^n m_i r_i}{\sum_{i=1}^n m_i} \\ &= \frac{1}{M} \sum_{i=1}^n m_i r_i \end{aligned}$$

Linear Momentum

$$P = \sum_{i=1}^n m_i v_i$$

F_L External Force

F_i' Internal Force

$$F_i' = \sum_{j=1}^n F_{ij} \quad F_{ii} = 0$$

$$F_{ji} = -F_{ij}$$

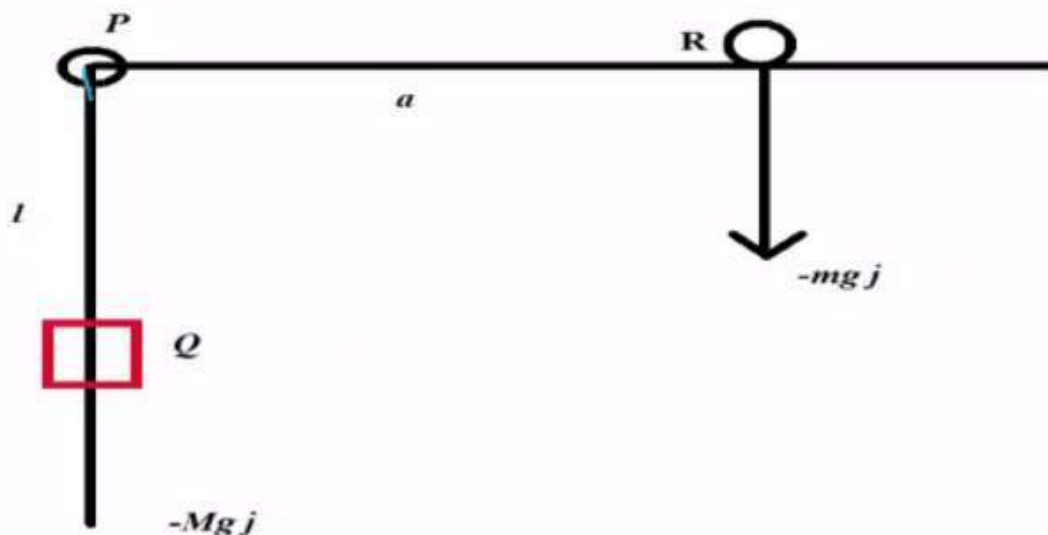
$$\sum_{i=1}^n F_i' = \sum_{i=1}^n \sum_{j=1}^n F_{ij}$$

$$\begin{aligned}
 \sum_{i=1}^n F_i' &= \sum_{i=1}^n \sum_{j=1}^n F_{ij} \\
 &= \sum_{j=1}^n F_{ji} \\
 &= - \sum_{i,j=1}^n F_{ij} \\
 &= - \sum_{i=1}^n \sum_{j=1}^n F_{ij} \\
 &= - \sum_{i=1}^n F_i'
 \end{aligned}$$

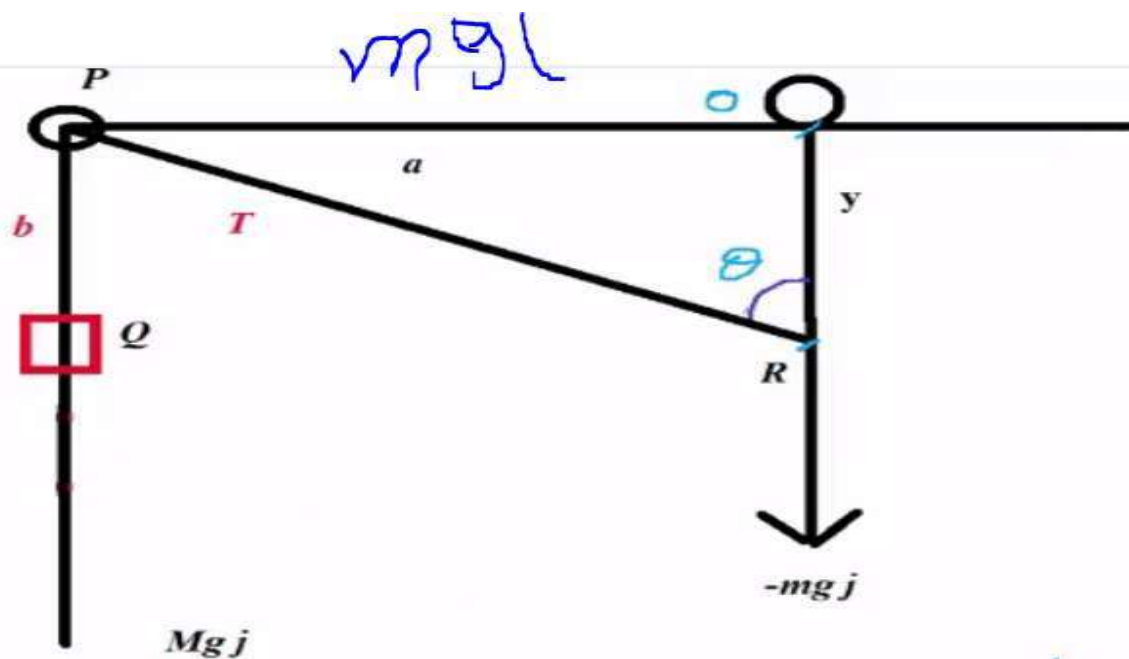
$$\begin{aligned}
 &= - \sum_{i=1}^n F_i' \\
 2 \sum_{i=1}^n F_i' &= 0 \\
 \sum_{i=1}^n F_i' &= 0
 \end{aligned}$$

Lecture 22:

A ring of mass m slides on a smooth vertical rod and is attached to a light string which passes over a small pulley distant a from the rod, and has mass $M > m$ fastened to its other end. Show that if the ring is dropped from the point in the rod in the same horizontal plane as the pulley, it will descend a distance $\frac{2amM}{M^2 - m^2}$ before coming to the rest.



$$-Mgl$$



$$\underline{\underline{\frac{1}{2} M v_Q^2 - Mgb}}$$

$$\underline{\underline{\frac{1}{2} m v_R^2 - mgy}}$$

1

$Mg\mathbf{j}$

$$\left(\underline{\underline{\frac{1}{2} M v_Q^2 - Mgb}} + \underline{\underline{\frac{1}{2} m v_R^2 - mgy}} \right)$$

$$\frac{1}{2} M v_Q^2 + \frac{1}{2} m v_R^2 - Mgb - mgy = -Mgl$$

$$a + l = \sqrt{a^2 + y^2} + b$$

$$M v_Q^2 + m v_R^2 - 2Mg(l + a - \sqrt{a^2 + y^2}) - 2mgy + 2Mgl = 0$$

$$- 2mgy + 2Mgl = 0$$

$$\underline{M V_D^2} + \underline{m V_A^2} = 2Mg(a - \sqrt{a^2 + h^2}) + 2mgy$$

$$2Mg(a - \sqrt{a^2 + h^2}) + 2mgh = 0$$

$$2Mga - 2Mg\sqrt{a^2 + h^2} + 2mgh = 0$$

$$Ma - M(\sqrt{a^2 + h^2}) + mh = 0$$

$$\Rightarrow h = \frac{2amM}{M^2 - m^2}$$

$$2Mga - 2Mg\sqrt{a^2 + h^2} + 2mgh = 0$$

$$Ma - M(\sqrt{a^2 + h^2}) + mh = 0$$

$$Ma + mh = M\sqrt{a^2 + h^2}$$

$$M^2a^2 + m^2h^2 + 2Mmah = M^2a^2 + M^2h^2$$

$$m^2h^2 - M^2h^2 = -2Mmah$$