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COMPLETE SLIDES

Tensor Analysis

MTH623

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Introduction to Tensors

Introduction

The origin of tensors...

Why we need tensors?

What tensors are actually?

The origin of tensors

- The word **tensor** has its root *tensus* in Latin, meaning stretch or tension.
- **J. W. Gibbs in (1884)** developed the concept of dyadics (object with two parts or elements) and polyadics (an object comprising of many elements) which are actually tensors in modern sense but only different by names.

The origin of tensors

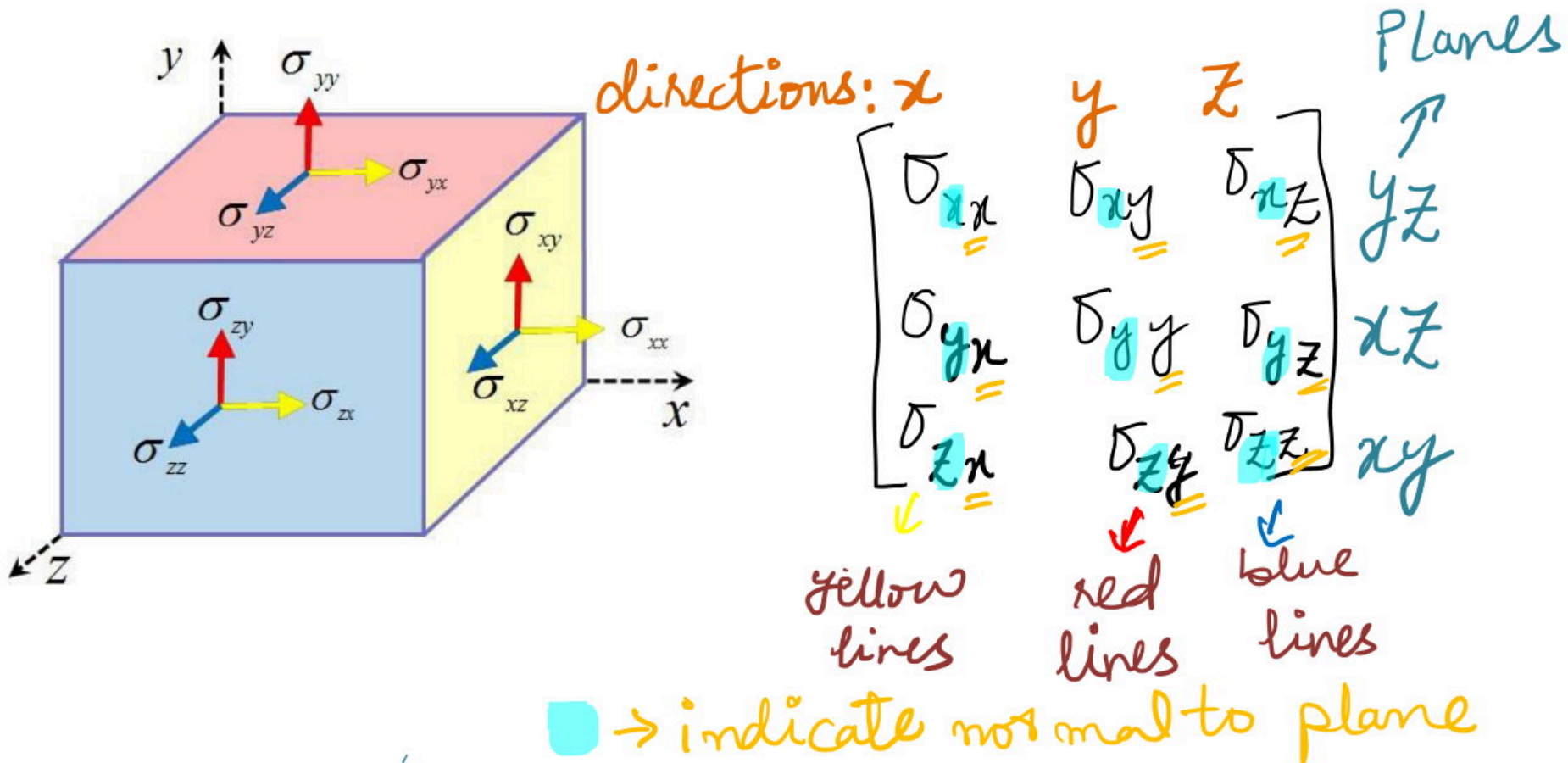
- W. Voigt in (1898) introduced the term tensor in his study of stress and strain of crystals.
- Later on many physicists including Albert Einstein and Marcel Grossmann in (1913) used the Ricci's concepts with the name tensors instead of systems.

Why we need tensors?

- A **scalar** is a physical quantity that is represented by a **number** at a particular point in space. For example, mass, density and temperature.
- A **vector** is a tool to keep track of **two pieces of information**, typically magnitude and direction, for a physical quantity. For example position, force and velocity.

Why we need tensors?

What happens when we need to keep track of **three pieces of information** for a given physical quantity? Say, three pieces of information are magnitude, direction and plane. For example stress, strain and inertia tensors.



Why we need tensors?

In short:

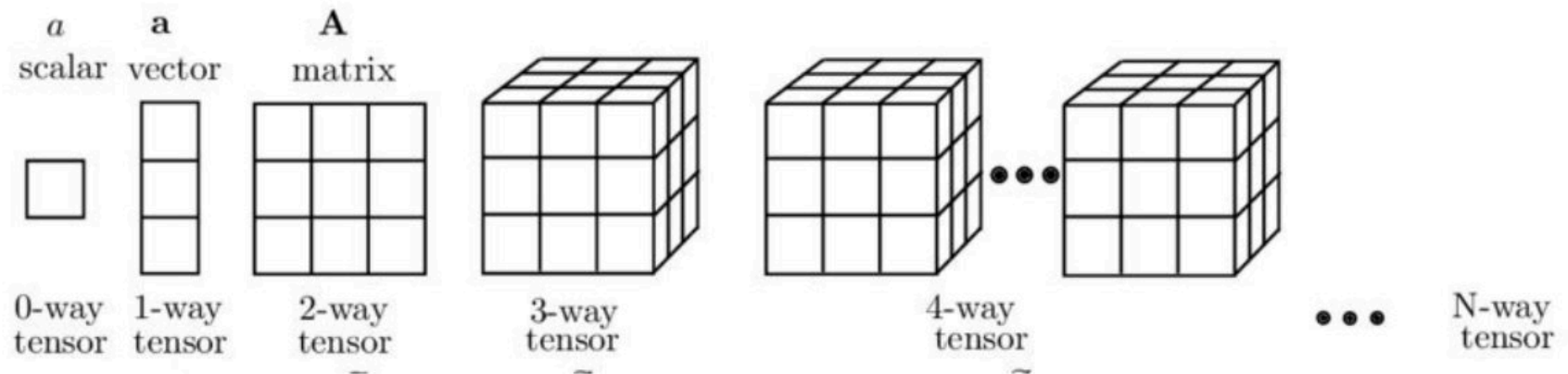
A scalar is a quantity whose specification requires just one number.

On the other hand, a vector is a quantity whose specification requires three numbers, namely its components with respect to some basis.

Scalars and vectors are both special cases of a more general term called a tensor.

What is a tensor?

Tensor can be regarded as a general name of multi-way array data.



Tensor Definition:

In an m dimensional space, a tensor of rank n is a mathematical object that has n indices, m^n components and obey certain transformation rule.

Rank of a tensor

The rank of a tensor also known as order or degree of tensor is the total number of free indices in a tensor.

For example,

1) m (scalar) tensor of rank zero

2) v_α (vector) tensor of rank 1,

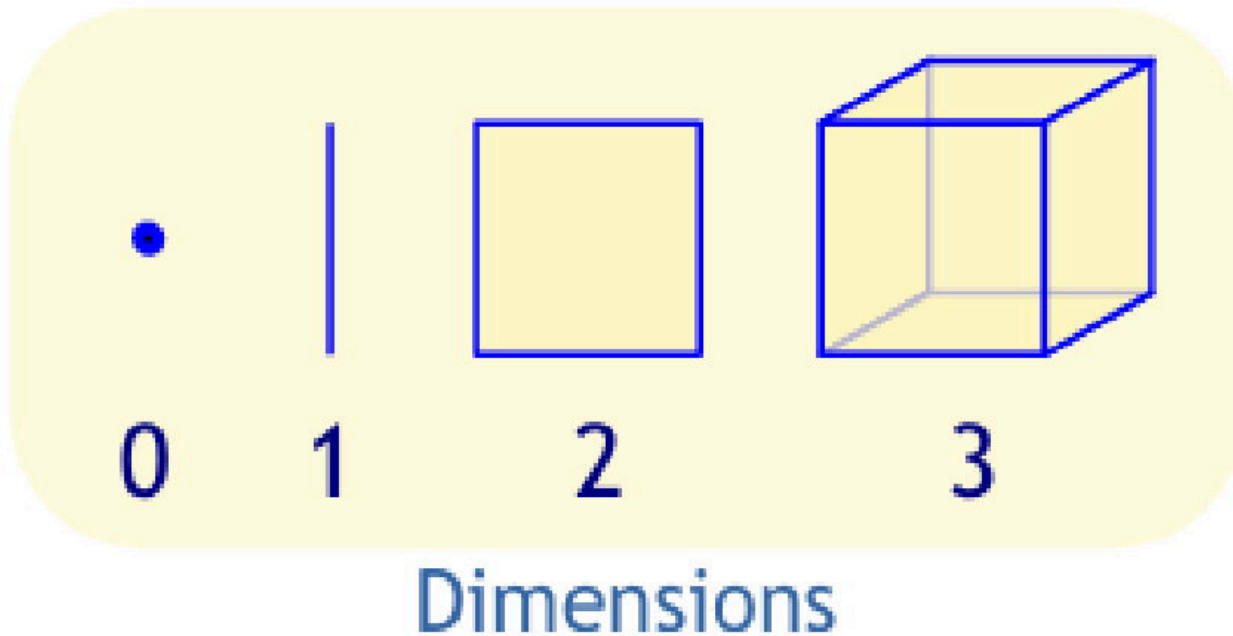
v^β (vector) tensor of rank 1

3) $T_{\mu\delta}$ (tensor) tensor of rank 2

Higher Dimensional Spaces

Dimensions:

The **dimensions** of a mathematical space (or object) is informally defined as the **minimum number of coordinates** needed to specify any point within it.



2-Dimensional space

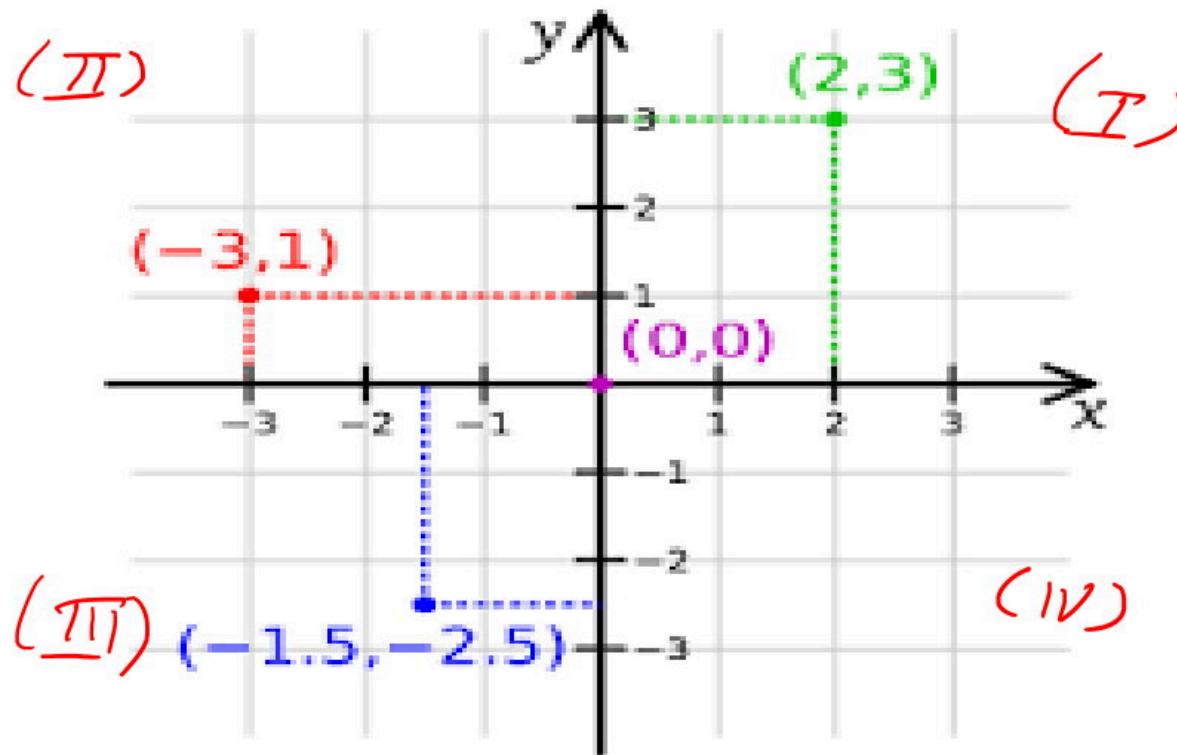
A coordinate system in which the exact position of a point is determined with two numerical values, is called the two dimensional space.

In Euclidean geometry, two perpendicular number-lines are intersected at their middle points for splitting the whole space into four equal parts. The number lines are used as base line to measure the distance of every point in two perpendicular directions from their bisection point.

2-Dimensional Space

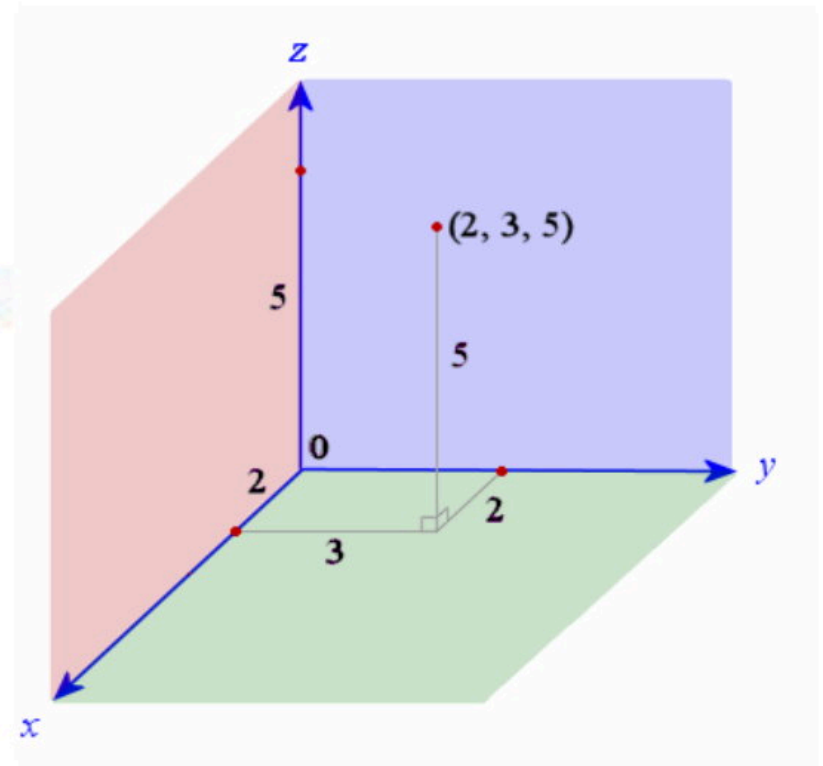
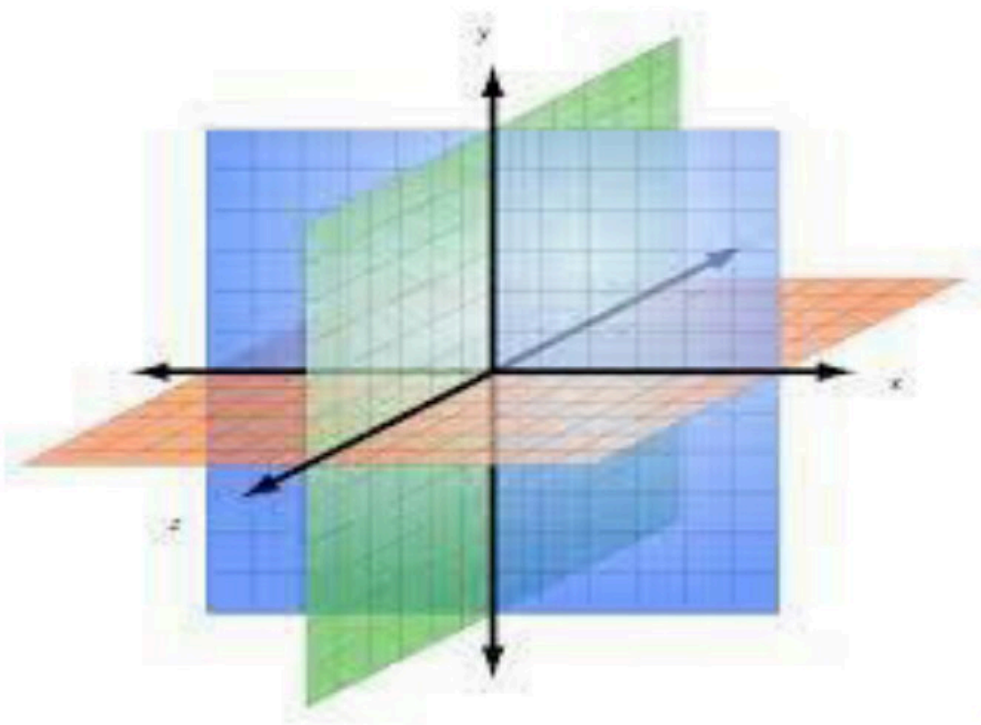
There are four fundamental concepts related to two dimensional Cartesian coordinate system:

Coordinate axes, origin, coordinates of a point and quadrants.



3-Dimensional space

We can expand our 2-dimensional (xy) coordinate system into a **3-dimensional coordinate system**, using x , y , and z -axes. In 3-dimensional space the exact position of a point is determined with three numerical values, called the coordinates.

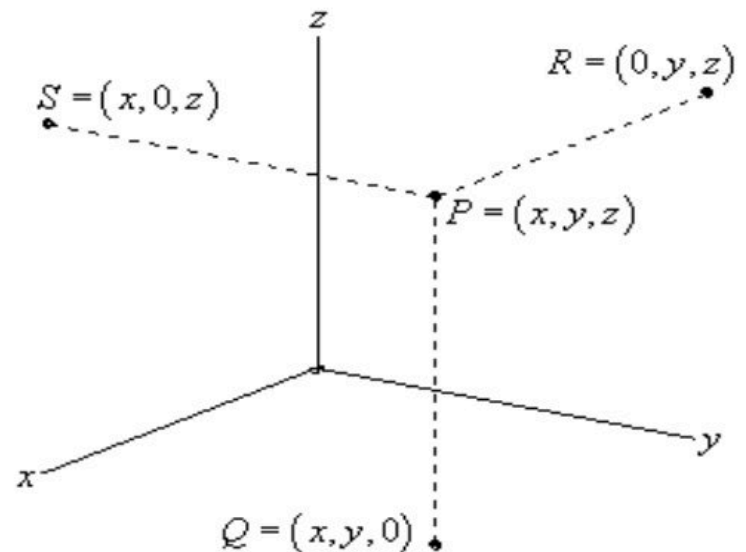
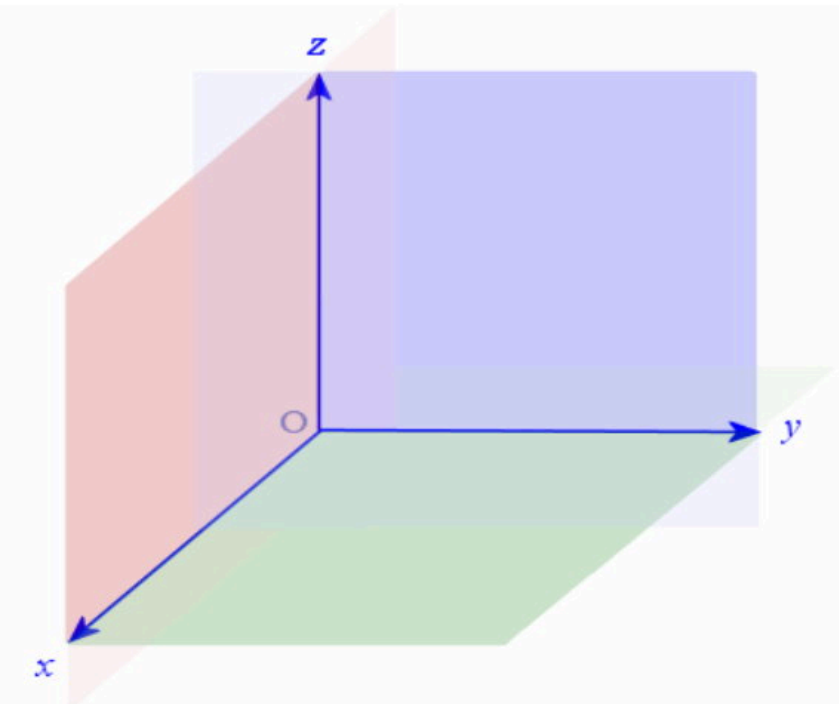


3-Dimensional space

The **xy-plane** is horizontal in diagram below and shaded green. It can also be described using the equation $z = 0$,

The **xz-plane** is vertical and shaded pink above. This plane can be described using the equation $y=0$.

The **yz-plane** is also vertical and shaded blue. The yz-plane can be described using the equation $x=0$.



Higher Dimensional Spaces

More than three dimensional spaces are treated as higher dimensional spaces.

For example, we can think of dimensions as ways in which we can move up and down, left and right, forward and backwards, which leads us to believe we live in a three-dimensional world. Every point in this world is defined by three coordinates. We can't imagine living in a four-dimensional world because we simply can't conceive of another direction to move in.

Higher Dimensional Spaces

However, now imagine you are arranging to meet someone. Not only do you need to give them the three coordinates of the place you are meeting, you also need to tell them the time you are meeting. Thus, time and space together form a four-dimensional space: you need four numbers to specify a point.

Generally, a point in an n -dimensional space is given by n numbers, or coordinates. Even though we can't visualize higher dimensional spaces and objects within them as geometric objects. Algebra gives the tools to deal with these objects.

Applications of Tensors

Applications of tensors

Tensors have a wide range of applications in different fields. Following are different fields of Mathematics and Physics where tensors are used.

- Mechanics

Stress tensor, Strain tensor, Elasticity tensor, Inertia tensor

- Electromagnetics

Maxwell equations and related tensors

- Special theory of relativity

Metric tensor, four vector formalism

- General theory of relativity

Riemannian tensor, Ricci tensor, Einstein tensor

Applications of tensors

- Following are some applications of tensors in **Engineering**
- To represent mechanical stress as the stress tensor in continuum mechanics
- To measure deformations through deformation tensor and strain via strain tensor in continuum mechanics
- To represent diffusion as a tensor in diffusion tensor imaging.
- To describe moment of inertia as inertia tensor.

Applications of tensors

Tensor Flow

Tensor flow is an open-source end-to-end platform for creating machine learning applications. It is a symbolic math library that uses dataflow and differentiable programming to perform various tasks focused on training and inference of deep neural networks.

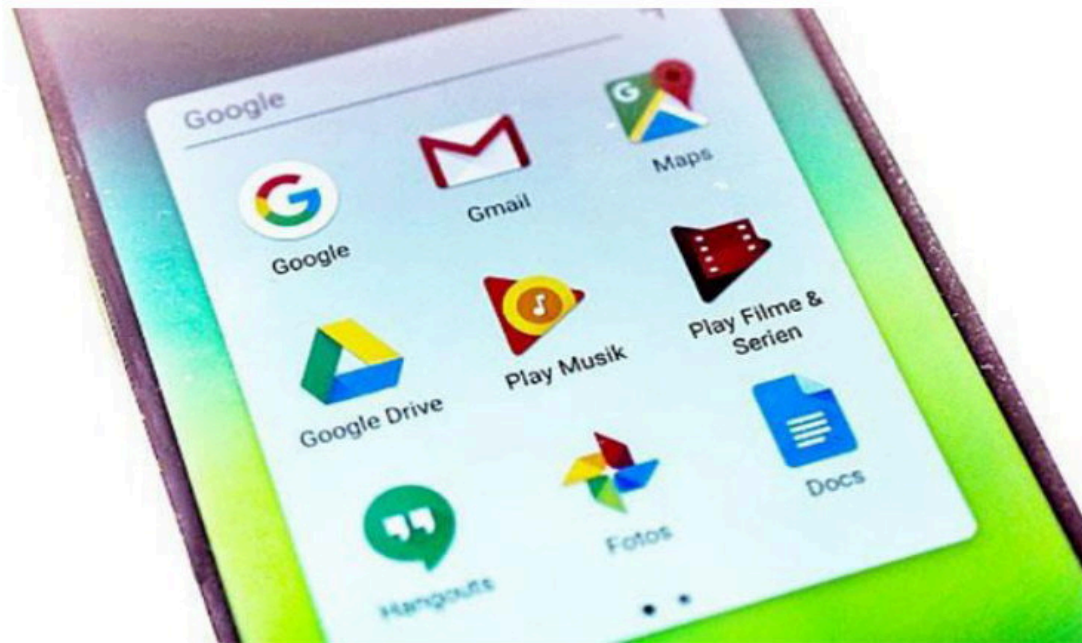


How Tensor Flow Works

Tensor Flow enables you to build dataflow graphs and structures to define how data moves through a graph by taking inputs as a multi-dimensional array called Tensor. It allows you to construct a flowchart of operations that can be performed on these inputs, which goes at one end and comes at the other end as output.

Applications of tensor flow

Currently, tensor flow is used in a variety of Google applications like Google photos, Google translate, Gmail, search engine, translation, image captioning or recommendations.



Free and Dummy Indices

Tensor Notation:

Tensor notation provides a convenient and unified system for describing physical quantities.

In the most general representation, a tensor is denoted by a symbol followed by a collection of subscripts (or superscripts). For example

$$X_i, \sigma_{ij}, U_l, B_{klm}$$

$$X^{\dot{i}} \delta^{\dot{j}}$$

Subscript and Superscript:

A subscript is a character (such as a number or letter) that is written slightly below and a superscript is written above the normal line of type. It is usually smaller than the rest of the text.

Subscripts appear **below** the baseline, while **superscripts** are **above**.

For example: A_i, u_j  **subscripts**
 B^k, C^{ij}  **superscripts**

These subscripts and superscripts are also called **indices**.

Indices:

Indices assume values from an index set. They successively assume each value from the given index set.

For example, if i takes the values from 1 to 4, then a_i denotes the set (a_1, a_2, a_3, a_4) and we say that the index i runs from 1 to 4.

Example:

$$\begin{array}{l} a_j; j = 1, 2, 3 \\ \neg b_k; k = 0, 1, 2, 3 \end{array} \quad a_1, a_2, a_3$$

$\hookrightarrow b_0, b_1, b_2, b_3$

Free and dummy Indices:

In the index notation, indices are categorized into two groups: free indices and dummy indices.

- An index which may take on any particular value from the index set is called a free index. For example

$$\begin{array}{l} A_i; i = 1, 2, 3 \\ \underline{B_{ij}}; i = 1, 2, 3 \text{ \& } j = 1, 2, 3 \end{array} \quad A_1, A_2, A_3$$

- A dummy index means summation. Dummy indices are repeated indices in an expression. For example

$$\begin{array}{l} \sum_i \underline{A_{ii}}; i = 1, 2, 3 \quad \rightarrow \quad A_{11} + A_{22} + A_{33} \\ \sum_i A_i B_{ij}; i = 1, 2, 3 \text{ \& } j = 1, 2, 3 \\ \quad \hookrightarrow A_1 B_{1j} + A_2 B_{2j} + A_3 B_{3j} \end{array}$$

Example 1:

1) Write down explicitly the equations represented

by the expression $y_i = \sum_{j=1}^3 a_{ij} x_j$ where $i = 1, 2, 3$ and $j = 1, 2, 3$

Solution: i) First differentiate the free and dummy indices in expression.

Here i is free index and j is dummy index.

ii) Then apply the sum over the dummy index.

$$y_i = a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3$$

iii) After that, put the values of free index one by one which yield a set of equations.

$$y_1 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3$$

$$y_2 = a_{21}x_1 + a_{22}x_2 + a_{23}x_3$$

$$y_3 = a_{31}x_1 + a_{32}x_2 + a_{33}x_3$$

Example 2:

Show that if $i = 1, 2, 3, \dots, n$ and $j = 1, 2, 3, \dots, n$ then $\sum_{i=1}^n a_i x^i = \sum_{j=1}^n a_j x^j$.

Solution: First consider

$$\text{L.H.S.} = \sum_{i=1}^n a_i x^i = a_1 x^1 + a_2 x^2 + a_3 x^3 + \dots + a_n x^n$$

$$\text{R.H.S.} = \sum_{j=1}^n a_j x^j = a_1 x^1 + a_2 x^2 + a_3 x^3 + \dots + a_n x^n$$

Note: So any dummy index can be replaced by other index ranging the same numbers.

Example 3: Consider the expression $T_{ij} = \sum_m A_{ijm} x_m + \sum_p B_{ip} C_{jp}$

- 1) Which are the free indices?
- 2) Which are the dummy indices?
- 3) How many terms are on the right hand side when m and p has variation 1,2.
- 4) How many equations are described by the given expression when i and j has variation 1,2?

1) $T_{ij} = \sum_m A_{ijm} x_m + \sum_p B_{ip} C_{jp}$ } i and j are free indices

2) $T_{ij} = \left(\sum_m A_{ijm} x_m \right) + \left(\sum_p B_{ip} C_{jp} \right)$

$m, p \rightarrow$ dummy indices

Example 3: Consider the expression $T_{ij} = \sum_m A_{ijm} x_m + \sum_p B_{ip} C_{jp}$

3) How many terms are on the right hand side when m and p has variation 1,2.

4) How many equations are described by the given expression when i and j has variation 1,2?

$$\begin{aligned} 3) \quad T_{ij} &= \sum_m A_{ijm} x_m + \sum_p B_{ip} C_{jp} \\ &= A_{ij1} x_1 + A_{ij2} x_2 \\ &\quad + B_{i1} C_{j1} + B_{i2} C_{j2} \end{aligned}$$

So, there are four terms
on R.H.S.

Example 3: Consider the expression $T_{ij} = \sum_m A_{ijm} x_m + \sum_p B_{ip} C_{jp}$

3) How many terms are on the right hand side when m and p has variation 1,2.

4) How many equations are described by the given expression when i and j has variation 1,2?

$$4) \quad T_{ij} = \sum_m A_{ijm} x_m + \sum_p B_{ip} C_{jp}$$

here $i = 1, 2$ and $j = 1, 2$ implies

$T_{1j} \Rightarrow T_{11}, T_{12} \Rightarrow 2$ equations

$T_{2j} \Rightarrow T_{21}, T_{22} \Rightarrow 2$ equations

$\Rightarrow$ The given expression describe a set of four equations.

Tensor Definition:

In simple words, tensors are simply arrays of numbers or functions that transform according to certain rules under a change of coordinates.

OR

In an m dimensional space, a tensor of rank n is a mathematical object that has n indices, m^n components and obey certain transformation rule.

Rank/order of a tensor

The rank of a tensor is the number of indices required to uniquely select each element of the tensor. Rank is also known as order, degree. In simple words, It is the total number of free indices in a tensor.

For example,

1) m (scalar) tensor of rank zero

2) v_α (vector) tensor of rank 1,

v^β (vector) tensor of rank 1

3) $T_{\mu\delta}$ (tensor) tensor of rank 2

4)

$A^\mu_{\nu\mu}$ (tensor) tensor of rank 1

Einstein Summation Convention

Einstein Summation Convention:

This convention was introduced with Einstein's words:

"I have made a great discovery in mathematics; I have suppressed the summation sign every time that the summation must be made over an index which occurs twice..."

Einstein summation is a notational convention for simplifying expressions including summations of vectors, matrices and tensors. Following are the rules of Einstein summation notation.

1. Repeated indices are summed over.

$$a_i a_i = \sum_{i=1}^n a_i a_i$$

$$a_{ik} a_{ij} = \sum_{i=1}^n a_{ik} a_{ij}$$

2. Each index can appear at most twice in any term.
3. Each term must contain identical non-repeated indices.

So,

$$M_{ij}v_j = \sum_j M_{ij}v_j$$

is a valid expression whereas

$$M_{ij}u_jv_j + w_i \quad \text{and} \quad T_{ijk}u_k + M_{ip}$$

are not valid expressions.

Question: Check which of the following is a valid expression.

1) $a_{ms} = b_m(c_r - d_r)$

not valid

2) $a_{ms} = b_m(c_s - d_s)$

valid

3) $t_i = a_{ji}n_j$

valid

4) $t_i = a_{ji}n_i$

not valid

5) $\sigma_{ij} = x_i x_{ij}$

not valid

6) $b_{ij}c_j = 3$

not valid

Example 1: Expand the summation, $c_i(x_i + y_i)$ for $n = 4$

$$\begin{aligned} c_i(x_i + y_i) &= c_1(x_1 + y_1) \\ &\quad + c_2(x_2 + y_2) \\ &\quad + c_3(x_3 + y_3) \\ &\quad + c_4(x_4 + y_4) \end{aligned}$$

Double Sums:

- An expression can involve more than one summation index.
- For example $a_{ij}x_iy_j$ indicates a summation taking place on both i and j simultaneously.
- If an expression has two summation indices varying from 1 to n , there will be a total of n^2 terms in the sum. If there are three summation indices, there will be n^3 terms in the sum and so on.

Example 2

Simplify $a_{ij}x_iy_j$, where $i = 1, 2, 3$ and $j = 1, 2, 3$.

Solution: First sum over i

$$a_{ij}x_iy_j = a_{1j}x_1y_j + a_{2j}x_2y_j + a_{3j}x_3y_j$$

Then sum over j

$$\begin{aligned} a_{ij}x_iy_j &= a_{11}x_1y_1 + a_{12}x_1y_2 + a_{13}x_1y_3 \\ &\quad + a_{21}x_2y_1 + a_{22}x_2y_2 + a_{23}x_2y_3 \\ &\quad + a_{31}x_3y_1 + a_{32}x_3y_2 + a_{33}x_3y_3 \end{aligned}$$

Example 3: Write down the equations for $n=2$ corresponding to the expression: $y_i = c_i^r a_{rs} x_s$

Solution:

$$y_i = c_i^r a_{rs} x_s \quad ; \quad n=2$$

r and s are dummy, i is free

and all i, r, s have values 1 and 2.

$$y_1 = c_1^r a_{rs} x_s = c_1^r a_{r1} x_1 + c_1^r a_{r2} x_2$$

$$y_1 = c_1^1 a_{11} x_1 + c_1^2 a_{21} x_1 + c_1^1 a_{12} x_2 + c_1^2 a_{22} x_2$$

Similarly:

$$y_2 = c_2^r a_{rs} x_s = c_2^1 a_{11} x_1 + c_2^2 a_{21} x_1 + c_2^1 a_{12} x_2 + c_2^2 a_{22} x_2$$

Substitutions:

Suppose it is required to substitute $y_i = a_{ij}x_j$ in the equation $Q = b_{ij}y_ix_j$ it would lead to $Q = b_{ij}a_{ij}x_jx_j$ which is an invalid indicial expression.

The correct method is that first we identify any **dummy indices in the expression to be substituted** that **coincide** with indices occurring in the main expression. Changing these dummy indices to characters not found in the main expression one can proceed substitution in the usual fashion.

Example 4

If $y_i = a_{ij}x_j$, express the quadratic form $Q = g_{ij}y_iy_j$ in terms of x -variable.

Solution:

First replace j with k in $y_i = a_{ij}x_j$ for y_i : $y_i = a_{ik}x_k$
and i with j replace j with s in $y_i = a_{ij}x_j$ for y_j : $y_j = a_{js}x_s$

Then by substitution it becomes

$$Q = g_{ij}a_{ik}x_k a_{js}x_s = g_{ij}a_{ik}a_{js}x_kx_s$$

$$Q = h_{rs}x_kx_s; \quad h_{rs} = g_{ij}a_{ik}a_{js}$$

Example 5: Express $b^{ij}y_i y_j$ in terms of x variable, if

$$y_i = c_{ij}x_j \text{ and } b^{ij}c_{ik} = \delta_k^j.$$

Solution:

consider $b^{ij} \underline{y_i} \underline{y_j}$
 $\underline{y_i} = c_{ir} x_r$ and $\underline{y_j} = c_{js} x_s$

$$\begin{aligned} \Rightarrow b^{ij} y_i y_j &= b^{ij} c_{ir} x_r c_{js} x_s \\ &= b^{ij} c_{ir} x_r c_{js} x_s \\ &= \delta_r^j x_r c_{js} x_s = x_j c_{js} x_s \\ &= c_{js} x_j x_s = c_{ij} x_i x_j \end{aligned}$$

$$\Rightarrow b^{ij} y_i y_j = c_{ij} x_i x_j$$

Kronecker Delta Symbol

Kronecker Delta

The Kronecker delta is defined as

$$\delta_{ij} = \delta_i^j = \delta^{ij} = \begin{cases} 1; & i = j \\ 0; & i \neq j \end{cases}$$

Clearly $\delta_{ij} = \delta_{ji}$ for all i, j .

The Kronecker- δ can be thought of as the identity matrix, for example for $i = 1, 2, 3$ and $j = 1, 2, 3$

$$\delta_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example: If $i = 1, 2, 3$ and $j = 1, 2, 3$ simplify $\delta_{ij}x_i x_j$.

Solution:

$$\begin{aligned}\delta_{ij}x_i x_j &= \delta_{1j}x_1 x_j + \delta_{2j}x_2 x_j + \delta_{3j}x_3 x_j \\&= \delta_{11}x_1 x_1 + \delta_{12}x_1 x_2 + \delta_{13}x_1 x_3 \\&\quad + \delta_{21}x_2 x_1 + \delta_{22}x_2 x_2 + \delta_{23}x_2 x_3 \\&\quad + \delta_{31}x_3 x_1 + \delta_{32}x_3 x_2 + \delta_{33}x_3 x_3 \\&= (1)x_1 x_1 + (0)x_1 x_2 + (0)x_1 x_3 \\&\quad + (0)x_2 x_1 + (1)x_2 x_2 + (0)x_2 x_3 \\&\quad + (0)x_3 x_1 + (0)x_3 x_2 + (1)x_3 x_3\end{aligned}$$

$$\delta_{ij}x_i x_j = x_1 x_1 + x_2 x_2 + x_3 x_3 = x_i x_i$$

$$\text{So, } \delta_{ij}x_i x_j = x_i x_i \quad \text{OR} \quad \delta_{ij}x_i x_j = x_j x_j$$

Hence in general

$$\delta_{ij} x_i x_j = x_i x_i$$

$$\delta_j^r a_{ir} x_i = a_{ij} x_i$$

Kronecker delta is a much used symbol in tensor calculus. It has the effect of annihilating the off diagonal terms.

$$\text{Also, } \delta_{ik} \delta_{kj} = \delta_{ij}$$

$$\delta_{ik} \delta_{kj} = \delta_{i1} \delta_{1j} + \delta_{i2} \delta_{2j} + \delta_{i3} \delta_{3j}$$

$$\delta_{1k} \delta_{1k} = \delta_{11} \delta_{11} + 0 + 0 = \delta_{11} = 1$$

$$\delta_{2k} \delta_{2k} = \delta_{22} = 1$$

$$\delta_{3k} \delta_{3k} = \delta_{33} = 1$$

Example: For arbitrary n , evaluate δ_{ii} .

Solution:

Consider δ_{ii} ; $i = 1, 2, \dots, n$.

i is dummy.

$$\begin{aligned}\delta_{ii} &= \delta_{11} + \delta_{22} + \delta_{33} + \dots + \delta_{nn} \\ &= \underbrace{1 + 1 + 1 + \dots + 1}_{n \text{ times}}\end{aligned}$$

$$\boxed{\delta_{ii} = n}$$

Example: For $n = 3$, evaluate $\delta_{ij} \delta_{ij}$.

Solution:

Consider $\delta_{ij} \delta_{ij} \rightarrow \begin{cases} i \text{ and } j \\ \text{dummy indices.} \end{cases}$

$$\delta_{ij} \delta_{ij} = \boxed{\delta_{1j} \delta_{1j}} + \boxed{\delta_{2j} \delta_{2j}} + \boxed{\delta_{3j} \delta_{3j}}$$

$$= \delta_{11} \delta_{11} + \delta_{12} \delta_{12} + \delta_{13} \delta_{13}$$

$$+ \delta_{21} \delta_{21} + \delta_{22} \delta_{22} + \delta_{23} \delta_{23}$$

$$+ \delta_{31} \delta_{31} + \delta_{32} \delta_{32} + \delta_{33} \delta_{33}$$

$$= 1 + 1 + 1 = 3$$

$$\Rightarrow \delta_{ij} \delta_{ij} = 3$$

Alternatively:

$$\underline{\delta_{ij} \delta_{ij} = \delta_{ii}}$$

and $\delta_{ii} = 3$
if $n = 3$ and

$$n, \quad \delta_{ii} = n$$

$$\delta_{ik} \delta_{kj} = \delta_{ij}$$

$$\delta_{ik} \delta_{jk} = \delta_{ij}$$

$$\delta_{ik} \delta_{ik} = \delta_{ii}$$

$$\delta_{ij} \delta_{ij} = \delta_{ii}$$

for arbitrary

Example: For arbitrary n , evaluate $\delta_{ij} \delta_k^j c_{ik}$.

Solution: In $\delta_{ij} \delta_k^j c_{ik}$ all i, j, k are dummy.

Recall first example: $(\delta_{ij} x_i x_j = x_i x_i)$

$$\delta_{ij} \delta_k^j c_{ik} = \delta_{ij} c_{ij}$$

Again keeping in mind property of Kronecker delta:

$$\delta_{ij} \delta_k^j c_{ik} = c_{ii}$$

Permutation symbol

The Permutation Symbol or Levi-Civita Symbol

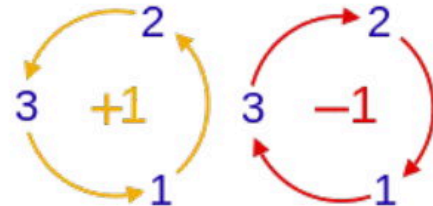
In three dimensions, the permutation symbol is a three index object also called the Levi-Civita symbol and alternating tensor. It is defined by

$$\varepsilon_{ijk} = \begin{cases} 0 & \text{for } i = j, j = k \text{ or } k = i \\ 1 & \text{for } (i, j, k) \in \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\} \\ -1 & \text{for } (i, j, k) \in \{(1, 3, 2), (3, 2, 1), (2, 1, 3)\} \end{cases}$$

OR

$$\varepsilon_{ijk} = \frac{1}{2}(i-j)(j-k)(k-i)$$

For the indices (i, j, k) in ε_{ijk} , the values 1, 2, 3 occurring in the cyclic order (1, 2, 3) correspond to $\varepsilon = +1$, while occurring in the reverse cyclic order correspond to $\varepsilon = -1$, otherwise $\varepsilon = 0$.



Levi-Civita Symbol

Example: The permutation symbol satisfies :

$$i) \delta_{ij} \varepsilon_{ijk} = 0$$

$$ii) \varepsilon_{ipq} \varepsilon_{jpq} = 2\delta_{ij}$$

$$iii) \varepsilon_{ijk} \varepsilon_{ijk} = 6$$

where δ_{ij} is the Kronecker delta.

$$i) \delta_{ij} \varepsilon_{ijk} = 0$$

$$\delta_{ij} \varepsilon_{ijk} = \varepsilon_{iik} = 0 \text{ (by definition of } \varepsilon_{ijk} \text{)}$$

Alternatively: let $n=3$, i.e. $i=1,2,3; j=1,2,3$
and $k=1,2,3$.

$$\begin{aligned} \delta_{ij} \varepsilon_{ijk} &= \delta_{1j} \varepsilon_{1jk} + \delta_{2j} \varepsilon_{2jk} + \delta_{3j} \varepsilon_{3jk} \\ &= \delta_{11} \varepsilon_{11k} + \delta_{12} \varepsilon_{12k} + \delta_{13} \varepsilon_{13k} \\ &\quad + \delta_{21} \varepsilon_{21k} + \delta_{22} \varepsilon_{22k} + \delta_{23} \varepsilon_{23k} \\ &\quad + \delta_{31} \varepsilon_{31k} + \delta_{32} \varepsilon_{32k} + \delta_{33} \varepsilon_{33k} \\ &= 0 \end{aligned}$$

$$ii) \epsilon_{ipq} \epsilon_{jpq} = 2\delta_{ij}$$

$\epsilon_{ipq} \epsilon_{jpq} \rightarrow i$ and j are free indices
 p and q are dummy indices.

Here, $n=3$. Then

$$\begin{aligned} \epsilon_{ipq} \epsilon_{jpq} &= \epsilon_{i1q} \epsilon_{j1q} + \epsilon_{i2q} \epsilon_{j2q} + \epsilon_{i3q} \epsilon_{j3q} \\ &= \epsilon_{i11} \epsilon_{j11} + \epsilon_{i12} \epsilon_{j12} + \epsilon_{i13} \epsilon_{j13} \\ &\quad + \epsilon_{i21} \epsilon_{j21} + \epsilon_{i22} \epsilon_{j22} + \epsilon_{i23} \epsilon_{j23} \\ &\quad + \epsilon_{i31} \epsilon_{j31} + \epsilon_{i32} \epsilon_{j32} + \epsilon_{i33} \epsilon_{j33} \\ &\stackrel{i=j=3}{=} \underbrace{\epsilon_{i12} \epsilon_{j12}}_{i=j=2} + \underbrace{\epsilon_{i13} \epsilon_{j13}}_{i=j=2} + \underbrace{\epsilon_{i21} \epsilon_{j21}}_{i=j=1} + \underbrace{\epsilon_{i23} \epsilon_{j23}}_{i=j=1} \\ &\quad + \underbrace{\epsilon_{i31} \epsilon_{j31}}_{i=j=2} + \underbrace{\epsilon_{i32} \epsilon_{j32}}_{i=j=2} \end{aligned}$$

$$\varepsilon_{ipq} \varepsilon_{j pq} =$$

$$= \overbrace{\varepsilon_{i12} \varepsilon_{j12}}^{i=j=3} + \overbrace{\varepsilon_{i13} \varepsilon_{j13}}^{i=j=2} + \overbrace{\varepsilon_{i21} \varepsilon_{j21}}^{i=j=3} + \overbrace{\varepsilon_{i23} \varepsilon_{j23}}^{i=j=1} + \underbrace{\varepsilon_{i31} \varepsilon_{j31}}_{i=j=2} + \underbrace{\varepsilon_{i32} \varepsilon_{j32}}_{i=j=1}$$

But i and j are free indices, so in one equation they can assume one value:

$$\begin{array}{l|l|l} \varepsilon_{1pq} \varepsilon_{1pq} = 2 & \varepsilon_{1pq} \varepsilon_{2pq} = 0 & \varepsilon_{1pq} \varepsilon_{3pq} = 0 \\ \varepsilon_{2pq} \varepsilon_{1pq} = 0 & \varepsilon_{2pq} \varepsilon_{2pq} = 2 & \varepsilon_{2pq} \varepsilon_{3pq} = 0 \\ \varepsilon_{3pq} \varepsilon_{1pq} = 0 & \varepsilon_{3pq} \varepsilon_{2pq} = 0 & \varepsilon_{3pq} \varepsilon_{3pq} = 2 \end{array}$$

$$iii) \varepsilon_{ijk} \varepsilon_{ijk} = 6$$

all indices are dummy.

$$\begin{aligned}
 \varepsilon_{ijk} \varepsilon_{ijk} &= \varepsilon_{1jk} \varepsilon_{1jk} + \varepsilon_{2jk} \varepsilon_{2jk} + \varepsilon_{3jk} \varepsilon_{3jk} \\
 &= \varepsilon_{12k} \varepsilon_{12k} + \varepsilon_{13k} \varepsilon_{13k} + \varepsilon_{21k} \varepsilon_{21k} + \varepsilon_{23k} \varepsilon_{23k} \\
 &\quad + \varepsilon_{31k} \varepsilon_{31k} + \varepsilon_{32k} \varepsilon_{32k} \\
 &= \varepsilon_{123} \varepsilon_{123} + \varepsilon_{132} \varepsilon_{132} + \varepsilon_{213} \varepsilon_{213} \\
 &\quad + \varepsilon_{231} \varepsilon_{231} + \varepsilon_{312} \varepsilon_{312} + \varepsilon_{321} \varepsilon_{321} \\
 &= (1)^2 + (-1)^2 + (-1)^2 + (1)^2 + (1)^2 + (-1)^2 \\
 &= 1 + 1 + 1 + 1 + 1 + 1
 \end{aligned}$$

$$\varepsilon_{ijk} \varepsilon_{ijk} = 6$$

Tensor Notation for Matrices

Tensor Notation for Matrices

In the matrix notation $[a_{ij}]_{mn}$, the first subscript i , tells what row the number a_{ij} lies in and the second j designates the column, m denotes the number of rows and n the number of columns.

Examples:

$$C = [c_{ij}]_{33} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$

$$A = [a_{ij}]_{mn}$$

a_{11} a_{32}

$$D = [d_{ij}]_{24} = \begin{bmatrix} d_{11} & d_{12} & d_{13} & d_{14} \\ d_{21} & d_{22} & d_{23} & d_{24} \end{bmatrix}$$

Basic Formulas:

The essential formulas involving matrices, vectors and determinants will be written in terms of summation convention.

▪ Matrix Multiplication

If $A = [a_{ij}]_{mn}$ and $B = [b_{ij}]_{nk}$, then

$$AB = [a_{ir}b_{rj}]_{mk}$$

▪ Identity Matrix

In terms of the kronecker delta, the identity matrix of order n is

$$I = [\delta_{ij}]_{nn}$$

Which has the property

$$A = AI = IA = [a_{ij}\delta_{ji}]_{nn}$$

For any square matrix of order n .

— Recall that

$$\delta_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

▪ Inverse of a square matrix

A square matrix $A = [a_{ij}]_{nn}$ is invertible if there exists a unique matrix $B = [b_{ij}]_{nn}$ called the inverse of A , such that

$$AB = BA = I$$

In terms of components:

$$a_{ir}b_{rj} = b_{ir}a_{rj} = \delta_{ij}$$

- **Transpose of a Matrix**

The transpose of a matrix in index notation is defined as

$$A^T = [a_{ij}]_{mn}^T = [a_{ij}^t]_{nm}, \text{ where } a_{ij}^t = a_{ji}$$

A matrix A is orthogonal if $A^T = A^{-1}$

Example: Display explicitly the matrices

(a) $[b_{ij}]_{42}$

(b) $[h_{ij}]_{33}$

a) $[b_{ij}]_{42} =$

rows columns

$$\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \\ b_{41} & b_{42} \end{bmatrix}$$

b) $[h_{ij}]_{33} =$

$$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$$

Example: Prove by using tensor notation and product rule for matrices that $(AB)^T = B^T A^T$.

$$\text{Let } A = [a_{ij}]_{mn}, \quad B = [b_{ij}]_{nk}$$

$$AB = [c_{ij}]_{mk} \quad \text{and for all } i \text{ and } j$$

$$a_{ij}^t = a_{ji}, \quad b_{ij}^t = b_{ji}, \quad c_{ij}^t = c_{ji}$$

$$\text{Now, } A^T = [a_{ij}^t]_{nm}, \quad B^T = [b_{ij}^t]_{kn} \quad \text{and}$$

$$(AB)^T = [c_{ij}^t]_{km}$$

By definition of matrix product:

$$B^T A^T = [b_{ir}^t a_{rj}^t]_{km} = [b_{ri} a_{jr}]_{km} = [c_{ij}^t]_{km}$$
$$b_{ri} a_{jr} = a_{jr} b_{ri} = c_{ji} = c_{ij}^t = (AB)^T$$

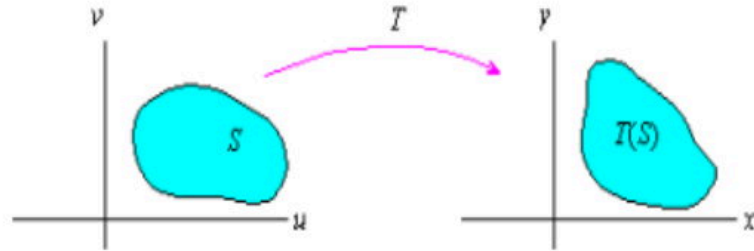
Coordinate Transformations

2-Dimensional Coordinate Transformations

A two dimensional coordinate transformation is a mapping of the form

$$T(u, v) = (x(u, v), y(u, v))$$

The transformation T maps a set S in the uv -plane to a set $T(S)$ in the xy -plane.



Example:

If P has the coordinates $(1,2)$ in uv -plane then find the coordinates in xy -plane if: $x = u + v$ and $y = 2u - v$.

Solution: $P: (u, v) = (1, 2)$
using the transformation:

$$x = u + v \quad \text{and} \quad y = 2u - v$$

$$\Rightarrow x = 1 + 2 = 3, \quad y = 2(1) - 2$$
$$y = 2 - 2 = 0$$

So, coordinates of P in xy -plane
are: $(3, 0)$

Example:

Transform the line PQ where P has coordinates $(1, -1)$ and Q has coordinates $(0, 1)$ from uv -plane to xy -plane if: $x = u + v$ and $y = 2u - v$.

Solution: To transform the line we find the new coordinates of both its end points.

$$P(u, v) = (1, -1),$$

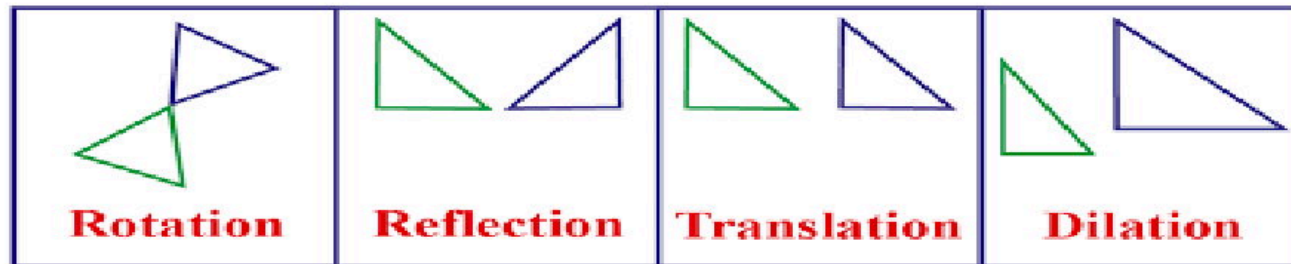
$$P(x, y) = (u + v, 2u - v) \\ = (0, 2 + 1)$$

$$P(x, y) = (0, 3)$$

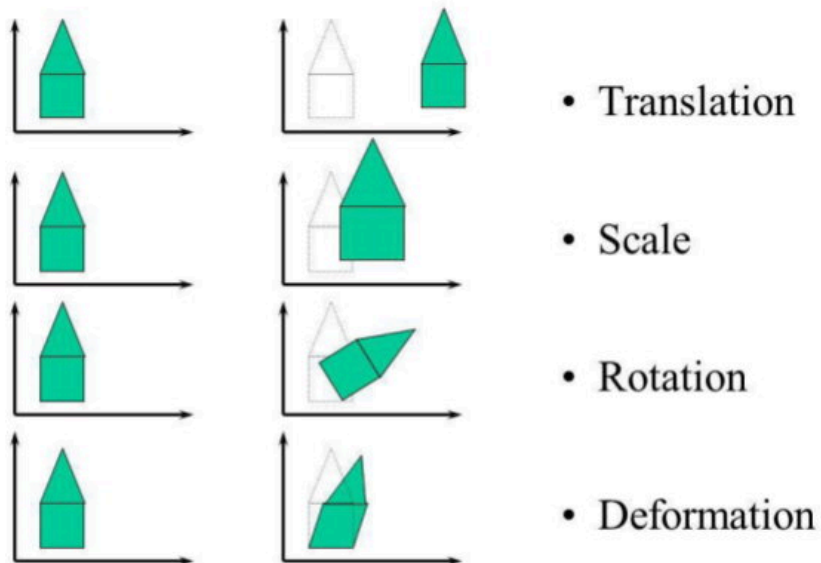
$$\left. \begin{array}{l} Q(u, v) = (0, 1) \\ Q(x, y) = (u + v, 2u - v) \\ Q(x, y) = (1, -1) \end{array} \right\}$$

$\Rightarrow$ The line PQ is transformed to a line joining $P(0, 3)$ and $Q(1, -1)$ in xy -plane.

Some Well-Known Transformations



2D Transformations

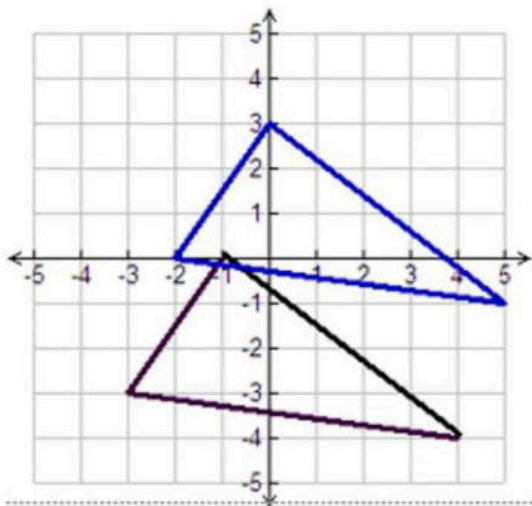


Coordinate Transformations

Translation

A translation slides an object a fixed distance in a given direction. The original object and its translation have the same shape and size, and they face in the same direction. To translate a shape every point of the shape must move the same distance and in the same direction.

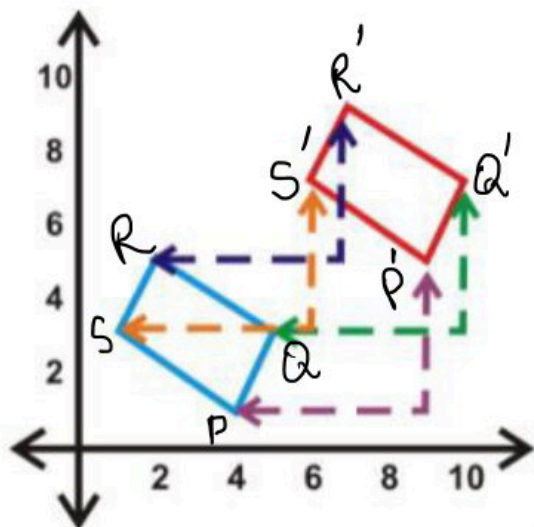
Translation



For example the transformation

$$(x, y) \rightarrow (x + 5, y + 3)$$

moves the parallelogram to the right 5 units and up 3 units.



$$P(x, y) = (4, 1)$$

$$Q(x, y) = (5, 3)$$

$$R(x, y) = (2, 5)$$

$$S(x, y) = (1, 3)$$

after translation:

$$P' = (9, 4)$$

$$Q' = (10, 6)$$

$$R' = (7, 8)$$

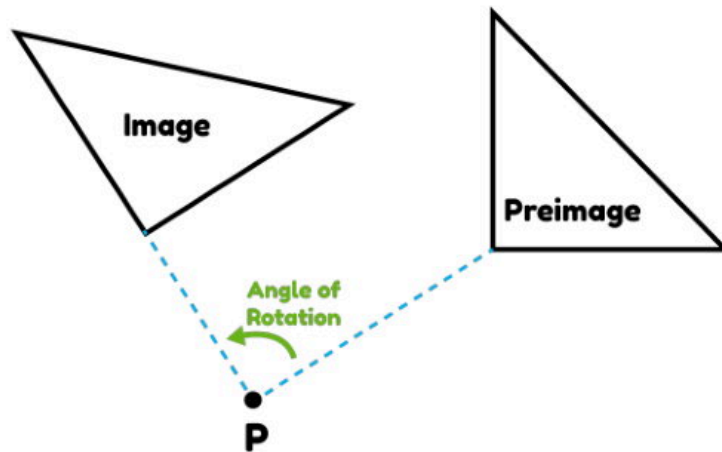
$$S' = (6, 6)$$

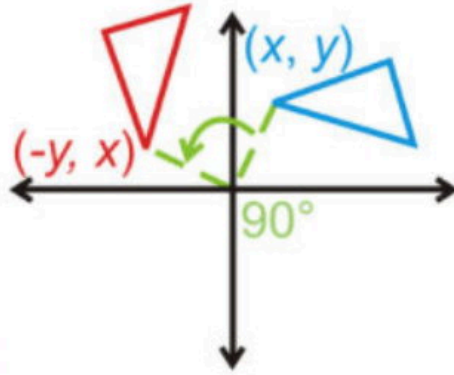
Rotation

A rotation is a transformation where a figure is turned around a fixed point to create an image. The two dimensions rotation matrix is

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

To perform the rotation on a plane point with coordinates (x, y) , it should be written as a column vector, and multiplied by the matrix R :



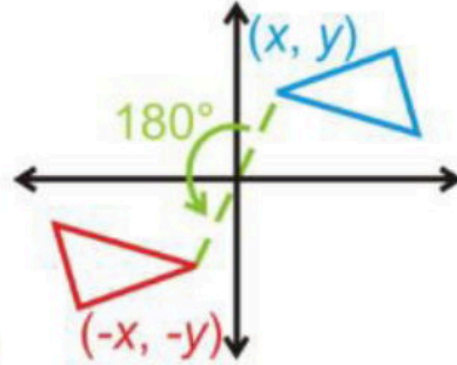


Rotation of 90° : $(x, y) \rightarrow (-y, x)$

Here $\theta = 90^\circ$, So

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ = \begin{bmatrix} 0 - y \\ x + 0 \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix}$$

$\Rightarrow$ under 90° rotation a point (x, y) is transformed to $(-y, x)$



Rotation of 180° : $(x, y) \rightarrow (-x, -y)$

for $\theta = 180^\circ$, rotation matrix becomes:

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x + 0 \\ 0 - y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix}$$

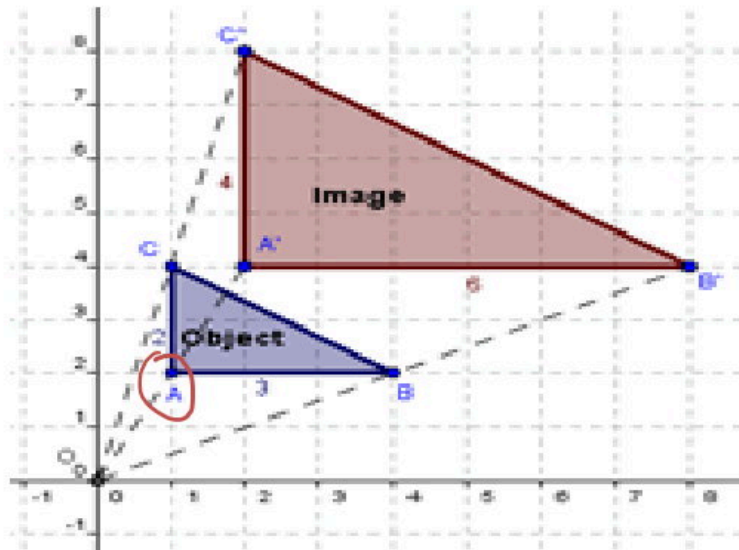
Implies under 180° rotation a point
 $(x, y) \rightarrow (-x, -y)$

Scaling

To change the size of an object, scaling transformation is used. In the scaling process, you either expand or compress the dimensions of the object. Scaling can be achieved by multiplying the original coordinates of the object with the scaling factor to get the desired result.

$$\begin{bmatrix} X_{\text{new}} \\ Y_{\text{new}} \end{bmatrix} = \begin{bmatrix} S_x & 0 \\ 0 & S_y \end{bmatrix} \times \begin{bmatrix} X_{\text{old}} \\ Y_{\text{old}} \end{bmatrix}$$

Scaling Matrix



consider point A:

A: (1, 2)

and Scaling matrix:

$$\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$$

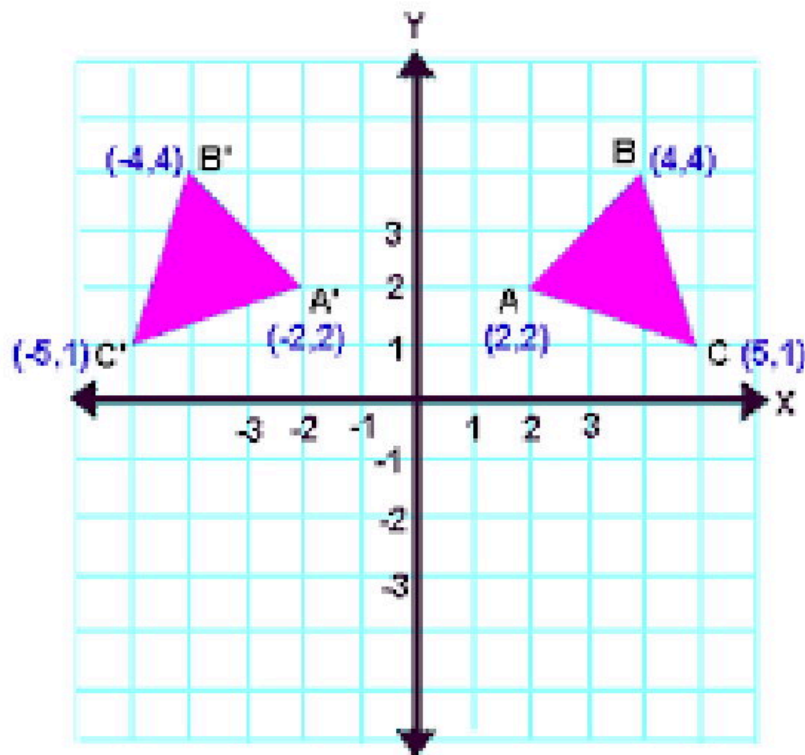
Then:

$$\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2+0 \\ 0+8 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \end{bmatrix}$$

$\Rightarrow A_{\text{new}}: (2, 8)$

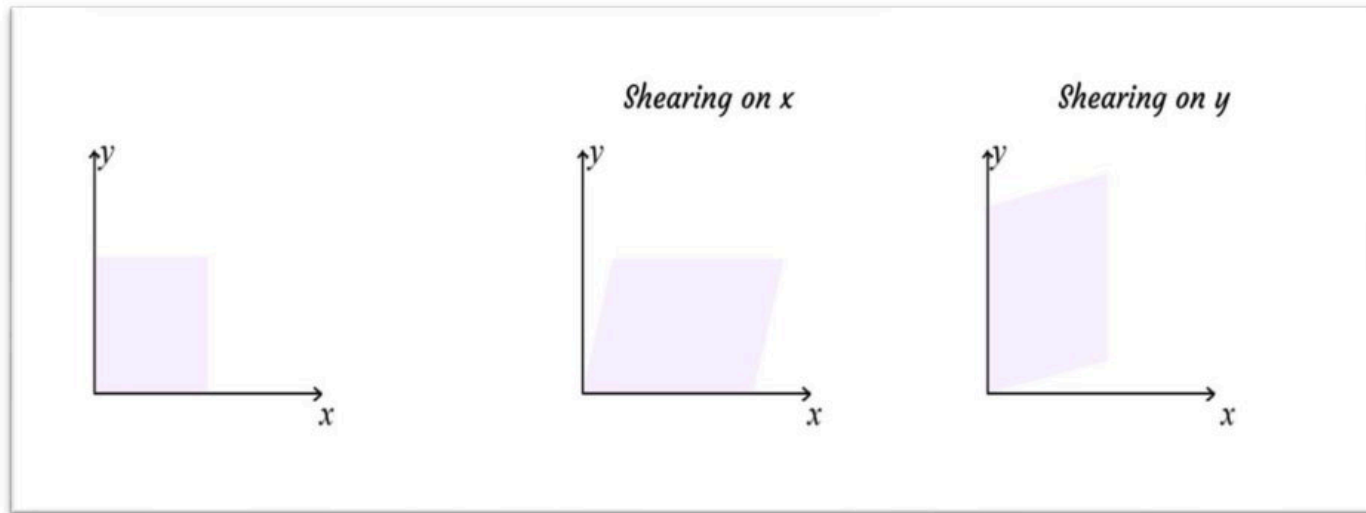
Reflection

A reflection is a transformation that turns a figure into its mirror image by flipping it over a line. Reflection is the mirror image of original object. In other words, we can say that it is a rotation operation with 180° . In reflection transformation, the size of the object does not change.



Shear

A transformation that slants the shape of an object is called the shear transformation. There are two shear transformations X-Shear and Y-Shear. One shifts X coordinate values and other shifts Y coordinate values. However; in both the cases only one coordinate changes its coordinates and other preserves its values. Shearing is also termed as Skewing.



A shear parallel to x-axis results in $x' = x + \alpha y$ and $y' = y$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Similarly a shear parallel to y-axis results in $x' = x$ and $y' = y + \alpha x$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ \alpha & 1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

If $A: (1, 2)$ and $x' = x + 2y$, $y' = y$
then $x' = 1 + 2(2) = 5$ and $y' = 2$
 $\Rightarrow (1, 2) \rightarrow (5, 2)$

Applications of Coordinate Transformations

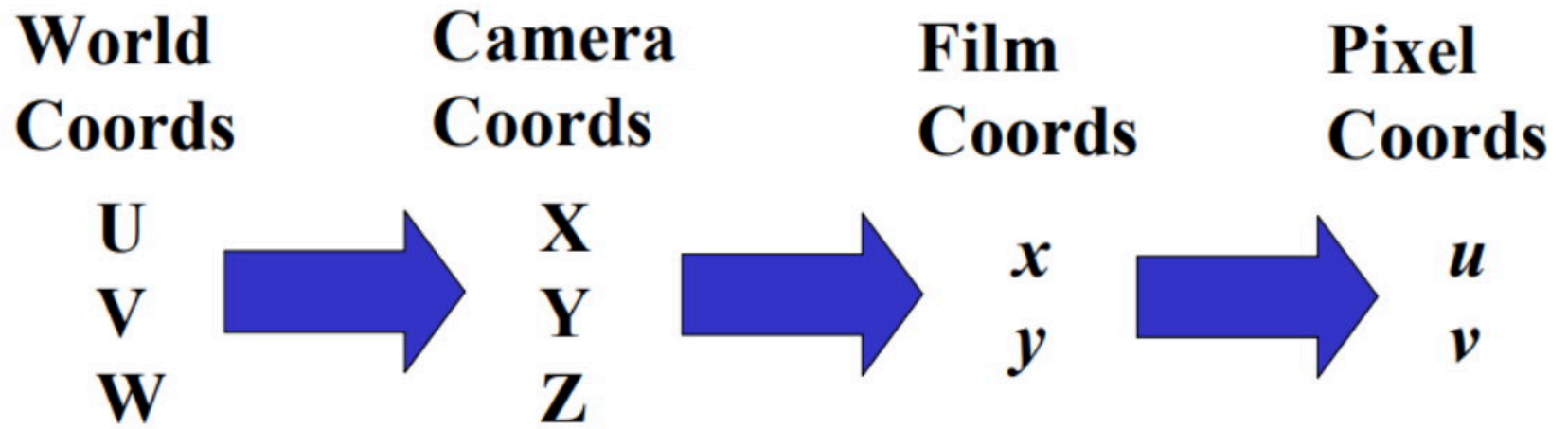
Applications of coordinate transformations

Transformations can represent geometric operations, which are caused by movement or action, as well as changes of coordinates, which are caused by changes of interpretation. Many common spatial transformations, including translations, rotations, and scaling are represented by matrix operations.

Here we will discuss some applications of transformations.

- Camera image processing
- Robotics
- Computer graphics

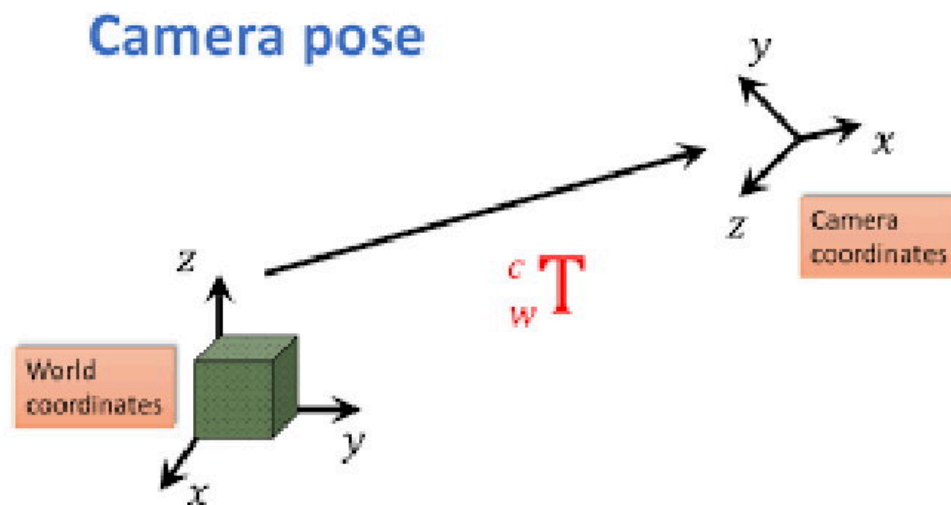
Camera image processing



Consider the three coordinate frames:

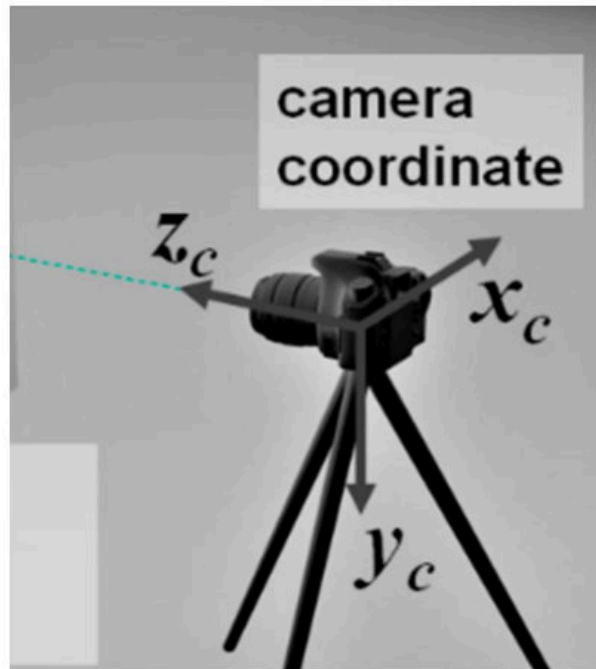
- **World coordinate frame:**

These are 3D coordinates fixed in the world, say with respect to one corner of the room.

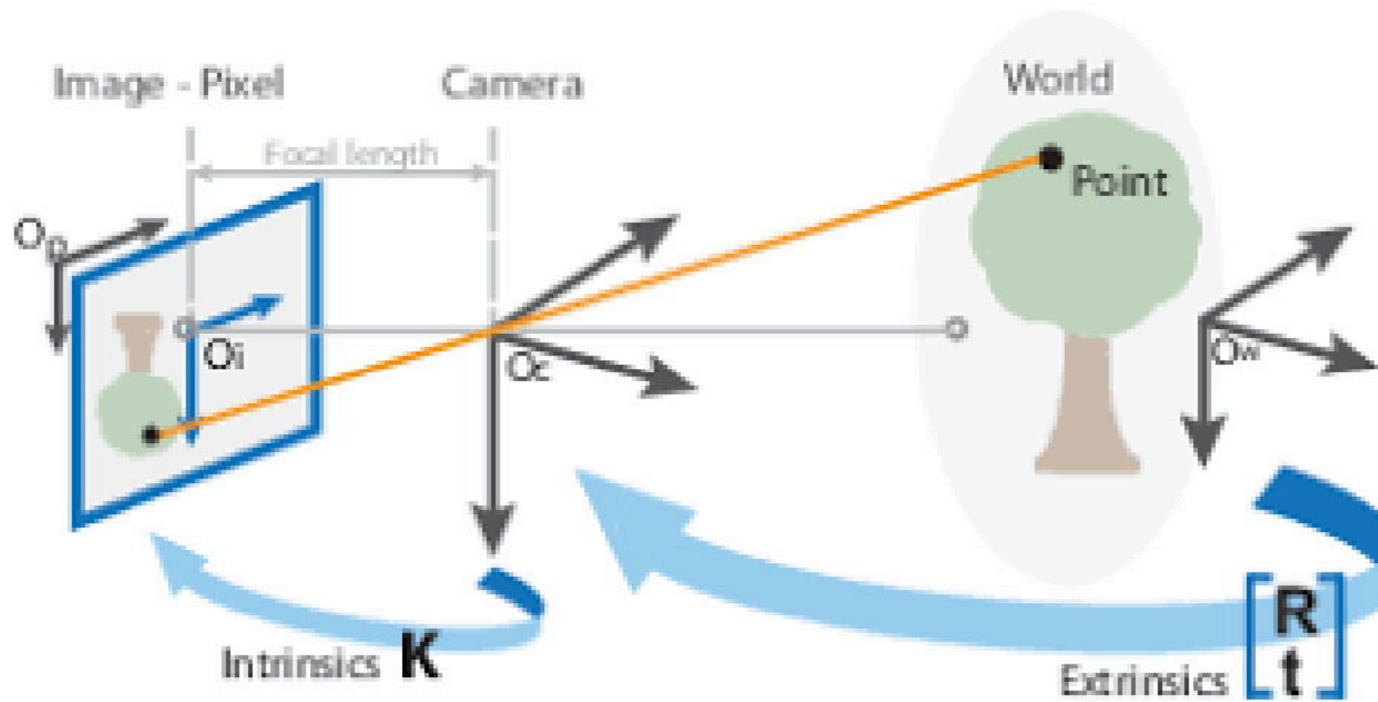


- **Camera coordinate frame:**

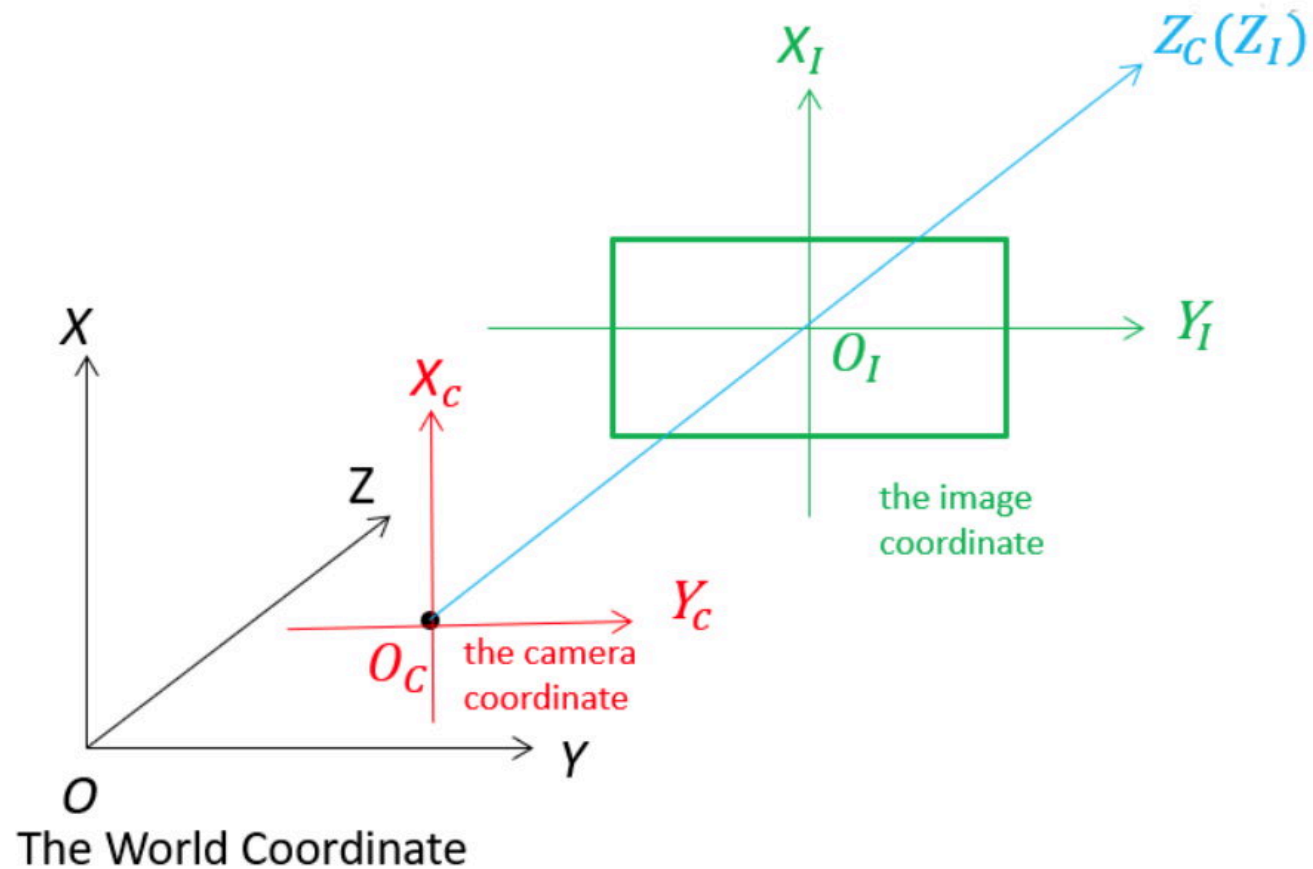
These are 3D coordinates fixed in the camera. The origin of the camera coordinates is at the center of projection of the camera. The z-axis is taken to be the optical axis of the camera (with points in front of the camera in the positive z direction).



- Image coordinate frame:

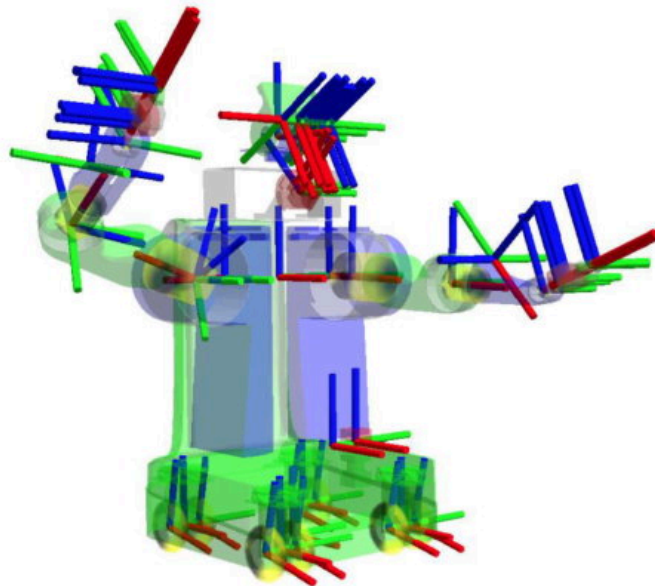


- **Coordinate transformations:**



Robotics:

For the purpose of controlling a robot, it is important to know the kinematics where transformations between the coordinate frames are performed. These frames are attached to different robot links of the robot. Vectors and matrices are used in robotics to represent 2D and 3D positions, directions, rigid body motion, and coordinate transformations.



Computer Graphics:

Computer graphics deals with generating images with the aid of computers. Today, computer graphics is a core technology in digital photography, film, video games, cell phone and computer displays, and many specialized applications.

A transformation is used to change the object's

- Position (translation)
- Size (scaling)
- Orientation (rotation)
- Shapes (shear)

Coordinate transformations

Position vector and point coordinates:

We will discuss the coordinates and position vector of a point in a three dimensional Cartesian coordinate system. The unit vector

$\hat{i}, \hat{j}, \hat{k}$ along x, y, z axes are represented by $\hat{e}_1, \hat{e}_2, \hat{e}_3$ in tensor notation or $\hat{e}_i, i=1, 2, 3$.

Similarly, the coordinates of a point (x, y, z) are represented as:

Coordinates of $P: (x, y, z) = (x_1, x_2, x_3) = x_i$; $i = 1, 2, 3$

and the position vector is represented as:

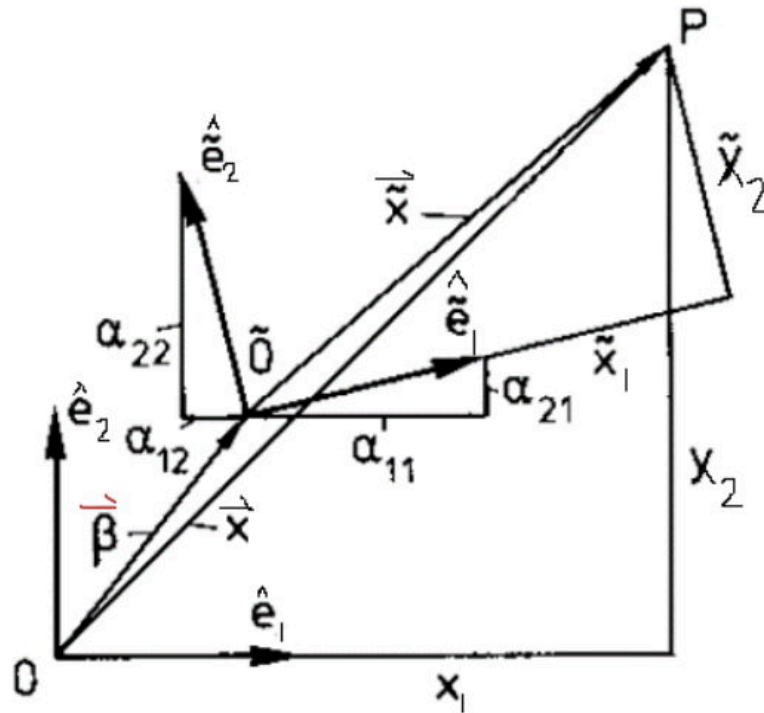
$$\vec{x} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{x} = x_1\hat{e}_1 + x_2\hat{e}_2 + x_3\hat{e}_3$$

$$\vec{x} = x_i\hat{e}_i$$

Transformation of Cartesian Coordinate Systems

Consider two Cartesian coordinate systems, in an arbitrary orientation relative to each other. We label one system by a tilde, to distinguish one from another, and call it the tilde coordinate system. The relative position of two coordinate systems in two dimensions is given in figure.



The origin of the tilde coordinate system has the coordinates w.r.t non-tilde system

$$\tilde{O} : (\beta_x, \beta_y, \beta_z) \quad (\beta_1, \beta_2, \beta_3)$$

So, its position vector can be written as

$$\begin{aligned} \bar{\beta} &= \beta_x \hat{e}_x + \beta_y \hat{e}_y + \beta_z \hat{e}_z = \beta_i \hat{e}_i \\ &= \beta_1 \hat{e}_1 + \beta_2 \hat{e}_2 + \beta_3 \hat{e}_3 = \beta_i \hat{e}_i \end{aligned}$$

The basis vectors $\hat{\tilde{e}}_i$ of the tilde coordinate system can be written in terms of the no-tilde system as

$$\hat{\tilde{e}}_x = \alpha_{11} \hat{e}_x + \alpha_{21} \hat{e}_y + \alpha_{31} \hat{e}_z = \alpha_{i1} \hat{e}_i,$$

$$\hat{\tilde{e}}_y = \alpha_{12} \hat{e}_x + \alpha_{22} \hat{e}_y + \alpha_{32} \hat{e}_z = \alpha_{i2} \hat{e}_i,$$

$$\hat{\tilde{e}}_z = \alpha_{13} \hat{e}_x + \alpha_{23} \hat{e}_y + \alpha_{33} \hat{e}_z = \alpha_{i3} \hat{e}_i,$$

$$\text{Implies } \hat{\tilde{e}}_j = \alpha_{ij} \hat{e}_i.$$

j is free index $\rightarrow i$ is dummy index

- The transformation between no-tilde and tilde coordinate systems is completely described by β_i and α_{ij} .
- The α_{ij} are called the transformation coefficients or the transformation matrix.

Properties of the Transformation Coefficients

- Let $\tilde{\gamma}_i$ be the coordinates of the vector $\hat{\tilde{e}}_j$, i.e.,

$$\hat{\tilde{e}}_j = \tilde{\gamma}_i \hat{e}_i$$

Comparing this expression with

$$\hat{\tilde{e}}_j = \alpha_{ij} \hat{e}_i$$

we have

$$\alpha_{ij} = \tilde{\gamma}_i$$

Implies that α_{ij} is the i th-component of the vector $\hat{\tilde{e}}_j$ with respect to basis $\hat{e}_i$.

- Another interpretation of α_{ij} is as follows. As

$$\hat{\tilde{e}}_j = \alpha_{ij} \hat{e}_i,$$

$$\hat{\tilde{e}}_j \cdot \hat{e}_k = \alpha_{ij} \hat{e}_i \cdot \hat{e}_k,$$

$$\hat{\tilde{e}}_j \cdot \hat{e}_k = \alpha_{ij} \delta_{ik} = \alpha_{kj}.$$

$$\alpha_{kj} = \hat{\tilde{e}}_j \cdot \hat{e}_k = |\hat{\tilde{e}}_j| |\hat{e}_k| \cos(\hat{e}_k \cdot \hat{\tilde{e}}_j)$$

Since $\hat{\tilde{e}}_j \cdot \hat{e}_k$ is the cosine of the angle between $\hat{\tilde{e}}_j$ and $\hat{e}_k$,
we get

$$\alpha_{ij} = \cos(\hat{e}_i, \hat{\tilde{e}}_j).$$

Where the α_{ij} are the so called direction cosines.

- α_{ij} is not the same as α_{ji} .

For example

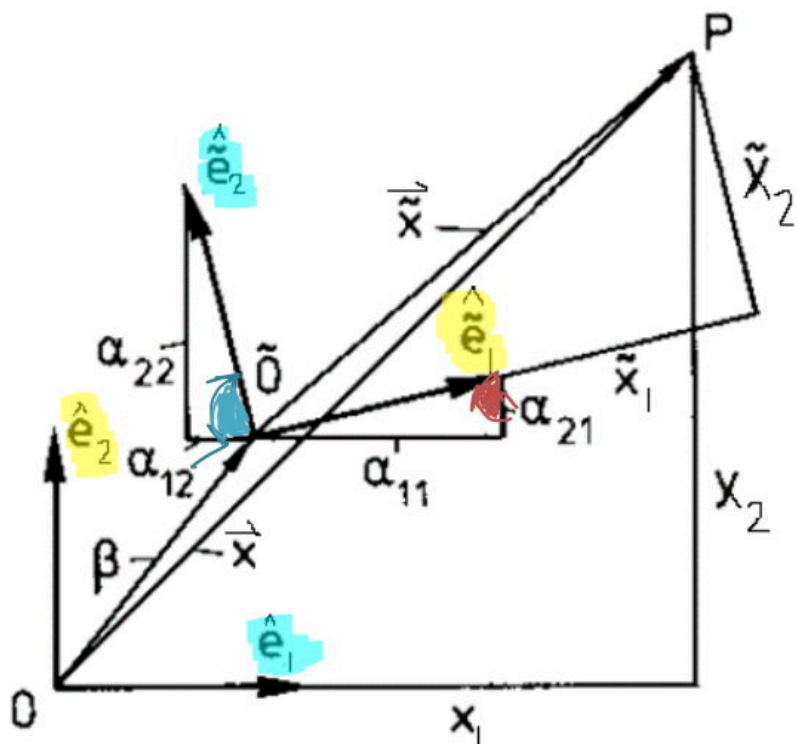
$$\alpha_{ij} = \cos(\hat{e}_i, \hat{\tilde{e}}_j) \text{ and}$$

$$\alpha_{ji} = \cos(\hat{e}_j, \hat{\tilde{e}}_i)$$

For example:

$$\alpha_{12} = \cos(\hat{e}_1, \hat{\tilde{e}}_2)$$

$$\alpha_{21} = \cos(\hat{e}_2, \hat{\tilde{e}}_1)$$



- Since both the $\hat{e}_i$ and the $\hat{\tilde{e}}_i$ are pairwise perpendicular unit vectors, they satisfy the relations

$$\hat{e}_i \cdot \hat{e}_j = \delta_{ij} \quad \text{and} \quad \hat{\tilde{e}}_i \cdot \hat{\tilde{e}}_j = \delta_{ij}$$

This gives

$$\begin{aligned} \delta_{ij} &= \hat{\tilde{e}}_i \cdot \hat{\tilde{e}}_j = \alpha_{mi} \hat{e}_m \cdot \alpha_{nj} \hat{e}_n, \\ &= \alpha_{mi} \alpha_{nj} \hat{e}_m \cdot \hat{e}_n, \\ &= \alpha_{mi} \alpha_{nj} \delta_{mn}, \\ \delta_{ij} &= \alpha_{mi} \alpha_{mj}. \end{aligned}$$

Hence the transformation coefficients form an orthogonal matrix and

$$\alpha_{ik}\alpha_{jk} = \delta_{ij}, \quad \alpha_{ki}\alpha_{kj} = \delta_{ij}$$

These are called orthogonality relations.

Where orthogonal matrices satisfy the relation

$$\det[\alpha_{ij}] = \pm 1$$

Transformation Law for Basis Vectors

Since we have:

$$\hat{\tilde{e}}_j = \alpha_{ij} \hat{e}_i.$$

Multiply the above equation with α_{kj} :

$$\begin{aligned} \alpha_{kj} \hat{\tilde{e}}_j &= \alpha_{kj} \alpha_{ij} \hat{e}_i \\ \alpha_{kj} \hat{\tilde{e}}_j &= \delta_{ki} \hat{e}_i = \hat{e}_k \Rightarrow \hat{\tilde{e}}_k = \alpha_{kj} \hat{\tilde{e}}_j \end{aligned}$$

After renaming the indices, we obtain the following transformation law for basis vectors:

$$\hat{e}_i = \alpha_{ij} \hat{\tilde{e}}_j,$$

$$\hat{\tilde{e}}_i = \alpha_{ji} \hat{e}_j$$

Example: let $\hat{e}_1 = (1, 0, 0)$, $\hat{e}_2 = (0, 1, 0)$, $\hat{e}_3 = (0, 0, 1)$
and $a_{ij} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is the transformation matrix

then find the new basis vectors.

Solution: Assume the transformation equation
for basis: $\tilde{e}_j = \alpha_{ji} \hat{e}_i$

$$\tilde{e}_1 = \alpha_{1i} \hat{e}_i = \alpha_{11} \hat{e}_1 + \alpha_{12} \hat{e}_2 + \alpha_{13} \hat{e}_3$$

$$\tilde{e}_1 = (-2)(1, 0, 0) + 0 + 0 = (-2, 0, 0) = \tilde{e}_1$$

$$\tilde{e}_2 = \alpha_{2i} \hat{e}_i = \alpha_{21} \hat{e}_1 + \alpha_{22} \hat{e}_2 + \alpha_{23} \hat{e}_3$$

$$= 0 + (-2)(0, 1, 0) + 0 = (0, -2, 0) = \tilde{e}_2$$

$$\tilde{e}_3 = \alpha_{3i} \hat{e}_i = \alpha_{31} \hat{e}_1 + \alpha_{32} \hat{e}_2 + \alpha_{33} \hat{e}_3 = (0, 0, 1) = \tilde{e}_3$$

Coordinate Transformations

Transformation law for point coordinates

The position vector to a point P can be expressed as

$$\vec{x} = x_i \hat{e}_i \rightarrow (\text{in no-tilde system})$$

$$\vec{x} = \tilde{x}_i \tilde{\hat{e}}_i \rightarrow (\text{in tilde system})$$

And following is the relation between them

Implies

$$\vec{x} = \vec{\beta} + \vec{\tilde{x}}$$

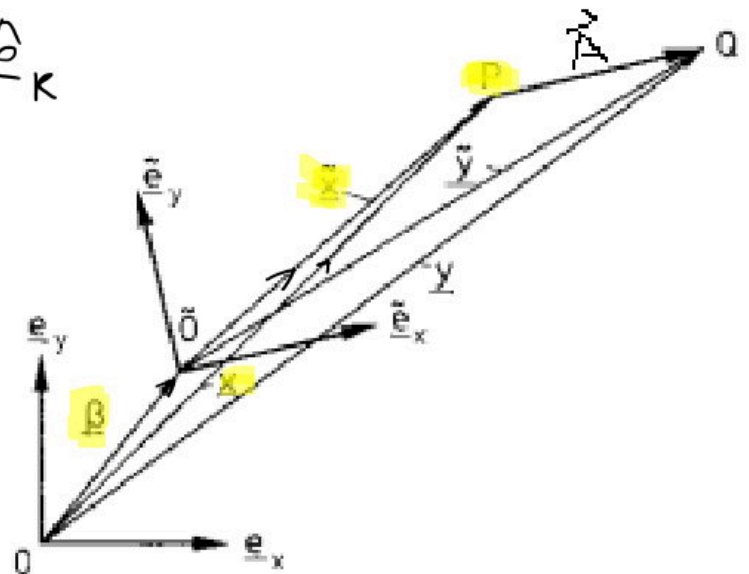
$$\vec{\beta} = \beta_i \hat{e}_i, \quad \tilde{\hat{e}}_i = \alpha_{ki} \hat{e}_k$$

$$\vec{x} = \beta_i \hat{e}_i + \tilde{x}_i \alpha_{ki} \hat{e}_k$$

$$x_i \hat{e}_i = \beta_i \hat{e}_i + \tilde{x}_i \alpha_{ki} \hat{e}_k$$

$$x_i \hat{e}_i = \beta_i \hat{e}_i + \tilde{x}_j \alpha_{ij} \hat{e}_i$$

$$x_i \hat{e}_i = (\beta_i + \tilde{x}_j \alpha_{ij}) \hat{e}_i$$



Since the representation of a position vector with respect to a basis is unique, we have

$$x_i = \beta_i + \tilde{x}_j \alpha_{ij}$$

Similarly, the inverse relation is

$$\tilde{x}_i = \alpha_{ji} x_j - \alpha_{ji} \beta_j$$

Consider $x_i = \beta_i + \tilde{x}_j \alpha_{ij}$ (multiply by α_{ik})

$$\alpha_{ik} x_i = \alpha_{ik} \beta_i + \alpha_{ik} \tilde{x}_j \alpha_{ij}$$

$$\Rightarrow \alpha_{ik} x_i = \alpha_{ik} \beta_i + \underbrace{\alpha_{ik} \alpha_{ij}}_{\delta_{kj}} \tilde{x}_j \quad (\because \alpha_{ik} \alpha_{ij} = \delta_{kj})$$

$$\Rightarrow \alpha_{ik} x_i = \alpha_{ik} \beta_i + \delta_{kj} \tilde{x}_j \quad (\delta_{kj} = 1 \text{ if } k=j)$$

$$\Rightarrow \alpha_{ik} x_i - \alpha_{ik} \beta_i = \tilde{x}_k$$

$$\underline{\text{OR}} \quad \tilde{x}_i = \alpha_{ji} x_j - \alpha_{ji} \beta_j$$

(just replacing k with i and dummy i with j)

The Transformation Law for Vector Coordinates

A vector $\vec{A}$, in a Cartesian coordinate system with basis vectors $\hat{e}_i$, can be written as

$$\vec{A} = A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3 = A_i \hat{e}_i$$

We can obtain the transformation law for the coordinates of a vector from the transformation law for point coordinates. From the figure

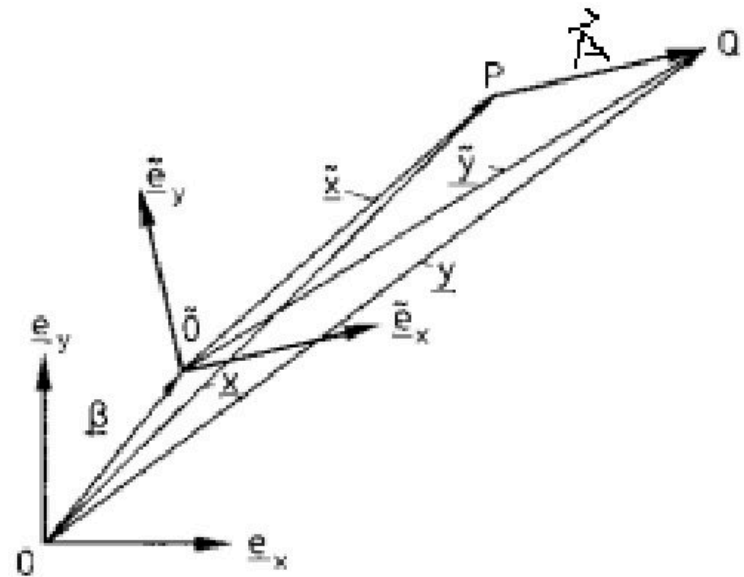
$$\vec{A} = \vec{y} - \vec{x} \quad \text{or} \quad \tilde{\vec{A}} = \tilde{\vec{y}} - \tilde{\vec{x}}$$

so $\vec{A} = \vec{y} - \vec{x}$ implies

$$A_i \hat{e}_i = (y_i - x_i) \hat{e}_i$$

and $\tilde{\vec{A}} = \tilde{\vec{y}} - \tilde{\vec{x}}$

$$\tilde{A}_i \hat{\tilde{e}}_i = (\tilde{y}_i - \tilde{x}_i) \hat{\tilde{e}}_i$$



So we obtain

$$A_i \hat{e}_i = (y_i - x_i) \hat{e}_i$$

implies $A_i = y_i - x_i$

and similarly

$$\tilde{A}_i = \tilde{y}_i - \tilde{x}_i$$

After using the transformation law for coordinates of points we obtain

$$A_i = \alpha_{ij} \tilde{A}_j$$

and $\tilde{A}_i = \alpha_{ji} A_j$

Thus we say that a vector is a quantity which in a coordinate system can be represented by

$$\vec{A} = A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3 = A_i \hat{e}_i$$

And whose coordinates transform according to following rule

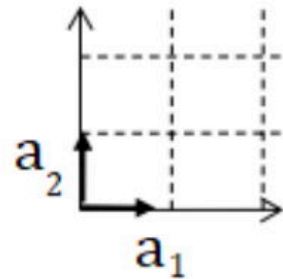
$$A_i = \alpha_{ij} \tilde{A}_j$$

and $\tilde{A}_i = \alpha_{ji} A_j$

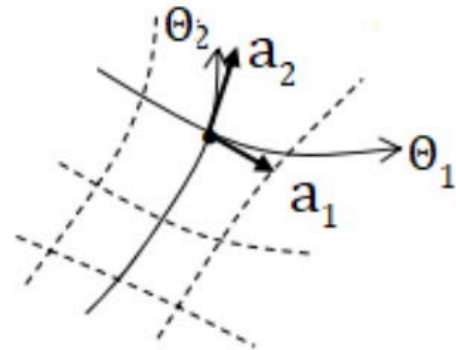
Curvilinear Coordinate Systems

Curvilinear Coordinate System

- In Curvilinear Coordinate system, the coordinate lines may be curved and this is not allowed in Cartesian Coordinate system.
- The basis set in Curvilinear coordinate system may not be orthogonal everywhere in space i.e. the basis vectors don't have to be at 90 degrees to each other, and they don't have a magnitude of 1.



Cartesian



Curvilinear

Orthogonal Curvilinear Coordinate System

A curvilinear coordinate system is called orthogonal when the basis vectors are always orthogonal.

Coordinate Lines	Basis Vectors	Example
Straight	Orthogonal	Cartesian coordinate systems
Straight	Not Orthogonal	Skew coordinate systems
Curved	Orthogonal	Plane polar coordinate system, Cylindrical coordinate system, Spherical coordinate system
Curved	Not orthogonal	Elliptical cylindrical coordinates

Transformation of Coordinates

Let the rectangular coordinates (x, y, z) of any point can be expressed as the function of (u_1, u_2, u_3) so that

$$x = x(u_1, u_2, u_3)$$

$$y = y(u_1, u_2, u_3)$$

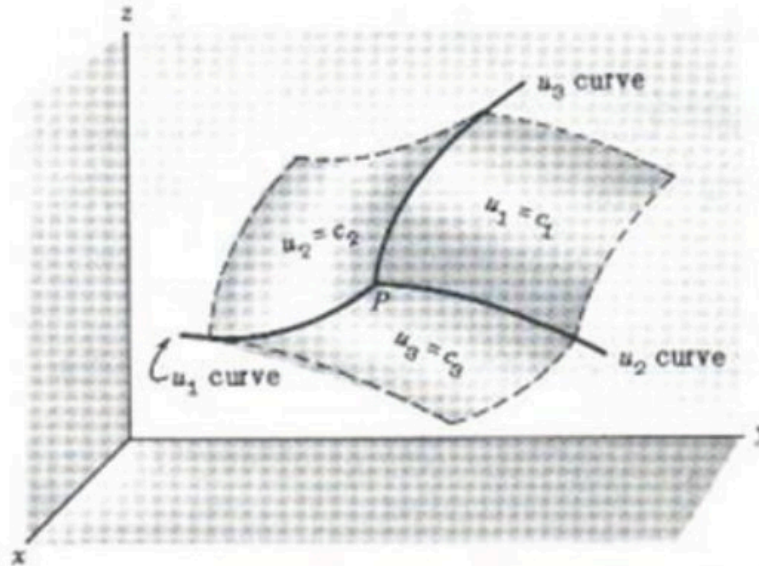
$$z = z(u_1, u_2, u_3)$$

and

$$u_1 = u_1(x, y, z)$$

$$u_2 = u_2(x, y, z)$$

$$u_3 = u_3(x, y, z)$$



Given a point P with rectangular coordinates (x, y, z) , we can associate a unique set of coordinates (u_1, u_2, u_3) called the curvilinear coordinates of the point P . Hence any point P can then be defined in space not only by rectangular coordinates (x, y, z) , but also by curvilinear coordinates (u_1, u_2, u_3) .

Unit vectors

Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

be the position vector of a point P. Then using the transformation between cartesian and curvilinear coordinates we have

$$\vec{r} = \vec{r}(u_1, u_2, u_3)$$

A tangent vector to the u_1 curve at P (for which u_2 and u_3 are constants) is $\frac{\partial \vec{r}}{\partial u_1}$. Then a unit tangent vector in this direction is

$$\hat{e}_1 = \frac{\frac{\partial \vec{r}}{\partial u_1}}{\left| \frac{\partial \vec{r}}{\partial u_1} \right|}$$

Similarly if $\hat{e}_2$ and $\hat{e}_3$ are unit tangent vectors to the u_2 and u_3 curves at P, respectively, then

$$\hat{e}_2 = \frac{\frac{\partial \vec{r}}{\partial u_2}}{\left| \frac{\partial \vec{r}}{\partial u_2} \right|}, \quad \hat{e}_3 = \frac{\frac{\partial \vec{r}}{\partial u_3}}{\left| \frac{\partial \vec{r}}{\partial u_3} \right|}$$

Scale factors

From the unit tangents along u_1 , u_2 and u_3 curve we have

$$\frac{\partial \vec{r}}{\partial u_1} = \left| \frac{\partial \vec{r}}{\partial u_1} \right| \hat{e}_1 = h_1 \hat{e}_1 \text{ where } h_1 = \left| \frac{\partial \vec{r}}{\partial u_1} \right|$$

And similarly

$$\frac{\partial \vec{r}}{\partial u_2} = \left| \frac{\partial \vec{r}}{\partial u_2} \right| \hat{e}_2 = h_2 \hat{e}_2 \text{ where } h_2 = \left| \frac{\partial \vec{r}}{\partial u_2} \right|$$

$$\frac{\partial \vec{r}}{\partial u_3} = \left| \frac{\partial \vec{r}}{\partial u_3} \right| \hat{e}_3 = h_3 \hat{e}_3 \text{ where } h_3 = \left| \frac{\partial \vec{r}}{\partial u_3} \right|$$

The quantities h_1 , h_2 and h_3 are called scale factors. The scale factor gives a measure of how a change in the coordinate, changes the position of a point.

Unit normal vectors:

∇u_1 is a vector at P normal to the surface $u_1 = c_1$, a unit vector in this direction is given by

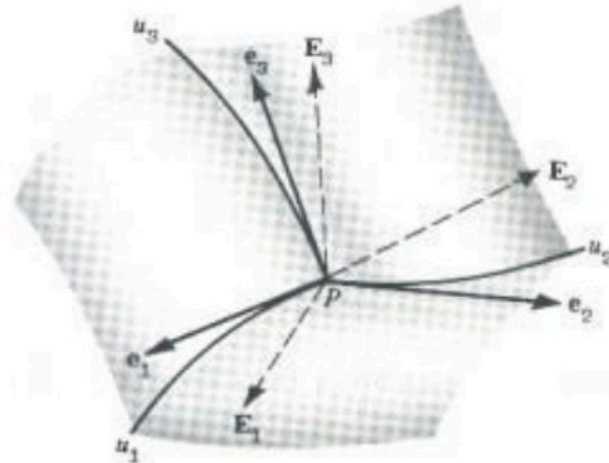
$$\hat{E}_1 = \frac{\nabla u_1}{|\nabla u_1|}$$

Similarly, the unit vectors

$$\hat{E}_2 = \frac{\nabla u_2}{|\nabla u_2|} \text{ and}$$

$$\hat{E}_3 = \frac{\nabla u_3}{|\nabla u_3|}$$

at P are normal to the surfaces $u_2 = c_2$ and $u_3 = c_3$, respectively.



- Thus at each point P of a curvilinear coordinate system there exist two sets of unit vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$ tangent to the coordinate curves and $\hat{E}_1, \hat{E}_2, \hat{E}_3$ normal to the coordinate surface.
- The two sets become identical if and only if the curvilinear system is orthogonal.
- Both sets are analogous to $\hat{i}, \hat{j}, \hat{k}$ unit vectors in rectangular Cartesian coordinates but are unlike in the way that they may change directions from point to point.
- These tangent vector and normal vectors constitute reciprocal system of vectors. (A vector having the same direction as that of a given vector but magnitude equal to the reciprocal of the given vector is known as the reciprocal of vector.)

Orthogonal curvilinear coordinate system

If the coordinate curves intersect at right angles, the curvilinear coordinate system is called **orthogonal**. The u_1, u_2 and u_3 coordinate curves of an orthogonal curvilinear system are similar to the x, y and z coordinate axes of a rectangular Cartesian system. In an orthogonal curvilinear coordinate system, the unit vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$ are mutually orthogonal.

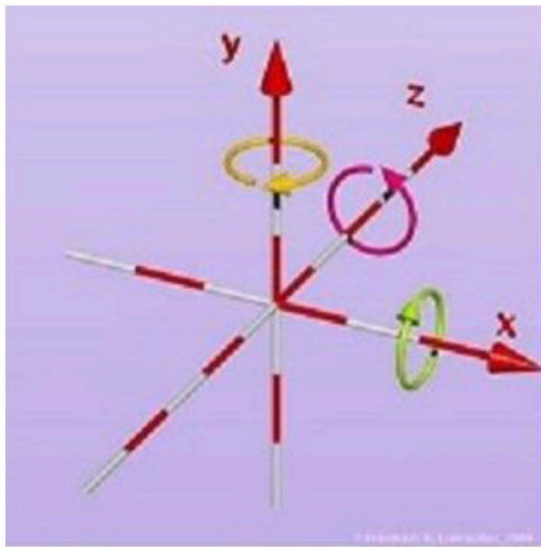
$$\hat{e}_1 \cdot \hat{e}_1 = \hat{e}_2 \cdot \hat{e}_2 = \hat{e}_3 \cdot \hat{e}_3 = 1, \text{ and } \hat{e}_1 \cdot \hat{e}_2 = \hat{e}_2 \cdot \hat{e}_3 = \hat{e}_3 \cdot \hat{e}_1 = 0$$

If the system is **right-handed** then

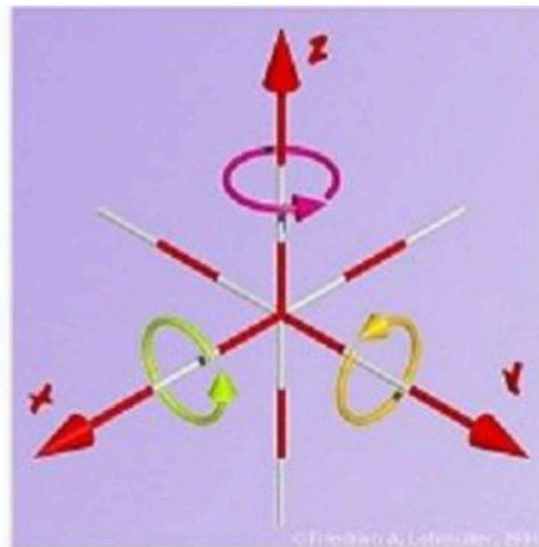
$$\hat{e}_1 \times \hat{e}_2 = -\hat{e}_2 \times \hat{e}_1 = \hat{e}_3, \quad \hat{e}_2 \times \hat{e}_3 = \hat{e}_3 \times \hat{e}_2 = \hat{e}_1, \quad \hat{e}_3 \times \hat{e}_1 = -\hat{e}_1 \times \hat{e}_3 = \hat{e}_2$$

If the system is **left-handed** then

$$\hat{e}_1 \times \hat{e}_2 = -\hat{e}_2 \times \hat{e}_1 = -\hat{e}_3, \quad \hat{e}_2 \times \hat{e}_3 = \hat{e}_3 \times \hat{e}_2 = -\hat{e}_1, \quad \hat{e}_3 \times \hat{e}_1 = -\hat{e}_1 \times \hat{e}_3 = -\hat{e}_2$$



Left-handed system



Right-handed system

Curvilinear Coordinate Systems

A vector in curvilinear coordinate system:

A vector $\vec{A}$ can be represented in terms of the unit basis vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$ or $\hat{E}_1, \hat{E}_2, \hat{E}_3$ in the form

$$\vec{A} = A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3$$

$$\vec{A} = a_1 \hat{E}_1 + a_2 \hat{E}_2 + a_3 \hat{E}_3$$

Where A_1, A_2, A_3 and a_1, a_2, a_3 are the respective components of vector $\vec{A}$ in each system.

We can also represent $\vec{A}$ in terms of $\frac{\partial \vec{r}}{\partial u_1}, \frac{\partial \vec{r}}{\partial u_2}, \frac{\partial \vec{r}}{\partial u_3}$ or $\nabla u_1, \nabla u_2, \nabla u_3$ as

$$\vec{A} = C_1 \frac{\partial \vec{r}}{\partial u_1} + C_2 \frac{\partial \vec{r}}{\partial u_2} + C_3 \frac{\partial \vec{r}}{\partial u_3}$$

$$\vec{A} = c_1 \nabla u_1 + c_2 \nabla u_2 + c_3 \nabla u_3$$

Where C_1, C_2, C_3 are called the contravariant components of the vector $\vec{A}$ and c_1, c_2, c_3 are called the covariant components of $\vec{A}$.

Note: If u_1, u_2, u_3 are general coordinates.
then $\frac{\partial \vec{r}}{\partial u_1}, \frac{\partial \vec{r}}{\partial u_2}, \frac{\partial \vec{r}}{\partial u_3}$ and $\nabla u_1, \nabla u_2, \nabla u_3$
are reciprocal system of vectors.

i.e. $\frac{\partial \vec{r}}{\partial u_i} \cdot \nabla u_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

where $i = 1, 2, 3$

$j = 1, 2, 3$

Arc length in general curvilinear coordinates

The square of the element of arc length in general curvilinear coordinates can be expressed by

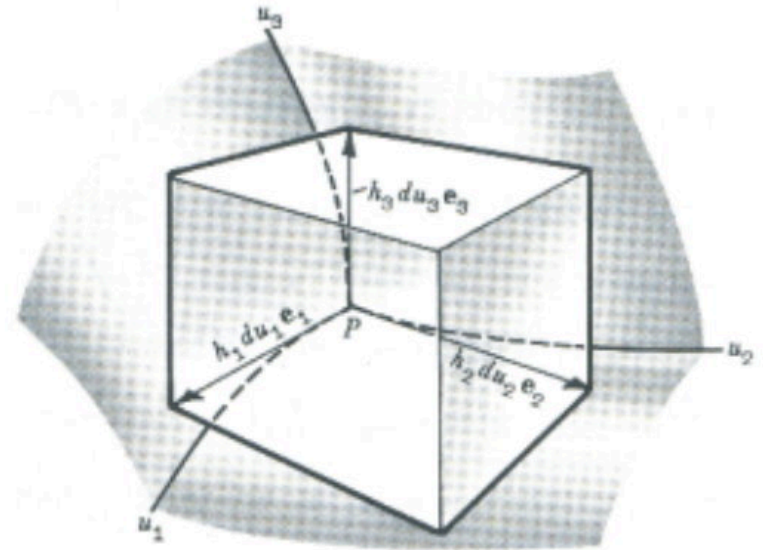
$$ds^2 = \sum_{i=1}^3 \sum_{j=1}^3 h_i h_j (\hat{e}_i \cdot \hat{e}_j) du_i du_j$$

$$\vec{r} = \vec{r}(u_1, u_2, u_3)$$

$$d\vec{r} = \frac{\partial \vec{r}}{\partial u_1} du_1 + \frac{\partial \vec{r}}{\partial u_2} du_2 + \frac{\partial \vec{r}}{\partial u_3} du_3$$

$$d\vec{r} = h_1 du_1 \hat{e}_1 + h_2 du_2 \hat{e}_2 + h_3 du_3 \hat{e}_3$$

$$ds^2 = d\vec{r} \cdot d\vec{r}$$



Arc length element in orthogonal curvilinear coordinates

The arc length ds is the length of the infinitesimal vector $d\vec{r}$:

$$(ds)^2 = d\vec{r} \cdot d\vec{r}$$

In curvilinear coordinates, we have

$$\vec{r} = \vec{r}(u_1, u_2, u_3)$$

$$d\vec{r} = \frac{\partial \vec{r}}{\partial u_1} du_1 + \frac{\partial \vec{r}}{\partial u_2} du_2 + \frac{\partial \vec{r}}{\partial u_3} du_3$$

$$d\vec{r} = h_1 du_1 \hat{e}_1 + h_2 du_2 \hat{e}_2 + h_3 du_3 \hat{e}_3$$

$$ds^2 = h_1^2 du_1^2 + h_2^2 du_2^2 + h_3^2 du_3^2$$

Area element in an orthogonal curvilinear system:

Along a u_1 curve, u_2 and u_3 are constants so that $d\vec{r} = h_1 du_1 \hat{e}_1$. Then the differential of arc length ds_1 along u_1 at P is $h_1 du_1$.

The area elements are given by

$$dA_1 = |(h_2 du_2 \hat{e}_2) \times (h_3 du_3 \hat{e}_3)|$$

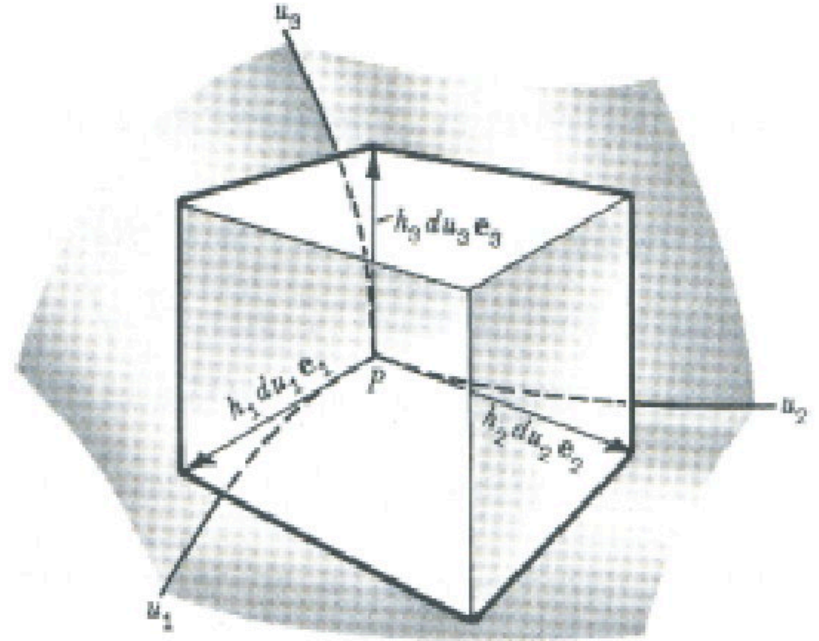
$$dA_1 = h_2 h_3 |\hat{e}_2 \times \hat{e}_3| du_2 du_3 = h_2 h_3 du_2 du_3$$

$$\text{As } |\hat{e}_2 \times \hat{e}_3| = |\hat{e}_1| = 1$$

similarly

$$dA_2 = |(h_1 du_1 \hat{e}_1) \times (h_3 du_3 \hat{e}_3)| = h_1 h_3 du_1 du_3$$

$$dA_3 = |(h_1 du_1 \hat{e}_1) \times (h_2 du_2 \hat{e}_2)| = h_1 h_2 du_1 du_2$$



Volume element:

We know that the absolute value of the scalar triple product gives the **volume of the parallelepiped**.

$$dv = |(h_1 du_1 \hat{e}_1) \cdot (h_2 du_2 \hat{e}_2) \times (h_3 du_3 \hat{e}_3)|$$

$$dv = h_1 h_2 h_3 du_1 du_2 du_3 |\hat{e}_1 \cdot \hat{e}_2 \times \hat{e}_3| = h_1 h_2 h_3 du_1 du_2 du_3$$

$$\text{As } |\hat{e}_1 \cdot \hat{e}_2 \times \hat{e}_3| = 1$$

Jacobian of transformation

If u_1, u_2 and u_3 are the orthogonal curvilinear coordinates and

$$x = x(u_1, u_2, u_3), \quad y = y(u_1, u_2, u_3), \quad z = z(u_1, u_2, u_3)$$

then the Jacobian of x, y, z is denoted by J or $J\left(\frac{x, y, z}{u_1, u_2, u_3}\right)$ is defined as

$$J = J\left(\frac{x, y, z}{u_1, u_2, u_3}\right) = \frac{\partial(x, y, z)}{\partial(u_1, u_2, u_3)} = \begin{vmatrix} \frac{\partial x}{\partial u_1} & \frac{\partial y}{\partial u_1} & \frac{\partial z}{\partial u_1} \\ \frac{\partial x}{\partial u_2} & \frac{\partial y}{\partial u_2} & \frac{\partial z}{\partial u_2} \\ \frac{\partial x}{\partial u_3} & \frac{\partial y}{\partial u_3} & \frac{\partial z}{\partial u_3} \end{vmatrix}$$

From the definition of a scalar triple product, the above determinant can be written as

$$J\left(\frac{x, y, z}{u_1, u_2, u_3}\right) = \left(\frac{\partial x}{\partial u_1} \hat{i} + \frac{\partial y}{\partial u_1} \hat{j} + \frac{\partial z}{\partial u_1} \hat{k} \right) \cdot \left(\frac{\partial x}{\partial u_2} \hat{i} + \frac{\partial y}{\partial u_2} \hat{j} + \frac{\partial z}{\partial u_2} \hat{k} \right) \times \left(\frac{\partial x}{\partial u_3} \hat{i} + \frac{\partial y}{\partial u_3} \hat{j} + \frac{\partial z}{\partial u_3} \hat{k} \right)$$

$$J\left(\frac{x, y, z}{u_1, u_2, u_3}\right) = \frac{\partial \vec{r}}{\partial u_1} \cdot \frac{\partial \vec{r}}{\partial u_2} \times \frac{\partial \vec{r}}{\partial u_3} = h_1 \hat{e}_1 \cdot h_2 \hat{e}_2 \times h_3 \hat{e}_3$$

$$J\left(\frac{x, y, z}{u_1, u_2, u_3}\right) = h_1 h_2 h_3 (\hat{e}_1 \cdot \hat{e}_2 \times \hat{e}_3)$$

$$J\left(\frac{x, y, z}{u_1, u_2, u_3}\right) = h_1 h_2 h_3 \quad (\text{since } \hat{e}_1 \cdot \hat{e}_2 \times \hat{e}_3 = 1)$$

If the Jacobian equals to zero identically then $\frac{\partial \vec{r}}{\partial u_1}, \frac{\partial \vec{r}}{\partial u_2}, \frac{\partial \vec{r}}{\partial u_3}$ are coplanar vectors and the linear coordinate transformation breaks down, i.e., there is a relation between x, y, z having the form $F(x, y, z) = 0$. We shall therefore require the Jacobian to be different from zero.

Curvilinear Coordinate Systems

Example: Find the scale factors, arc length and Jacobian in rectangular Cartesian coordinates.

Solution: If $\vec{r}$ is the position vector of any point P in a coordinate system then the scale factors are defined by:

$$h_1 = \left| \frac{\partial \vec{r}}{\partial u_1} \right|, h_2 = \left| \frac{\partial \vec{r}}{\partial u_2} \right|, h_3 = \left| \frac{\partial \vec{r}}{\partial u_3} \right|$$

also,

$$\hat{e}_1 = \frac{\left(\frac{\partial \vec{r}}{\partial u_1} \right)}{h_1}, \hat{e}_2 = \frac{\left(\frac{\partial \vec{r}}{\partial u_2} \right)}{h_2}, \hat{e}_3 = \frac{\left(\frac{\partial \vec{r}}{\partial u_3} \right)}{h_3}$$

Now, the position vector of a point 'P' in rectangular Cartesian Coordinate is:

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

and $(u_1, u_2, u_3) \rightarrow (x, y, z)$

Implies:

$$\frac{\partial \vec{r}}{\partial x} = \hat{i}, \quad \frac{\partial \vec{r}}{\partial y} = \hat{j}, \quad \frac{\partial \vec{r}}{\partial z} = \hat{k}$$

$$\left| \frac{\partial \vec{r}}{\partial x} \right| = 1, \quad \left| \frac{\partial \vec{r}}{\partial y} \right| = 1, \quad \left| \frac{\partial \vec{r}}{\partial z} \right| = 1$$

$$\Rightarrow h_1 = 1, \quad h_2 = 1, \quad h_3 = 1$$

$$\text{Also, } \hat{e}_1 = \hat{i}, \hat{e}_2 = \hat{j}, \hat{e}_3 = \hat{k}$$

$$ds^2 = d\vec{r} \cdot d\vec{r}$$

$$d\vec{r} = \frac{\partial \vec{r}}{\partial x} dx + \frac{\partial \vec{r}}{\partial y} dy + \frac{\partial \vec{r}}{\partial z} dz$$

$$d\vec{r} = dx \hat{i} + dy \hat{j} + dz \hat{k}$$

$$\Rightarrow ds^2 = (dx \hat{i} + dy \hat{j} + dz \hat{k}) \cdot (dx \hat{i} + dy \hat{j} + dz \hat{k})$$

$$ds^2 = (dx)^2 + (dy)^2 + (dz)^2$$

$$\text{Similarly: } J = h_1 h_2 h_3 = 1$$

Example: Consider the curvilinear coordinates system defined by

$$x = u_3, \quad y = e^{u_2} \cos u_1, \quad z = e^{u_2} \sin u_1$$

Find the unit vectors $\hat{e}_1, \hat{e}_2, \hat{e}_3$ and show that the system is orthogonal. Compute the scales factors h_1, h_2 , and h_3 .

Solution: Consider:

$$\vec{r} = x \hat{i} + y \hat{j} + z \hat{k}$$

$$\Rightarrow \vec{r} = u_3 \hat{i} + e^{u_2} \cos u_1 \hat{j} + e^{u_2} \sin u_1 \hat{k}$$

$$\text{Then } \frac{\partial \vec{r}}{\partial u_1} = -e^{u_2} \sin u_1 \hat{j} + e^{u_2} \cos u_1 \hat{k}$$

$$\frac{\partial \vec{r}}{\partial u_2} = e^{u_2} \cos u_1 \hat{j} + e^{u_2} \sin u_1 \hat{k}$$

$$\text{and } \frac{\partial \vec{r}}{\partial u_3} = \hat{i}$$

Now,

$$\begin{aligned} \left| \frac{\partial \vec{r}}{\partial u_1} \right| &= \sqrt{(-e^{u_2} \sin u_1)^2 + (e^{u_2} \cos u_1)^2} \\ &= \sqrt{e^{2u_2} (\sin^2 u_1 + \cos^2 u_1)} = \sqrt{e^{2u_2}} \end{aligned}$$

$$\Rightarrow \left| \frac{\partial \vec{r}}{\partial u_1} \right| = e^{u_2}$$

$$\begin{aligned} \left| \frac{\partial \vec{r}}{\partial u_2} \right| &= \sqrt{(e^{u_2} \cos u_1)^2 + (e^{u_2} \sin u_1)^2} \\ &= e^{u_2} \end{aligned}$$

$$\text{and } \left| \frac{\partial \vec{r}}{\partial u_3} \right| = |\hat{z}| = 1$$

$$\text{So, } \hat{e}_1 = \frac{\partial \vec{r} / \partial u_1}{|\partial \vec{r} / \partial u_1|} = \frac{-e^{u_2} \sin u_1 \hat{j} + e^{u_2} \cos u_1 \hat{k}}{e^{u_2}}$$

$$\hat{e}_1 = -\sin u_1 \hat{j} + \cos u_1 \hat{k}$$

Similarly:

$$\hat{e}_2 = \cos u_1 \hat{j} + \sin u_1 \hat{k}$$

$$\text{and } \hat{e}_3 = \hat{i}$$

Now, we show that the system is orthogonal. So consider:

$$\begin{aligned}\hat{e}_1 \cdot \hat{e}_2 &= (-\sin u, \hat{j} + \cos u, \hat{k}) \cdot (\cos u, \hat{j} + \sin u, \hat{k}) \\ &= -\sin u, \cancel{\cos u} + \cancel{\cos u}, \sin u = 0\end{aligned}$$

$$\Rightarrow \hat{e}_1 \cdot \hat{e}_2 = 0$$

$$\hat{e}_2 \cdot \hat{e}_3 = (\cos u, \hat{j} + \sin u, \hat{k}) \cdot \hat{i} = 0$$

$$\Rightarrow \hat{e}_2 \cdot \hat{e}_3 = 0$$

$$\text{and } \hat{e}_3 \cdot \hat{e}_1 = \hat{i} \cdot (-\sin u, \hat{j} + \cos u, \hat{k})$$

$$\hat{e}_3 \cdot \hat{e}_1 = 0$$

Scale factors:

$$h_1 = \left| \frac{\partial \vec{r}}{\partial u_1} \right| = e^{u_2}$$

$$h_2 = \left| \frac{\partial \vec{r}}{\partial u_2} \right| = e^{u_2}$$

$$h_3 = \left| \frac{\partial \vec{r}}{\partial u_3} \right| = 1$$

Example: Check that the following system of elliptical cylindrical coordinates are orthogonal or not?

$$\vec{r} = a p \cos \theta \hat{i} + b p \sin \theta \hat{j} + z \hat{k} ; a \neq b$$

Solution: First we have to find $\hat{e}_1, \hat{e}_2, \hat{e}_3$ for the given system.

$$\hat{e}_1 = \frac{\partial \vec{r} / \partial p}{|\partial \vec{r} / \partial p|} = \frac{(a \cos \theta \hat{i} + b \sin \theta \hat{j})}{\sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta}}$$

$$\begin{aligned} \hat{e}_2 &= \frac{\partial \vec{r} / \partial \theta}{|\partial \vec{r} / \partial \theta|} = \frac{(-a p \sin \theta \hat{i} + b p \cos \theta \hat{j})}{p \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta}} \\ &= \frac{-a \sin \theta \hat{i} + b \cos \theta \hat{j}}{\sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta}} \end{aligned}$$

$$\hat{e}_3 = \frac{\partial \vec{r} / \partial z}{|\partial \vec{r} / \partial z|} = \hat{k}$$

Now:

$$\hat{e}_1 \cdot \hat{e}_2 = \frac{-a^2 \sin \theta \cos \theta + b^2 \sin \theta \cos \theta}{\sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta} \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta}} \neq 0 \quad \text{b/c } a \neq b$$

$\Rightarrow$ The elliptical cylindrical coordinate system is not orthogonal.

Differential Operators in Orthogonal Curvilinear Coordinate Systems

Nabla operator

The differential operator nabla often appear in vector analysis. In the space of three variables it is defined as

$$\nabla = \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k}$$

- Gradient $\rightarrow \nabla \psi$
- Divergence $\rightarrow \nabla \cdot \vec{A}$
- Curl $\longrightarrow \nabla \times A$
- Laplacian $\rightarrow \nabla^2 \psi$

Gradient of a scalar field in curvilinear coordinates

Let u_1, u_2 and u_3 are the curvilinear coordinates.

$$\text{Let } \nabla \psi = f_1 \hat{e}_1 + f_2 \hat{e}_2 + f_3 \hat{e}_3$$

Where f_1, f_2 and f_3 are to be determined.

$$\text{Since } d\vec{r} = \frac{\partial \vec{r}}{\partial u_1} du_1 + \frac{\partial \vec{r}}{\partial u_2} du_2 + \frac{\partial \vec{r}}{\partial u_3} du_3$$

$$d\vec{r} = h_1 du_1 \hat{e}_1 + h_2 du_2 \hat{e}_2 + h_3 du_3 \hat{e}_3$$

we have

$$d\psi = \nabla \psi \cdot d\vec{r} = h_1 f_1 du_1 + h_2 f_2 du_2 + h_3 f_3 du_3$$

$$d\psi = \frac{\partial \psi}{\partial u_1} du_1 + \frac{\partial \psi}{\partial u_2} du_2 + \frac{\partial \psi}{\partial u_3} du_3$$

$$f_1 = \frac{1}{h_1} \frac{\partial \psi}{\partial u_1}, \quad f_2 = \frac{1}{h_2} \frac{\partial \psi}{\partial u_2}, \quad f_3 = \frac{1}{h_3} \frac{\partial \psi}{\partial u_3}$$

Then

$$\nabla \psi = \frac{1}{h_1} \frac{\partial \psi}{\partial u_1} \hat{e}_1 + \frac{1}{h_2} \frac{\partial \psi}{\partial u_2} \hat{e}_2 + \frac{1}{h_3} \frac{\partial \psi}{\partial u_3} \hat{e}_3$$

On comparing:

$$\nabla = \frac{1}{h_1} \frac{\partial}{\partial u_1} \hat{e}_1 + \frac{1}{h_2} \frac{\partial}{\partial u_2} \hat{e}_2 + \frac{1}{h_3} \frac{\partial}{\partial u_3} \hat{e}_3$$

Expression for Divergence of a vector field

Consider

$$\nabla \cdot \vec{A} = \nabla \cdot (A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3)$$

$$\nabla \cdot \vec{A} = \nabla \cdot (A_1 \hat{e}_1) + \nabla \cdot (A_2 \hat{e}_2) + \nabla \cdot (A_3 \hat{e}_3)$$

since $\hat{e}_1, \hat{e}_2$, and $\hat{e}_3$ form a right handed system, therefore

$$\hat{e}_1 = \hat{e}_2 \times \hat{e}_3 = h_2 h_3 \nabla u_2 \times \nabla u_3$$

$$\hat{e}_2 = \hat{e}_3 \times \hat{e}_1 = h_3 h_1 \nabla u_3 \times \nabla u_1$$

$$\hat{e}_3 = \hat{e}_1 \times \hat{e}_2 = h_1 h_2 \nabla u_1 \times \nabla u_2$$

$$\text{Now } \nabla \cdot (A_1 \hat{e}_1) = \nabla \cdot (A_1 h_2 h_3 \nabla u_2 \times \nabla u_3)$$

$$= \nabla (A_1 h_2 h_3) \cdot \nabla u_2 \times \nabla u_3 + A_1 h_2 h_3 \nabla \cdot (\nabla u_2 \times \nabla u_3)$$

$$\therefore \nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B}) \quad \text{and} \quad \nabla \times \nabla \phi = 0$$

$$\Rightarrow \nabla \cdot (\nabla u_2 \times \nabla u_3) = \nabla u_3 \cdot (\nabla \times \nabla u_2) - \nabla u_2 \cdot (\nabla \times \nabla u_3) = 0$$

$$\begin{aligned}
\Rightarrow \nabla.(A_1 \hat{e}_1) &= \nabla(A_1 h_2 h_3) \cdot \frac{\hat{e}_2}{h_2} \times \frac{\hat{e}_3}{h_3} + 0 \\
&= \bar{\nabla}(A_1 h_2 h_3) \cdot \frac{\hat{e}_1}{h_2 h_3} \\
&= \left[\frac{1}{h_1} \frac{\partial}{\partial u_1} (A_1 h_2 h_3) \hat{e}_1 + \frac{1}{h_2} \frac{\partial}{\partial u_2} (A_1 h_2 h_3) \hat{e}_2 + \frac{1}{h_3} \frac{\partial}{\partial u_3} (A_1 h_2 h_3) \hat{e}_3 \right] \cdot \frac{\hat{e}_1}{h_2 h_3} \\
\nabla.(A_1 \hat{e}_1) &= \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial u_1} (A_1 h_2 h_3)
\end{aligned}$$

similarly

$$\nabla.(A_2 \hat{e}_2) = \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial u_2} (A_2 h_3 h_1)$$

$$\text{and } \nabla.(A_3 \hat{e}_3) = \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial u_3} (A_3 h_1 h_2)$$

Hence,

$$\nabla.\bar{A} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} (A_1 h_2 h_3) + \frac{\partial}{\partial u_2} (A_2 h_3 h_1) + \frac{\partial}{\partial u_3} (A_3 h_1 h_2) \right]$$

Expression of curl

Consider

$$\nabla \times \vec{A} = \nabla \times (A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3)$$

$$= \nabla \times (A_1 \hat{e}_1) + \nabla \times (A_2 \hat{e}_2) + \nabla \times (A_3 \hat{e}_3)$$

$$\nabla \times (A_1 \hat{e}_1) = \nabla \times (A_1 h_1 \nabla u_1) \quad (\text{As, } \hat{e}_1 = h_1 \nabla u_1)$$

$$= \nabla (A_1 h_1) \times \nabla u_1 + A_1 h_1 \nabla \times \nabla u_1$$

$$= \nabla (A_1 h_1) \times \frac{\hat{e}_1}{h_1} + 0 \quad (\text{As, } \nabla \times \nabla u_1 = 0)$$

$$= \left[\frac{1}{h_1} \frac{\partial}{\partial u_1} (A_1 h_1) \hat{e}_1 + \frac{1}{h_2} \frac{\partial}{\partial u_2} (A_1 h_1) \hat{e}_2 + \frac{1}{h_3} \frac{\partial}{\partial u_3} (A_1 h_1) \hat{e}_3 \right] \times \frac{\hat{e}_1}{h_1}$$

$$\nabla \times (A_1 \hat{e}_1) = \frac{1}{h_3 h_1} \frac{\partial}{\partial u_3} (A_1 h_1) \hat{e}_2 - \frac{1}{h_1 h_2} \frac{\partial}{\partial u_2} (A_1 h_1) \hat{e}_3$$

similarly

$$\nabla \times (A_2 \hat{e}_2) = \frac{1}{h_1 h_2} \frac{\partial}{\partial u_1} (A_2 h_2) \hat{e}_3 - \frac{1}{h_2 h_3} \frac{\partial}{\partial u_3} (A_2 h_2) \hat{e}_1$$

$$\nabla \times (A_3 \hat{e}_3) = \frac{1}{h_2 h_3} \frac{\partial}{\partial u_2} (A_3 h_3) \hat{e}_1 - \frac{1}{h_3 h_1} \frac{\partial}{\partial u_1} (A_3 h_3) \hat{e}_2$$

Hence,

$$\begin{aligned} \nabla \times \vec{A} = & \frac{\hat{e}_1}{h_2 h_3} \left[\frac{\partial}{\partial u_2} (A_3 h_3) - \frac{\partial}{\partial u_3} (A_2 h_2) \right] + \frac{\hat{e}_2}{h_3 h_1} \left[\frac{\partial}{\partial u_3} (A_1 h_1) - \frac{\partial}{\partial u_1} (A_3 h_3) \right] \\ & + \frac{\hat{e}_3}{h_1 h_2} \left[\frac{\partial}{\partial u_1} (A_2 h_2) - \frac{\partial}{\partial u_2} (A_1 h_1) \right] \end{aligned}$$

This can be write as

$$\nabla \times \vec{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ A_1 h_1 & A_2 h_2 & A_3 h_3 \end{vmatrix}$$

Expression of Laplacian

Consider

$$\nabla \psi = \frac{1}{h_1} \frac{\partial \psi}{\partial u_1} \hat{e}_1 + \frac{1}{h_2} \frac{\partial \psi}{\partial u_2} \hat{e}_2 + \frac{1}{h_3} \frac{\partial \psi}{\partial u_3} \hat{e}_3. \text{ and}$$

$$\nabla \cdot \vec{A} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} (A_1 h_2 h_3) + \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial u_2} (A_2 h_3 h_1) + \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial u_3} (A_3 h_1 h_2) \right]$$

$$\text{Let, } \vec{A} = \nabla \psi \text{ then } A_1 = \frac{1}{h_1} \frac{\partial \psi}{\partial u_1}, \quad A_2 = \frac{1}{h_2} \frac{\partial \psi}{\partial u_2}, \quad A_3 = \frac{1}{h_3} \frac{\partial \psi}{\partial u_3}$$

$$\text{and } \nabla \cdot \vec{A} = \nabla \cdot \nabla \psi = \nabla^2 \psi$$

$$\nabla^2 \psi = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial \psi}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left(\frac{h_3 h_1}{h_2} \frac{\partial \psi}{\partial u_2} \right) + \frac{\partial}{\partial u_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial \psi}{\partial u_3} \right) \right]$$

Differential Operators in Orthogonal Curvilinear Coordinate Systems

Example: Consider the curvilinear coordinate system defined by

$$x = u_3, \quad y = e^{u_2} \cos u_1, \quad z = e^{u_2} \sin u_1$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

Find $\nabla\psi$ and $\nabla^2\psi$, If $\psi = u_1^2 + u_2^2 + u_3^2$.

Sol.

$$\frac{\partial \psi}{\partial u_1} = 2u_1, \quad \frac{\partial \psi}{\partial u_2} = 2u_2, \quad \frac{\partial \psi}{\partial u_3} = 2u_3$$

$$h_1 = \left| \frac{\partial \vec{r}}{\partial u_1} \right| = e^{u_2}$$

In an orthogonal coordinate system

$$\begin{aligned} \nabla \psi &= \frac{1}{h_1} \frac{\partial \psi}{\partial u_1} \hat{e}_1 + \frac{1}{h_2} \frac{\partial \psi}{\partial u_2} \hat{e}_2 + \frac{1}{h_3} \frac{\partial \psi}{\partial u_3} \hat{e}_3 \\ &= e^{-u_2} (2u_1) \hat{e}_1 + e^{-u_2} (2u_2) \hat{e}_2 + (2u_3) \hat{e}_3 \\ &= 2e^{-u_2} (u_1 \hat{e}_1 + u_2 \hat{e}_2) + (2u_3) \hat{e}_3 \end{aligned}$$

$$h_2 = \left| \frac{\partial \vec{r}}{\partial u_2} \right| = e^{u_2}$$

$$h_3 = \left| \frac{\partial \vec{r}}{\partial u_3} \right| = 1$$

and

$$\begin{aligned}\nabla^2\psi &= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\psi}{\partial u_1}\right)+\frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial\psi}{\partial u_2}\right)+\frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\psi}{\partial u_3}\right)\right] \\&= \frac{1}{(e^{u_2})(e^{u_2})(1)}\left[\frac{\partial}{\partial u_1}\left(\frac{e^{u_2}}{e^{u_2}}(2u_1)\right)+\frac{\partial}{\partial u_2}\left(\frac{e^{u_2}}{e^{u_2}}(2u_2)\right)+\frac{\partial}{\partial u_3}\left(e^{u_2}e^{u_2}(2u_3)\right)\right] \\&= e^{-2u_2}\left[\frac{\partial}{\partial u_1}(2u_1)+\frac{\partial}{\partial u_2}(2u_2)+\frac{\partial}{\partial u_3}(e^{2u_2}(2u_3))\right] \\&= e^{-2u_2}\left[2+2+2e^{2u_2}\right]=e^{-2u_2}\left[4+2e^{2u_2}\right]\end{aligned}$$

Example. Find $\nabla \cdot \vec{A}$ and $\nabla \times \vec{A}$ in the system defined in previous example for the vector field

$$\vec{A} = u_3 \hat{e}_1 - u_2 \hat{e}_3.$$

$$\begin{aligned} \nabla \cdot \vec{A} &= \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} (h_2 h_3 A_1) + \frac{\partial}{\partial u_2} (h_1 h_3 A_2) + \frac{\partial}{\partial u_3} (h_1 h_2 A_3) \right] \\ &= \frac{1}{(e^{u_2})(e^{u_2})(1)} \left[\frac{\partial}{\partial u_1} (u_3 (1) e^{u_2}) + \frac{\partial}{\partial u_2} ((0)(1) e^{u_2}) + \frac{\partial}{\partial u_3} (-u_2 (e^{u_2})(e^{u_2})) \right] \\ &= \frac{1}{e^{2u_2}} \left[\frac{\partial}{\partial u_1} (u_3 e^{u_2}) + \frac{\partial}{\partial u_2} (0) - \frac{\partial}{\partial u_3} (e^{3u_2}) \right] = \frac{1}{e^{2u_2}} \left[\frac{\partial}{\partial u_1} (0) + \frac{\partial}{\partial u_2} (0) - \frac{\partial}{\partial u_3} (0) \right] \end{aligned}$$

$$\nabla \cdot \vec{A} = \frac{1}{e^{2u_2}} [(0) + (0) - (0)]$$

$$\nabla \cdot \vec{A} = 0$$

and

$$\begin{aligned}
 \nabla \times \bar{A} &= \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ A_1 h_1 & A_2 h_2 & A_3 h_3 \end{vmatrix} \\
 &= \frac{1}{(e^{u_2})(e^{u_2})(1)} \begin{vmatrix} e^{u_2} \hat{e}_1 & e^{u_2} \hat{e}_2 & \hat{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ u_3 e^{u_2} & 0 & -e^{u_2} \end{vmatrix} \\
 &= \frac{1}{e^{2u_2}} \left[e^{u_2} \hat{e}_1 \frac{\partial}{\partial u_2} (-e^{u_2} - 0) + e^{u_2} \hat{e}_2 \frac{\partial}{\partial u_3} (u_3 e^{u_2} - 0) + \hat{e}_3 (0 - \frac{\partial}{\partial u_2} (u_3 e^{u_2})) \right] \\
 &= \frac{1}{e^{2u_2}} \left[e^{u_2} \hat{e}_1 (-e^{u_2}) + e^{u_2} \hat{e}_2 (e^{u_2}) - (u_3 \hat{e}_3) \right] \\
 \nabla \times \bar{A} &= \frac{1}{e^{u_2}} \left[-e^{u_2} \hat{e}_1 + e^{u_2} \hat{e}_2 - u_3 \hat{e}_3 \right]
 \end{aligned}$$

Example: Find the expressions of gradient, divergence and curl in parabolic cylindrical coordinates defined by

$$x = \frac{1}{2} (u^2 - v^2) , y = uv , z = z$$

Solution: First we find scale factors

$$\text{here } u_1 = u , u_2 = v , u_3 = z$$

$$\vec{r} = \frac{1}{2} (u^2 - v^2) \hat{i} + uv \hat{j} + z \hat{k}$$

$$\Rightarrow \frac{\partial \vec{r}}{\partial u_1} = u \hat{i} + v \hat{j} , \frac{\partial \vec{r}}{\partial u_2} = -v \hat{i} + u \hat{j} , \frac{\partial \vec{r}}{\partial u_3} = \hat{k}$$

$$\Rightarrow h_1 = \sqrt{u^2 + v^2} , h_2 = \sqrt{u^2 + v^2} , h_3 = 1$$

$$\nabla \psi = \frac{1}{h_1} \frac{\partial \psi}{\partial u_1} \hat{e}_1 + \frac{1}{h_2} \frac{\partial \psi}{\partial u_2} \hat{e}_2 + \frac{1}{h_3} \frac{\partial \psi}{\partial u_3} \hat{e}_3$$

$$\nabla \psi = \frac{1}{\sqrt{u^2 + v^2}} \frac{\partial \psi}{\partial u} \hat{e}_1 + \frac{1}{\sqrt{u^2 + v^2}} \frac{\partial \psi}{\partial v} \hat{e}_2 + \frac{\partial \psi}{\partial u_3} \hat{e}_3$$

$$\nabla \cdot \vec{A} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} (h_2 h_3 A_1) + \frac{\partial}{\partial u_2} (h_3 h_1 A_2) + \frac{\partial}{\partial u_3} (h_1 h_2 A_3) \right]$$

$$= \frac{1}{(u^2 + v^2)} \left[\frac{\partial}{\partial u_1} (A_1 \sqrt{u^2 + v^2}) + \frac{\partial}{\partial u_2} (\sqrt{u^2 + v^2} A_2) + \frac{\partial}{\partial u_3} (A_3 (u^2 + v^2)) \right]$$

$$\nabla \times \vec{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \partial/\partial u_1 & \partial/\partial u_2 & \partial/\partial u_3 \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$

$$= \frac{1}{(u^2 + v^2)} \begin{vmatrix} \sqrt{u^2 + v^2} \hat{e}_1 & \sqrt{u^2 + v^2} \hat{e}_2 & \hat{e}_3 \\ \partial/\partial u & \partial/\partial v & \partial/\partial z \\ \sqrt{u^2 + v^2} A_1 & \sqrt{u^2 + v^2} A_2 & A_3 \end{vmatrix}$$

Example: Using the expressions of the last example evaluate the gradient of scalar field

$$\psi = u^2 + v^2 + z^2$$

$$\nabla \psi = \frac{1}{\sqrt{u^2 + v^2}} \frac{\partial \psi}{\partial u} \hat{e}_1 + \frac{1}{\sqrt{u^2 + v^2}} \frac{\partial \psi}{\partial v} \hat{e}_2 + \frac{\partial \psi}{\partial z} \hat{e}_3$$

$$\nabla \psi = \frac{2u}{\sqrt{u^2 + v^2}} \hat{e}_1 + \frac{2v}{\sqrt{u^2 + v^2}} \hat{e}_2 + 2z \hat{e}_3$$

Example. Find $\nabla \cdot \vec{A}$ and $\nabla \times \vec{A}$ in parabolic cylindrical coordinates system for the vector field

$$\vec{A} = u_3 \hat{e}_1 - u_2 \hat{e}_3.$$

Solution: $\vec{A} = z \hat{e}_1 - v \hat{e}_3$

$$\nabla \cdot \vec{A} = \frac{1}{u^2 + v^2} \left[\frac{\partial}{\partial u} (\sqrt{u^2 + v^2} z) + 0 + \frac{\partial}{\partial z} (-(u^2 + v^2) v) \right]$$

$$= \frac{1}{u^2 + v^2} z \left(\frac{1}{z} \right) \frac{1}{\sqrt{u^2 + v^2}} (2u)$$

$$\nabla \cdot \vec{A} = \frac{zu}{(u^2 + v^2)^{3/2}}$$

$$\nabla \times \vec{A} = \frac{1}{(u^2+v^2)} \begin{vmatrix} \sqrt{u^2+v^2} \hat{e}_1 & \sqrt{u^2+v^2} \hat{e}_2 & \hat{e}_3 \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial z} \\ \sqrt{u^2+v^2} z & 0 & -v \end{vmatrix}$$

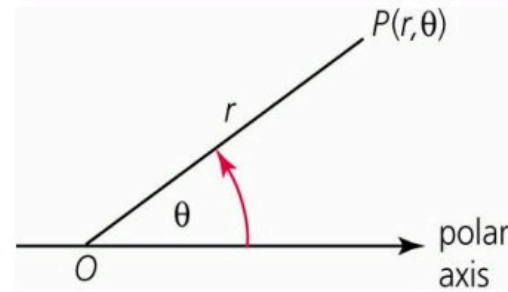
$$= \frac{1}{u^2+v^2} \left[\sqrt{u^2+v^2} \hat{e}_1 (-1) - \sqrt{u^2+v^2} \hat{e}_2 (-\sqrt{u^2+v^2}) + \hat{e}_3 \left(\frac{-zv}{\sqrt{u^2+v^2}} \right) \right]$$

$$= \frac{-1}{\sqrt{u^2+v^2}} \hat{e}_1 + \hat{e}_2 - \frac{zv}{(u^2+v^2)^{3/2}} \hat{e}_3$$

Polar Coordinate System

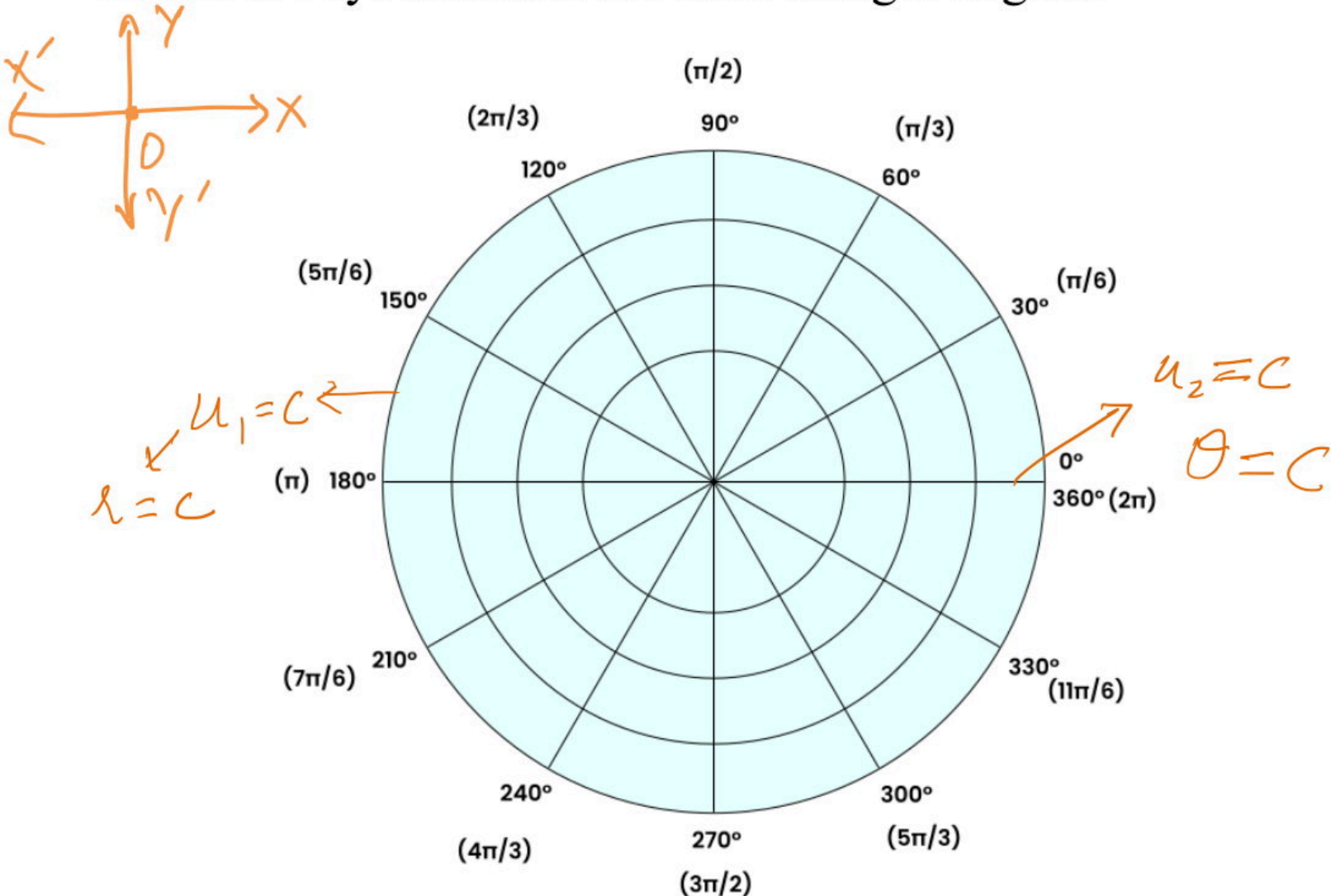
Polar coordinate system

The **polar coordinate** system is a two-dimensional coordinate system in which each point on a plane is determined by a distance “ r ” from the pole or reference point and an angle “ θ ”.



- The reference point is called **pole**.
- The distance r from the pole is called the **radial coordinate** and the angle is called the **angular coordinate**.

The coordinate lines for r are circles, those for θ are lines. The circles always intersect the lines at right angles.



Polar coordinates to rectangular coordinates

Following are the conversion formulas from polar to rectangular coordinates.

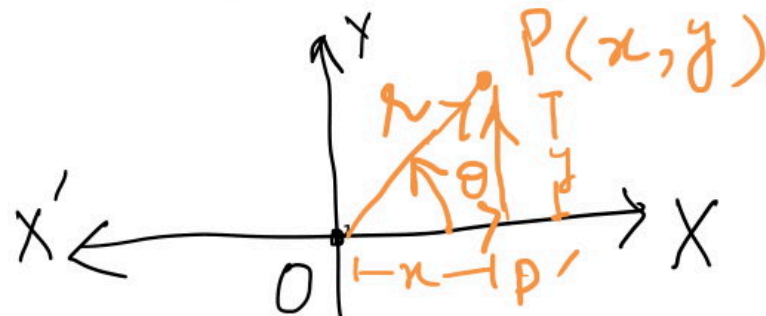
$$x = r \cos \theta$$

$$y = r \sin \theta$$

and

$$r^2 = x^2 + y^2 \Rightarrow r = \sqrt{x^2 + y^2}$$

$$\tan \theta = \frac{y}{x} \Rightarrow \theta = \tan^{-1} \frac{y}{x}$$



OPP' is right angle triangle.
 $\Rightarrow x = r \cos \theta$
 $y = r \sin \theta$

Example No 1:

Convert the polar coordinates $(3, \frac{\pi}{2})$ as rectangular coordinates.

$$\begin{aligned}x &= r \cos \theta \\&= 3 \cos \pi/2 \\&= 3(0) = 0\end{aligned}$$

$$\boxed{x = 0}$$

$$\begin{aligned}y &= r \sin \theta \\&= 3 \sin \pi/2 \\&= 3(1) = 3\end{aligned}$$

$$\boxed{y = 3}$$

The rectangular coordinates are (0,3).

Example No 2:

Convert the rectangular coordinates (3,3) to polar coordinate.

$$r^2 = x^2 + y^2$$

$$r^2 = (3)^2 + (3)^2 \quad \Rightarrow r^2 = 9 + 9$$

$$r = \sqrt{18} \quad \Rightarrow \boxed{r = 3\sqrt{2}}$$

$$\tan \theta = \frac{y}{x}$$

$$\tan \theta = \frac{3}{3} \quad \Rightarrow \tan \theta = 1$$

$$\theta = \tan^{-1}(1) \quad \Rightarrow \theta = \frac{\pi}{4}$$

Hence the polar coordinates are $(3\sqrt{2}, \frac{\pi}{4})$.

Example No 3

Convert the polar equation $4r \cos \theta + r \sin \theta = 8$ into a rectangular equation that expresses y in terms of x .

Solution: Using

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$4r \cos \theta + r \sin \theta = 8$$

$$4(x) + (y) = 8$$

$$4x + y = 8$$

$$y = 8 - 4x$$

Unit Vectors in Polar Coordinates:

The position vector in plane polar coordinates is:

$$\vec{r} = x\hat{i} + y\hat{j}$$

Using the transformation $x = r \cos \theta$, $y = r \sin \theta$

$$\vec{r} = r \cos \theta \hat{i} + r \sin \theta \hat{j}$$

$$\frac{\partial \vec{r}}{\partial r} = \cos \theta \hat{i} + \sin \theta \hat{j}$$

$$\frac{\partial \vec{r}}{\partial \theta} = -r \sin \theta \hat{i} + r \cos \theta \hat{j}$$

$$\hat{e}_1 = \frac{\frac{\partial \vec{r}}{\partial r}}{\left| \frac{\partial \vec{r}}{\partial r} \right|} = \cos \theta \hat{i} + \sin \theta \hat{j}$$

$$\hat{e}_2 = \frac{\frac{\partial \vec{r}}{\partial \theta}}{\left| \frac{\partial \vec{r}}{\partial \theta} \right|} = \frac{-r \sin \theta \hat{i} + r \cos \theta \hat{j}}{r} = -\sin \theta \hat{i} + \cos \theta \hat{j}$$

The scale factors are:

$$h_1 = \left| \frac{\partial \vec{r}}{\partial r} \right| = 1 \quad \text{and} \quad h_2 = \left| \frac{\partial \vec{r}}{\partial \theta} \right| = r$$

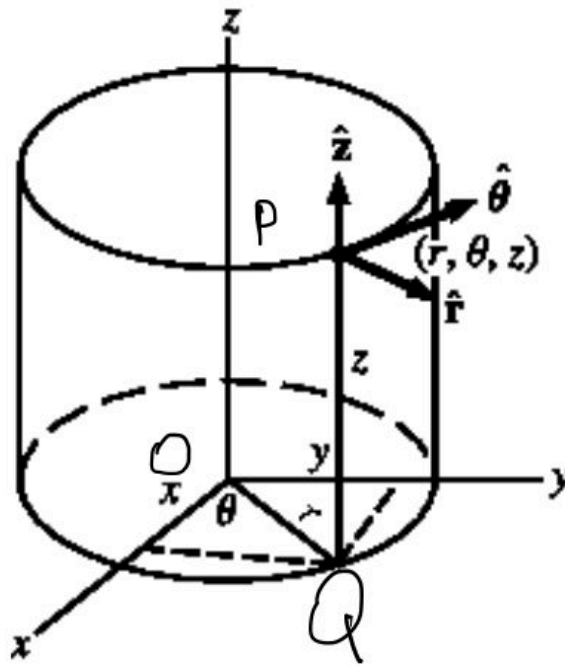
$$ds^2 = dr^2 + r^2 d\theta^2$$

Cylindrical Polar Coordinates

Cylindrical Polar Coordinates

Cylindrical coordinates simply combine the polar coordinates in the xy -plane with the usual z coordinate of Cartesian coordinate system. Hence cylindrical polar coordinates are represented by (r, θ, z) .

Let $P(x, y, z)$ be any point whose projection on the xy - plane is $Q(x, y)$. Then its cylindrical coordinates are (r, θ, z) .



Transformation Equations from Cylindrical coordinates to Cartesian coordinates:

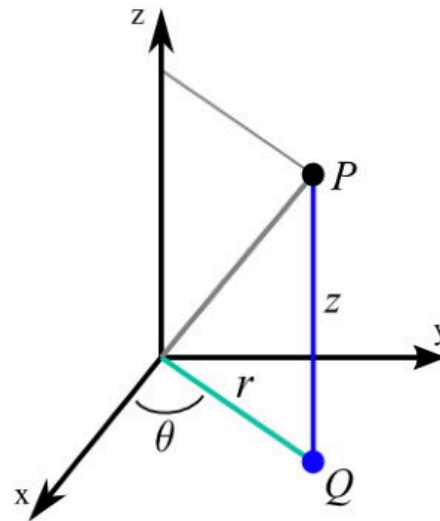
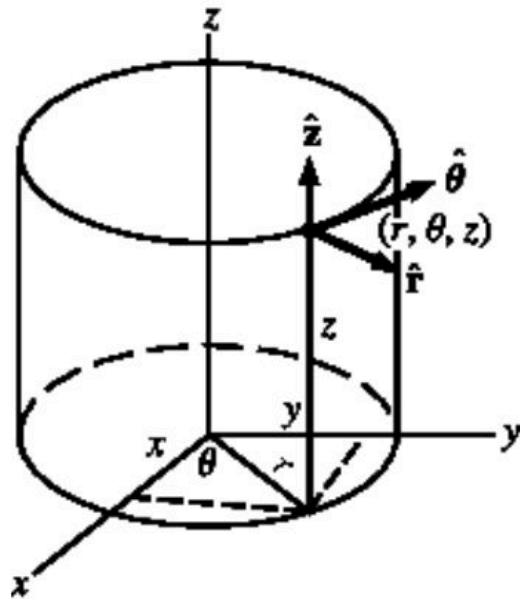
From this figure the transformation equations expressing the rectangular coordinates in terms of cylindrical coordinate:

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$z = z$$

Where $r \geq 0$, $0 \leq \theta \leq 2\pi$, and $-\infty < z < \infty$.



Transformation Equations from Cartesian coordinates to Cylindrical coordinates:

squaring and adding,

$$x = r \cos \theta$$

$$y = r \sin \theta$$

we have

$$x^2 + y^2 = r^2 (\cos^2 \theta + \sin^2 \theta) = r^2$$

$$\text{or} \quad r = \sqrt{x^2 + y^2}$$

Moreover:

$$\frac{y}{x} = \frac{r \cos \theta}{r \sin \theta} = \tan \theta$$

$$\text{or} \quad \theta = \tan^{-1} \frac{y}{x}$$

and $z = z$

Example. Convert the cylindrical coordinates $(4, \pi, 1)$ in rectangular coordinates.

Solution: Using the transformation:

$$x = r \cos \theta = 4 \cos \pi = -4$$

$$y = r \sin \theta = 4 \sin \pi = 0$$

$$z = z = 1$$

$$\Rightarrow (4, \pi, 1) \rightarrow (-4, 0, 1)$$

Example: Convert $(1, 1, -1)$ in
cylindrical coordinates.

using the inverse transformation:

$$\rho = \sqrt{x^2 + y^2} = \sqrt{1+1} = \sqrt{2}$$

$$\theta = \tan^{-1} \frac{y}{x} = \tan^{-1}(1) = \pi/4$$

$$z = -1$$

$$(1, 1, -1) \rightarrow (\sqrt{2}, \pi/4, -1)$$

Tangent vectors in cylindrical coordinate system:

The position vector of any point P in cylindrical polar coordinates is

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{r} = r \cos \theta \hat{i} + r \sin \theta \hat{j} + z\hat{k}$$

The tangent vectors in the direction of r, θ and z respectively, are given by

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\frac{\partial \vec{r}}{\partial r} = \cos \theta \hat{i} + \sin \theta \hat{j}$$

$$\frac{\partial \vec{r}}{\partial \theta} = -r \sin \theta \hat{i} + r \cos \theta \hat{j}$$

$$\frac{\partial \vec{r}}{\partial z} = \hat{k}$$

Unit tangent vectors in cylindrical coordinate system:

The unit vectors in these direction of r , θ and z are given by

$$\hat{e}_r = \frac{\frac{\partial \vec{r}}{\partial r}}{\left| \frac{\partial \vec{r}}{\partial r} \right|} = \frac{\cos \theta \hat{i} + \sin \theta \hat{j}}{\sqrt{\cos^2 \theta + \sin^2 \theta}} = \cos \theta \hat{i} + \sin \theta \hat{j}$$

$$\hat{e}_\theta = \frac{\frac{\partial \vec{r}}{\partial \theta}}{\left| \frac{\partial \vec{r}}{\partial \theta} \right|} = \frac{-r \sin \theta \hat{i} + r \cos \theta \hat{j}}{\sqrt{r^2 \sin^2 \theta + r^2 \cos^2 \theta}} = -\sin \theta \hat{i} + \cos \theta \hat{j}$$

$$\hat{e}_z = \frac{\frac{\partial \vec{r}}{\partial z}}{\left| \frac{\partial \vec{r}}{\partial z} \right|} = \hat{k}$$

In matrix notation, we have

$$\begin{bmatrix} \hat{e}_r \\ \hat{e}_\theta \\ \hat{e}_z \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix}$$

Scale factors:

The scale factor for the cylindrical coordinates are given by

$$h_1 = h_r = \left| \frac{\partial \vec{r}}{\partial r} \right| = \sqrt{\cos^2 \theta + \sin^2 \theta} = 1$$

$$\begin{aligned} h_2 = h_\theta &= \left| \frac{\partial \vec{r}}{\partial \theta} \right| = \sqrt{(-r \sin \theta)^2 + (r \cos \theta)^2} \\ &= r \sqrt{\cos^2 \theta + \sin^2 \theta} = r \end{aligned}$$

$$h_3 = h_z = \left| \frac{\partial \vec{r}}{\partial z} \right| = |k| = 1$$

hence the scale factor are:

$$\boxed{h_1 = h_r = 1}$$

$$\boxed{h_2 = h_\theta = r}$$

$$\boxed{h_3 = h_z = 1}$$

Question: In cylindrical polar coordinates, show that.

$$\frac{\partial \hat{e}_r}{\partial r} = 0, \frac{\partial \hat{e}_r}{\partial \theta} = \hat{e}_\theta, \frac{\partial \hat{e}_r}{\partial z} = 0$$

$$\frac{\partial \hat{e}_\theta}{\partial r} = 0, \frac{\partial \hat{e}_\theta}{\partial \theta} = -\hat{e}_r, \frac{\partial \hat{e}_\theta}{\partial z} = 0$$

$$\frac{\partial \hat{e}_z}{\partial r} = 0, \frac{\partial \hat{e}_z}{\partial \theta} = 0, \frac{\partial \hat{e}_z}{\partial z} = 0$$

Solution: In cylindrical coordinates

$$\hat{e}_r = \cos\theta \hat{i} + \sin\theta \hat{j}$$

$$\Rightarrow \frac{\partial \hat{e}_r}{\partial r} = 0, \frac{\partial \hat{e}_r}{\partial \theta} = -\sin\theta \hat{i} + \cos\theta \hat{j} = \hat{e}_\theta$$

$$\text{and } \frac{\partial \hat{e}_r}{\partial z} = 0$$

Question: Prove that in cylindrical polar coordinates

$$\frac{d}{dt}\hat{e}_r = \dot{\theta}\hat{e}_\theta, \quad \frac{d}{dt}\hat{e}_\theta = -\dot{\theta}\hat{e}_r, \quad \frac{d}{dt}\hat{e}_z = \bar{0}$$

Where dot denote differentiation w.r.t time t.

$$\hat{e}_r = \cos\theta\hat{i} + \sin\theta\hat{j}, \quad \hat{e}_\theta = -\sin\theta\hat{i} + \cos\theta\hat{j}, \quad \hat{e}_z = \hat{k}$$

$$\frac{d}{dt}\hat{e}_r = -\sin\theta\frac{d\theta}{dt}\hat{i} + \cos\theta\frac{d\theta}{dt}\hat{j} = (-\sin\theta\hat{i} + \cos\theta\hat{j})\frac{d\theta}{dt} = \dot{\theta}\hat{e}_\theta,$$

$$\frac{d}{dt}\hat{e}_\theta = -\cos\theta\frac{d\theta}{dt}\hat{i} - \sin\theta\frac{d\theta}{dt}\hat{j} = -(\cos\theta\hat{i} + \sin\theta\hat{j})\frac{d\theta}{dt} = -\dot{\theta}\hat{e}_r,$$

$$\frac{d}{dt}\hat{e}_z = \frac{d}{dt}\hat{k} = \bar{0}$$

$$\frac{d\hat{e}_r}{dt} = \frac{\partial\hat{e}_r}{\partial r}\frac{dr}{dt} + \frac{\partial\hat{e}_r}{\partial\theta}\frac{d\theta}{dt} + \frac{\partial\hat{e}_r}{\partial z}\frac{dz}{dt}$$

Arc length element:

In cylindrical polar coordinates, the elements of arc length are determined from

$$u_1 = r, \quad u_2 = \theta, \quad u_3 = z, \quad h_1 = 1, \quad h_2 = r, \quad h_3 = 1$$

$$(ds)^2 = h_1^2 (du_1)^2 + h_2^2 (du_2)^2 + h_3^2 (du_3)^2$$

In cylindrical polar coordinates, these becomes

$$(ds)^2 = (1)^2 (dr)^2 + (r)^2 (d\theta)^2 + (1)^2 (dz)^2$$

$$(ds)^2 = (dr)^2 + r^2 (d\theta)^2 + (dz)^2$$

Area elements

we know that the elements of area in an orthogonal curvilinear coordinates are:

$$dA_1 = h_2 h_3 du_2 du_3, \quad dA_2 = h_1 h_3 du_1 du_3, \quad dA_3 = h_1 h_2 du_1 du_2$$

In cylindrical polar coordinates, we have

$$dA_1 = (r)(1)d\theta dz = r d\theta dz$$

$$dA_2 = (1)(1)dr dz = dr dz$$

$$dA_3 = (1)(r)dr d\theta = r dr d\theta$$

Volume element

$$dv = h_1 h_2 h_3 du_1 du_2 du_3$$

In cylindrical polar coordinate, we *have*

$$dv = (1)(r)(1)dr d\theta dz = r dr d\theta dz$$

Expression for Jacobian in cylindrical polar coordinates:

We know that the Jacobian in orthogonal curvilinear coordinates

u_1, u_2, u_3 is given by

$$J = J\left(\frac{x, y, z}{u_1, u_2, u_3}\right) = h_1 h_2 h_3$$

$$h_1 = 1, \quad h_2 = r, \quad h_3 = 1 \quad \text{and} \quad u_1 = r, \quad u_2 = \theta, \quad u_3 = z$$

$$J\left(\frac{x, y, z}{r, \theta, z}\right) = (1)(r)(1) = r$$

Example: Represent the vector

$$\vec{A} = z\hat{i} - 2x\hat{j} + y\hat{k}$$

in terms of cylindrical coordinates.

Solution: Consider the transformation

$$x = r \cos \theta, \quad y = r \sin \theta, \quad z = z$$

and

$$\hat{e}_r = \cos \theta \hat{i} + \sin \theta \hat{j}, \quad \hat{e}_\theta = -\sin \theta \hat{i} + \cos \theta \hat{j}, \quad \hat{e}_z = \hat{k}$$

Implies:

$$\cos \theta \hat{e}_r - \sin \theta \hat{e}_\theta = \hat{i}$$

and

$$\sin \theta \hat{e}_r + \cos \theta \hat{e}_\theta = \hat{j}$$

Replacing the values of x, y, z and $\hat{i}, \hat{j}, \hat{k}$ in terms of cylindrical coordinates. we have

$$\vec{A} = z(\cos\theta \hat{e}_r - \sin\theta \hat{e}_\theta) - 2r\cos\theta(\sin\theta \hat{e}_r + \cos\theta \hat{e}_\theta) + r\sin\theta \hat{e}_z$$

$$\begin{aligned}\vec{A} &= (z\cos\theta - 2r\sin\theta\cos\theta)\hat{e}_r \\ &\quad - (z\sin\theta - 2r\cos^2\theta)\hat{e}_\theta \\ &\quad + (r\sin\theta)\hat{e}_z\end{aligned}$$

Cylindrical Coordinate System

Example: Find the expressions of $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ in cylindrical coordinate system.

$$x = r \cos \theta, \quad y = r \sin \theta, \quad z = z \quad \text{and}$$

$$r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1}\left(\frac{y}{x}\right), \quad z = z$$

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}} = \frac{r \cos \theta}{r} = \cos \theta$$

$$\frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}} = \frac{r \sin \theta}{r} = \sin \theta$$

$$\frac{\partial r}{\partial z} = 0$$

$$\frac{\partial \theta}{\partial x} = -\frac{y}{x^2 + y^2} = -\frac{r \sin \theta}{r^2} = -\frac{\sin \theta}{r}$$

$$\frac{\partial \theta}{\partial y} = \frac{x}{x^2 + y^2} = \frac{r \cos \theta}{r^2} = \frac{\cos \theta}{r}$$

$$\frac{\partial \theta}{\partial z} = 0$$

$$\frac{\partial z}{\partial x} = 0, \quad \frac{\partial z}{\partial y} = 0, \quad \frac{\partial z}{\partial z} = 1$$

Now by the chain rule, we have

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta} \frac{\partial \theta}{\partial x} + \frac{\partial}{\partial z} \frac{\partial z}{\partial x} = \cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} + (0) \frac{\partial}{\partial z}$$

$$= \cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta}$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial}{\partial \theta} \frac{\partial \theta}{\partial y} + \frac{\partial}{\partial z} \frac{\partial z}{\partial y} = \sin \theta \frac{\partial}{\partial r} + \frac{\cos \theta}{r} \frac{\partial}{\partial \theta} + (0) \frac{\partial}{\partial z}$$

$$= \sin \theta \frac{\partial}{\partial r} + \frac{\cos \theta}{r} \frac{\partial}{\partial \theta}$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial r} \frac{\partial r}{\partial z} + \frac{\partial}{\partial \theta} \frac{\partial \theta}{\partial z} + \frac{\partial}{\partial z} \frac{\partial z}{\partial z} = (0) \frac{\partial}{\partial r} + (0) \frac{\partial}{\partial \theta} + (1) \frac{\partial}{\partial z}$$

$$= \frac{\partial}{\partial z}$$

Expression for gradient, divergence, curl, and Laplacian in cylindrical polar coordinates:

Cylindrical polar coordinates are (r, θ, z) .

$$h_1 = 1, \quad h_2 = r, \quad h_3 = 1,$$

Gradient:

In orthogonal curvilinear coordinates, we have

$$\nabla \psi = \frac{1}{h_1} \frac{\partial \psi}{\partial u_1} \hat{e}_1 + \frac{1}{h_2} \frac{\partial \psi}{\partial u_2} \hat{e}_2 + \frac{1}{h_3} \frac{\partial \psi}{\partial u_3} \hat{e}_3$$

In cylindrical polar coordinates, this becomes

$$\nabla \psi = \frac{1}{1} \frac{\partial \psi}{\partial r} \hat{e}_r + \frac{1}{r} \frac{\partial \psi}{\partial \theta} \hat{e}_\theta + \frac{1}{1} \frac{\partial \psi}{\partial z} \hat{e}_z$$

$$\nabla \psi = \frac{\partial \psi}{\partial r} \hat{e}_r + \frac{1}{r} \frac{\partial \psi}{\partial \theta} \hat{e}_\theta + \frac{\partial \psi}{\partial z} \hat{e}_z$$

Divergence:

In orthogonal curvilinear coordinates, we have

$$\nabla \cdot \vec{A} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} (A_1 h_2 h_3) + \frac{\partial}{\partial u_2} (A_2 h_3 h_1) + \frac{\partial}{\partial u_3} (A_3 h_1 h_2) \right]$$

In cylindrical polar coordinates, we have

$$\nabla \cdot \vec{A} = \frac{1}{(1)(r)(1)} \left[\frac{\partial}{\partial r} ((r)(1)A_r) + \frac{\partial}{\partial \theta} ((1)(1)A_\theta) + \frac{\partial}{\partial z} ((1)(r)A_z) \right]$$

$$\nabla \cdot \vec{A} = \frac{1}{r} \left[\frac{\partial}{\partial r} (rA_r) + \frac{\partial A_\theta}{\partial \theta} + \frac{\partial}{\partial z} (rA_z) \right]$$

Curl:

In orthogonal curvilinear coordinates, we have

$$\nabla \times \vec{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ A_1 h_1 & A_2 h_2 & A_3 h_3 \end{vmatrix}$$

In cylindrical polar coordinates, we have

$$\nabla \times \vec{A} = \frac{1}{r} \begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ A_r & rA_\theta & A_z \end{vmatrix}$$

$$\nabla \times \vec{A} = \frac{1}{r} \left[\left(\frac{\partial A_z}{\partial \theta} - \frac{\partial}{\partial z} (rA_\theta) \right) \hat{e}_r + \left(r \frac{\partial A_r}{\partial z} - r \frac{\partial A_z}{\partial r} \right) \hat{e}_\theta + \left(\frac{\partial}{\partial r} (rA_\theta) - \frac{\partial A_r}{\partial z} \right) \hat{e}_z \right]$$

Laplacian:

In orthogonal curvilinear coordinates, we have

$$\nabla^2 \psi = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial \psi}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left(\frac{h_3 h_1}{h_2} \frac{\partial \psi}{\partial u_2} \right) + \frac{\partial}{\partial u_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial \psi}{\partial u_3} \right) \right]$$

In cylindrical polar coordinates, it becomes

$$\nabla^2 \psi = \frac{1}{(1)(r)(1)} \left[\frac{\partial}{\partial r} \left(\frac{(r)(1)h_3}{(1)} \frac{\partial \psi}{\partial r} \right) + \frac{\partial}{\partial \theta} \left(\frac{(1)(1)}{(r)} \frac{\partial \psi}{\partial \theta} \right) + \frac{\partial}{\partial z} \left(\frac{(1)(r)}{(1)} \frac{\partial \psi}{\partial z} \right) \right]$$

$$\nabla^2 \psi = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} + \frac{\partial^2 \psi}{\partial z^2}$$

Example: Express the velocity $\vec{v}$ and acceleration $\vec{a}$ of a particle in cylindrical coordinates.

Solution:

$$\vec{r} = x \hat{i} + y \hat{j} + z \hat{k}$$

$$\Rightarrow \vec{r} = r \cos \theta (\cos \theta \hat{e}_r - \sin \theta \hat{e}_\theta) + r \sin \theta (\sin \theta \hat{e}_r + \cos \theta \hat{e}_\theta) + z \hat{e}_z$$

$$= (r \cos^2 \theta + r \sin^2 \theta) \hat{e}_r + (r \cos \theta \sin \theta - r \cos \theta \sin \theta) \hat{e}_\theta + z \hat{e}_z$$

$$\Rightarrow \vec{r} = r \hat{e}_r + z \hat{e}_z$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dr}{dt} \hat{e}_r + r \frac{d}{dt} (\hat{e}_r) + \frac{dz}{dt} \hat{e}_z + z \frac{d\hat{e}_z}{dt}$$

$$\vec{v} = \dot{r} \hat{e}_r + r (\dot{\theta} \hat{e}_\theta) + \dot{z} \hat{e}_z$$

Now consider:

$$\vec{v} = \dot{r} \hat{e}_r + r \dot{\theta} \hat{e}_\theta + \dot{z} \hat{e}_z$$

$$\Rightarrow \vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt} (\dot{r} \hat{e}_r + r \dot{\theta} \hat{e}_\theta + \dot{z} \hat{e}_z)$$

$$\vec{a} = \ddot{r} \hat{e}_r + \dot{r} \frac{d}{dt} (\hat{e}_r) + \dot{r} \dot{\theta} \hat{e}_\theta + r \ddot{\theta} \hat{e}_\theta + r \dot{\theta} \frac{d}{dt} (\hat{e}_\theta) + \ddot{z} \hat{e}_z$$

$$= \ddot{r} \hat{e}_r + \dot{r} \dot{\theta} \hat{e}_\theta + \dot{r} \dot{\theta} \hat{e}_\theta + r \ddot{\theta} \hat{e}_\theta + r \dot{\theta} (-\dot{\theta} \hat{e}_r) + \ddot{z} \hat{e}_z$$

$$\vec{a} = (\ddot{r} - r \dot{\theta}^2) \hat{e}_r + (r \ddot{\theta} + 2\dot{r}\dot{\theta}) \hat{e}_\theta + \ddot{z} \hat{e}_z$$

Example: Express heat equation in cylindrical coordinates.

$$\frac{\partial u}{\partial t} = \alpha \nabla^2 u$$

Solution:

Replacing ∇^2 in cylindrical coordinates we have:

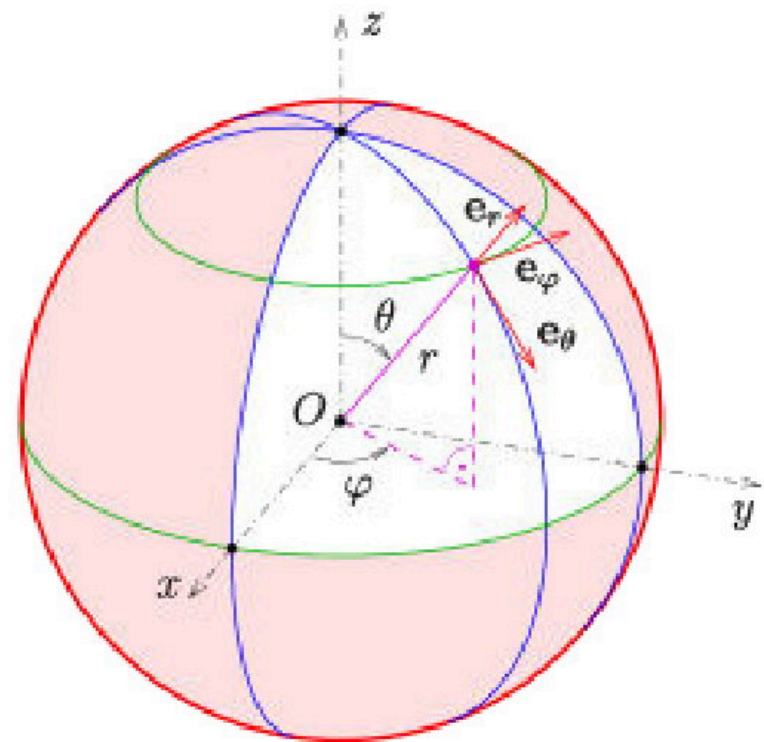
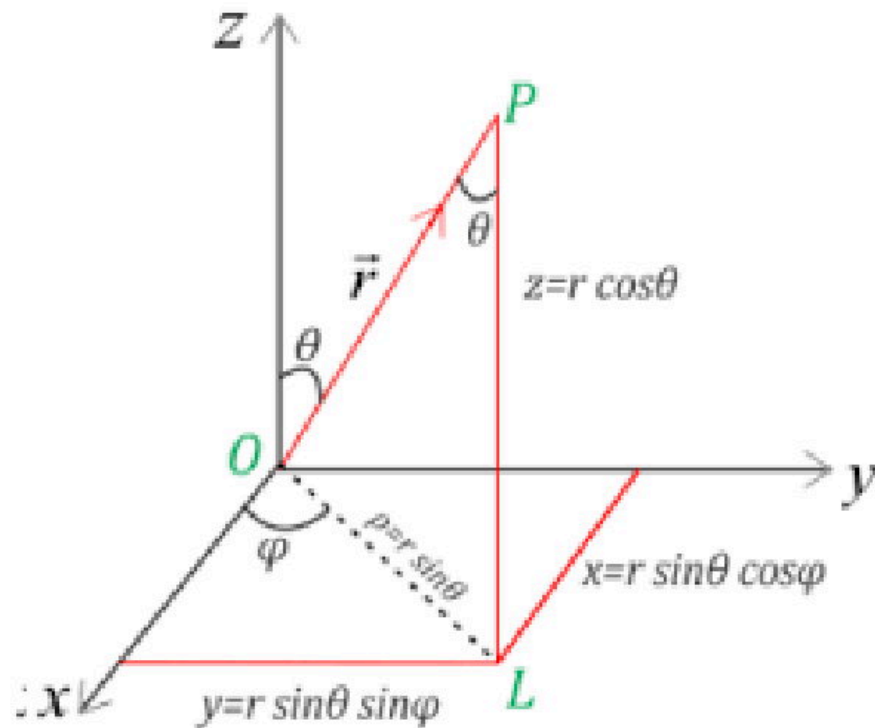
$$\frac{\partial u}{\partial t} = \alpha \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} \right]$$

Spherical Coordinate System

Spherical Polar Coordinates

Let $P(x, y, z)$ be any point whose projection on the xy - plane is $L(x, y, 0)$.

Then the spherical polar coordinates of P are (r, θ, ϕ) in which $\vec{r} = \overrightarrow{OP}$.



Transformation equations between spherical and rectangular coordinates

The transformation equations expressing the rectangular coordinates in terms of spherical polar coordinates are

$$x = r \sin \theta \cos \phi$$

$$y = r \sin \theta \sin \phi$$

$$z = r \cos \theta$$

squaring and adding above equations

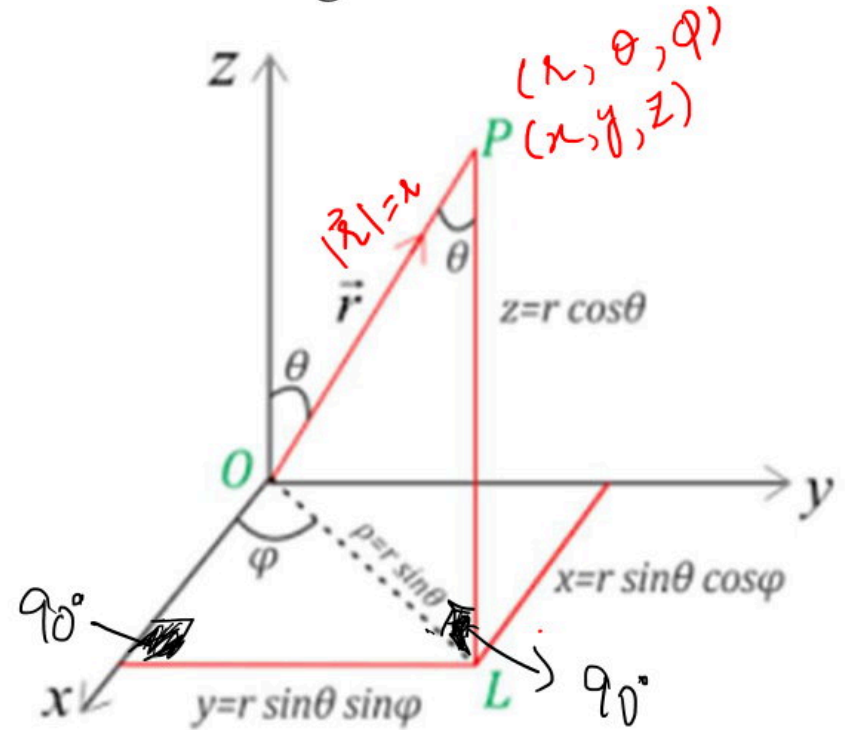
$$x^2 + y^2 + z^2 = r^2 \sin^2 \theta \cos^2 \phi + r^2 \sin^2 \theta \sin^2 \phi + r^2 \cos^2 \theta$$

$$= r^2 \sin^2 \theta (\cos^2 \phi + \sin^2 \phi) + r^2 \cos^2 \theta$$

$$= r^2 \sin^2 \theta + r^2 \cos^2 \theta = r^2 (\sin^2 \theta + \cos^2 \theta) = r^2$$

$\Rightarrow$

$$\boxed{r = \sqrt{x^2 + y^2 + z^2}}$$



squaring and adding first two equations, we get

$$\begin{aligned}x^2 + y^2 &= r^2 \sin^2 \theta \cos^2 \phi + r^2 \sin^2 \theta \sin^2 \phi \\&= r^2 \sin^2 \theta (\cos^2 \phi + \sin^2 \phi) = r^2 \sin^2 \theta\end{aligned}$$

or $r \sin \theta = \sqrt{x^2 + y^2}$

Divide this equation, by third equation, we *have*

$$\frac{r \sin \theta}{r \cos \theta} = \frac{\sqrt{x^2 + y^2}}{z} \quad \text{or} \quad \tan \theta = \frac{\sqrt{x^2 + y^2}}{z}$$

$$\Rightarrow \boxed{\theta = \tan^{-1} \frac{\sqrt{x^2 + y^2}}{z}}$$

Now Dividing first two equations

$$\frac{r \sin \theta \sin \phi}{r \sin \theta \cos \phi} = \frac{y}{x} \quad \text{or} \quad \tan \phi = \frac{y}{x}$$

$$\Rightarrow \boxed{\phi = \tan^{-1} \frac{y}{x}}$$

Where $r \geq 0, 0 \leq \theta \leq \pi, 0 \leq \phi \leq 2\pi$

Unit vectors :

The position vector of point $P(x, y, z)$ in spherical polar coordinates is given by

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{r} = r \sin \theta \cos \phi \hat{i} + r \sin \theta \sin \phi \hat{j} + r \cos \theta \hat{k}$$

The tangent vectors in the direction of r, θ and ϕ respectively are given by

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{r} = r \sin \theta \cos \phi \hat{i} + r \sin \theta \sin \phi \hat{j} + r \cos \theta \hat{k}$$

$$\frac{\partial \vec{r}}{\partial r} = \sin \theta \cos \phi \hat{i} + \sin \theta \sin \phi \hat{j} + \cos \theta \hat{k}$$

$$\frac{\partial \vec{r}}{\partial \theta} = r \cos \theta \cos \phi \hat{i} + r \cos \theta \sin \phi \hat{j} - r \sin \theta \hat{k}$$

$$\frac{\partial \vec{r}}{\partial \phi} = -r \sin \theta \sin \phi \hat{i} + r \sin \theta \cos \phi \hat{j}$$

$$\hat{e}_r = \frac{\frac{\partial \bar{r}}{\partial r}}{\left| \frac{\partial \bar{r}}{\partial r} \right|} = \frac{\sin \theta \cos \phi \hat{i} + \sin \theta \sin \phi j + \cos \theta k}{\sqrt{\sin^2 \theta \cos^2 \phi + \sin^2 \theta \sin^2 \phi + \cos^2 \theta}}$$

$$\hat{e}_r = \sin \theta \cos \phi \hat{i} + \sin \theta \sin \phi j + \cos \theta k$$

$$\hat{e}_\theta = \frac{\frac{\partial \bar{r}}{\partial \theta}}{\left| \frac{\partial \bar{r}}{\partial \theta} \right|} = \frac{r \cos \theta \cos \phi \hat{i} + r \cos \theta \sin \phi j - r \sin \theta k}{\sqrt{r^2 \cos^2 \theta \cos^2 \phi + r^2 \cos^2 \theta \sin^2 \phi + r^2 \sin^2 \theta}}$$

$$\hat{e}_\theta = \cos \theta \cos \phi \hat{i} + \cos \theta \sin \phi j - \sin \theta k$$

$$\hat{e}_\phi = \frac{\frac{\partial \bar{r}}{\partial \phi}}{\left| \frac{\partial \bar{r}}{\partial \phi} \right|} = \frac{-r \sin \theta \sin \phi \hat{i} + r \sin \theta \cos \phi j}{\sqrt{r^2 \sin^2 \theta \sin^2 \phi + r^2 \sin^2 \theta \cos^2 \phi}}$$

$$\hat{e}_\phi = -\sin \phi \hat{i} + \cos \phi j$$

In matrix notation

$$\begin{bmatrix} \hat{e}_r \\ \hat{e}_\theta \\ \hat{e}_\phi \end{bmatrix} = \begin{bmatrix} \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\ \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\ -\sin \phi & \cos \phi & 0 \end{bmatrix} \begin{bmatrix} \hat{i} \\ j \\ k \end{bmatrix}$$

Scale factors:

The scales factors for the spherical coordinate system are given by.

$$\begin{aligned}h_1 = h_r &= \left| \frac{\partial \vec{r}}{\partial r} \right| = \sqrt{\sin^2 \theta \cos^2 \phi + \sin^2 \theta \sin^2 \phi + \cos^2 \theta} \\&= \sqrt{\sin^2 \theta (\cos^2 \phi + \sin^2 \phi) + \cos^2 \theta} \\&= \sqrt{\sin^2 \theta + \cos^2 \theta} = 1\end{aligned}$$

$$\boxed{h_1 = h_r = 1}$$

$$\begin{aligned}h_2 = h_\theta &= \sqrt{r^2 \cos^2 \theta \cos^2 \phi + r^2 \cos^2 \theta \sin^2 \phi + r^2 \sin^2 \theta} \\&= \sqrt{r^2 \cos^2 \theta (\cos^2 \phi + \sin^2 \phi) + r^2 \sin^2 \theta} \\&= \sqrt{r^2 \cos^2 \theta + r^2 \sin^2 \theta} = r\end{aligned}$$

$$\boxed{h_2 = h_\theta = r}$$

$$\begin{aligned}h_3 = h_\phi &= \sqrt{r^2 \sin^2 \theta \sin^2 \phi + r^2 \sin^2 \theta \cos^2 \phi} \\&= \sqrt{r^2 \sin^2 \theta (\sin^2 \phi + \cos^2 \phi)} \\&= \sqrt{r^2 \sin^2 \theta} = r \sin \theta\end{aligned}$$

$$\boxed{h_3 = h_\phi = r \sin \theta}$$

Arc length element:

In orthogonal curvilinear coordinates, the elements of arc length is determined by

$$(ds)^2 = h_1^2 (du_1)^2 + h_2^2 (du_2)^2 + h_3^2 (du_3)^2$$

In cylindrical polar coordinates, these becomes

$$\begin{aligned}(ds)^2 &= (1)^2 (dr)^2 + (r)^2 (d\theta)^2 + (r \sin \theta)^2 (d\phi)^2 \\ &= (dr)^2 + r^2 (d\theta)^2 + r^2 \sin^2 \theta (d\phi)^2\end{aligned}$$

Area element

we know that the elements of area in an orthogonal curvilinear coordinates are :

$$dA_1 = h_2 h_3 du_2 du_3, \quad dA_2 = h_1 h_3 du_1 du_3, \quad dA_3 = h_1 h_2 du_1 du_2$$

In cylindrical polar coordinates, these becomes

$$dA_1 = (r)(r \sin \theta) d\theta d\phi = r^2 \sin \theta d\theta d\phi$$

$$dA_2 = (1)(r \sin \theta) dr d\phi = r \sin \theta dr d\phi$$

$$dA_3 = (1)(r) dr d\theta = r dr d\theta$$

Volume element

$$dv = h_1 h_2 h_3 du_1 du_2 du_3$$

In spherical polar coordinate, this becomes

$$dv = (1)(r)(r \sin \theta) dr d\theta d\phi = r^2 \sin \theta dr d\theta dz$$

Expression for Jacobian in spherical polar coordinates:

We know that the Jacobian in orthogonal curvilinear coordinates u_1, u_2, u_3 is given by

$$J\left(\frac{x, y, z}{u_1, u_2, u_3}\right) = h_1 h_2 h_3$$

In spherical polar coordinates $h_1 = 1, h_2 = r, h_3 = r \sin \theta$ and $u_1 = r, u_2 = \theta, u_3 = \phi$

$$\text{Therefore } J\left(\frac{x, y, z}{u_1, u_2, u_3}\right) = (1)(r)(r \sin \theta) = r^2 \sin \theta$$

Expression for gradient, divergence, curl and Laplacian for spherical polar coordinates:

we know that for spherical polar coordinates (r, θ, ϕ)

$$\begin{aligned} u_1 = r, & \quad u_2 = \theta, & \quad u_3 = \phi; & \quad \hat{e}_1 = \hat{e}_r, & \quad \hat{e}_2 = \hat{e}_\theta, & \quad \hat{e}_3 = \hat{e}_\phi \\ h_1 = h_r = 1, & \quad h_2 = h_\theta = r, & \quad h_3 = h_\phi = r \sin \theta \\ A_1 = A_r & \quad A_2 = A_\theta & \quad A_3 = A_\phi \end{aligned}$$

Gradient:

In orthogonal curvilinear coordinates, we have

$$\nabla \psi = \frac{1}{h_1} \frac{\partial \psi}{\partial u_1} \hat{e}_1 + \frac{1}{h_2} \frac{\partial \psi}{\partial u_2} \hat{e}_2 + \frac{1}{h_3} \frac{\partial \psi}{\partial u_3} \hat{e}_3$$

In spherical polar coordinates it becomes

$$\nabla \psi = \frac{1}{r} \frac{\partial \psi}{\partial r} \hat{e}_r + \frac{1}{r} \frac{\partial \psi}{\partial \theta} \hat{e}_\theta + \frac{1}{r \sin \theta} \frac{\partial \psi}{\partial \phi} \hat{e}_\phi$$

$$\nabla \psi = \frac{\partial \psi}{\partial r} \hat{e}_r + \frac{1}{r} \frac{\partial \psi}{\partial \theta} \hat{e}_\theta + \frac{1}{r \sin \theta} \frac{\partial \psi}{\partial \phi} \hat{e}_\phi$$

Divergence:

In orthogonal curvilinear coordinates, we have

$$\nabla \cdot \vec{A} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} (A_1 h_2 h_3) + \frac{\partial}{\partial u_2} (A_2 h_3 h_1) + \frac{\partial}{\partial u_3} (A_3 h_1 h_2) \right]$$

In spherical polar coordinates this becomes

$$\nabla \cdot \vec{A} = \frac{1}{(1)(r)(r \sin \theta)} \left[\frac{\partial}{\partial r} (r)(r \sin \theta) A_r + \frac{\partial}{\partial \theta} ((r \sin \theta)(1) A_\theta) + \frac{\partial}{\partial \phi} (1)(r)(A_\phi) \right]$$

$$\nabla \cdot \vec{A} = \frac{1}{(r^2 \sin \theta)} \left[\frac{\partial}{\partial r} (r^2 \sin \theta) A_r + \frac{\partial}{\partial \theta} (r \sin \theta A_\theta) + \frac{\partial}{\partial \phi} (r A_\phi) \right]$$

$$\nabla \cdot \vec{A} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (r \sin \theta A_\theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} (r A_\phi)$$

Curl:

In orthogonal curvilinear coordinates, we have

$$\nabla \times \bar{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{e}_1 & h_2 \hat{e}_2 & h_3 \hat{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ A_1 h_1 & A_2 h_2 & A_3 h_3 \end{vmatrix}$$

In spherical polar coordinates, this becomes

$$\nabla \times \bar{A} = \frac{1}{(1)(r)(r \sin \theta)} \begin{vmatrix} \hat{e}_r & r \hat{e}_\theta & r \sin \theta \hat{e}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ A_r & r A_\theta & r \sin \theta A_\phi \end{vmatrix}$$

$$\nabla \times \bar{A} = \frac{1}{(r^2 \sin \theta)} \left[\begin{aligned} & \hat{e}_r \left\{ \frac{\partial}{\partial \theta} (r \sin \theta A_\phi) - \frac{\partial}{\partial \phi} (r A_\theta) \right\} - r \hat{e}_\theta \left\{ \frac{\partial}{\partial r} (r \sin \theta A_\phi) - \frac{\partial}{\partial \phi} (A_r) \right\} \\ & + \left\{ \frac{\partial}{\partial r} (r A_\theta) - \frac{\partial A_r}{\partial \phi} \right\} r \sin \theta \hat{e}_\phi \end{aligned} \right]$$

Laplacian:

In orthogonal curvilinear coordinates, we have

$$\nabla^2 \psi = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial \psi}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left(\frac{h_3 h_1}{h_2} \frac{\partial \psi}{\partial u_2} \right) + \frac{\partial}{\partial u_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial \psi}{\partial u_3} \right) \right]$$

In spherical polar coordinates, this becomes

$$\nabla^2 \psi = \frac{1}{(1)(r)(r \sin \theta)} \left[\frac{\partial}{\partial r} \left(\frac{(r)(r \sin \theta)}{(1)} \frac{\partial \psi}{\partial r} \right) + \frac{\partial}{\partial \theta} \left(\frac{(r \sin \theta)(1)}{r} \frac{\partial \psi}{\partial \theta} \right) + \frac{\partial}{\partial \phi} \left(\frac{(1)(r)}{(r \sin \theta)} \frac{\partial \psi}{\partial \phi} \right) \right]$$

$$\nabla^2 \psi = \frac{1}{(r^2 \sin \theta)} \left[(r \sin \theta) \frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi}{\partial r} \right) + \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{(\sin \theta)} \frac{\partial^2 \psi}{\partial \phi^2} \right]$$

$$\boxed{\nabla^2 \psi = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial^2 \psi}{\partial \phi^2}}$$

Spherical Coordinate System

Transformation equations from cylindrical to spherical system:

If point P has cylindrical polar coordinates (ρ, φ, z) then

$$\rho = \sqrt{x^2 + y^2} \quad \varphi = \tan^{-1} \frac{y}{x}$$

$$z = z$$

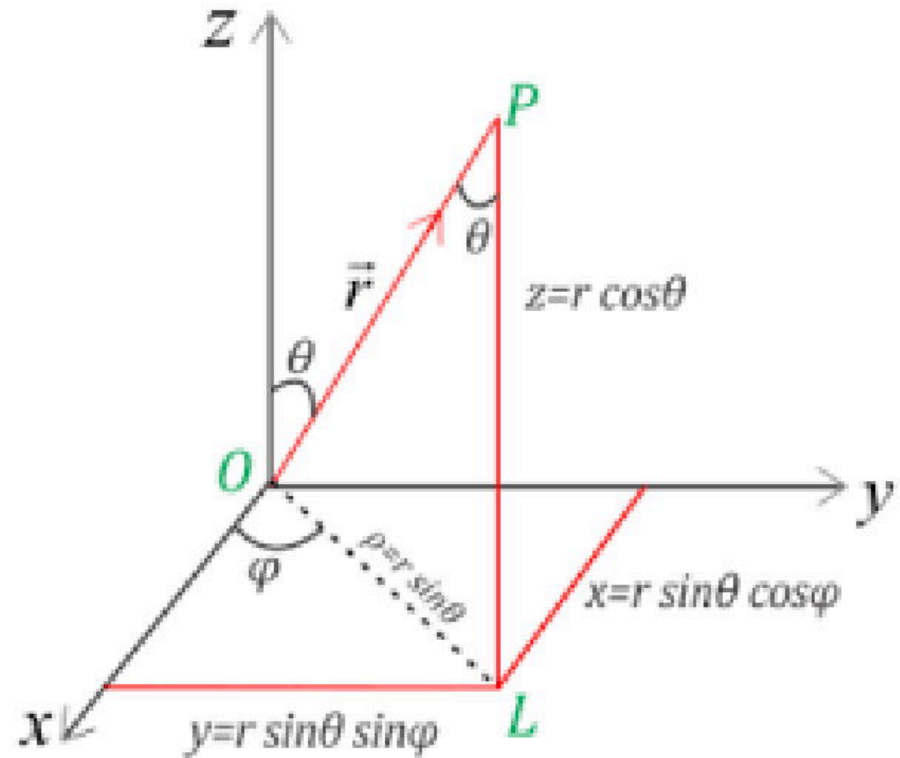
If spherical polar coordinates of P are (r, θ, φ) then

$$r = \sqrt{x^2 + y^2 + z^2} \quad \theta = \tan^{-1} \frac{\sqrt{x^2 + y^2}}{z}$$

$$\varphi = \tan^{-1} \frac{y}{x}$$

Then from the figure and above equations we have

$$r = \sqrt{\rho^2 + z^2} \quad \theta = \tan^{-1} \frac{\rho}{z} \quad \varphi = \varphi$$



Example: Prove that the spherical coordinate system is orthogonal.

Solution: Consider the basis vectors:

$$\hat{e}_r = \sin\theta \cos\phi \hat{i} + \sin\theta \sin\phi \hat{j} + \cos\theta \hat{k}$$

$$\hat{e}_\theta = \cos\theta \cos\phi \hat{i} + \cos\theta \sin\phi \hat{j} - \sin\theta \hat{k}$$

$$\hat{e}_\phi = -\sin\phi \hat{i} + \cos\phi \hat{j}$$

$$\begin{aligned}\hat{e}_r \cdot \hat{e}_\theta &= \sin\theta \cos\theta \cos^2\phi + \sin\theta \cos\theta \sin^2\phi \\ &\quad - \sin\theta \cos\theta\end{aligned}$$

$$= \sin\theta \cos\theta (\cos^2\phi + \sin^2\phi) - \sin\theta \cos\theta$$

$$= \cancel{\sin\theta \cos\theta} - \cancel{\sin\theta \cos\theta}$$

$$\hat{e}_r \cdot \hat{e}_\theta = 0 = \hat{e}_\theta \cdot \hat{e}_r$$

Now,

$$\hat{e}_\theta \cdot \hat{e}_\phi = -\sin\phi \cos\phi \cos\theta + \sin\phi \cos\phi \cos\theta$$

$$\Rightarrow \hat{e}_\theta \cdot \hat{e}_\phi = 0 = \hat{e}_\phi \cdot \hat{e}_\theta$$

$$\hat{e}_\phi \cdot \hat{e}_r = -\sin\theta \sin\phi \cos\phi + \sin\theta \sin\phi \cos\phi$$

$$\Rightarrow \hat{e}_\phi \cdot \hat{e}_r = 0 = \hat{e}_r \cdot \hat{e}_\phi$$

Implies that the spherical polar coordinate system is orthogonal.

Example: Express the unit vectors $\hat{i}, \hat{j}, \hat{k}$ in terms of $\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi$.

Solution:

we have the expressions of $\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi$ in terms of $\hat{i}, \hat{j}, \hat{k}$. We consider those equations and solve them simultaneously to evaluate $\hat{i}, \hat{j}, \hat{k}$ in terms of $\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi$.

We have:

$$\hat{e}_r = \sin \theta \cos \phi \hat{i} + \sin \theta \sin \phi \hat{j} + \cos \theta \hat{k}$$

$$\hat{e}_\theta = \cos \theta \cos \phi \hat{i} + \cos \theta \sin \phi \hat{j} - \sin \theta \hat{k}$$

$$\hat{e}_\phi = -\sin \phi \hat{i} + \cos \phi \hat{j}$$

$$\sin \theta \hat{e}_r + \cos \theta \hat{e}_\theta = \sin^2 \theta (\cos \phi \hat{i} + \sin \phi \hat{j}) + \cos^2 \theta (\cos \phi \hat{i} + \sin \phi \hat{j})$$

$$\sin \theta \hat{e}_r + \cos \theta \hat{e}_\theta = \cos \phi \hat{i} + \sin \phi \hat{j}$$

also

$$\hat{e}_\phi = -\sin \phi \hat{i} + \cos \phi \hat{j}$$

From last two equations, we have

$$\hat{j} = \sin\phi \sin\theta \hat{e}_r + \sin\phi \cos\theta \hat{e}_\theta + \cos\phi \hat{e}_\phi$$

and

$$\hat{i} = \cos\phi \sin\theta \hat{e}_r + \cos\phi \cos\theta \hat{e}_\theta - \sin\phi \hat{e}_\phi$$

and similarly, from the expressions of $\hat{e}_r$ and $\hat{e}_\theta$:

$$\hat{k} = \cos\theta \hat{e}_r - \sin\theta \hat{e}_\theta$$

Example: Find

$$\frac{d}{dt}(\hat{e}_r), \frac{d}{dt}(\hat{e}_\theta) \text{ and } \frac{d}{dt}(\hat{e}_\phi).$$

Solution:

As

$$\hat{e}_r = \sin\theta \cos\phi \hat{i} + \sin\theta \sin\phi \hat{j} + \cos\theta \hat{k}$$

$$\frac{d}{dt}(\hat{e}_r) = \frac{d}{dt}(\sin\theta \cos\phi) \hat{i} + \frac{d}{dt}(\sin\theta \sin\phi) \hat{j} + \frac{d}{dt}(\cos\theta) \hat{k}$$

$$= [\cos\theta \dot{\theta} \cos\phi - \sin\theta \sin\phi \dot{\phi}] \hat{i}$$

$$+ [\cos\theta \dot{\theta} \sin\phi + \sin\theta \cos\phi \dot{\phi}] \hat{j}$$

$$+ (-\sin\theta \dot{\theta}) \hat{k}$$

Separating the terms of $\dot{\theta}$ and $\dot{\phi}$,
we have:

$$\frac{d}{dt}(\hat{e}_r) = [\cos\theta\cos\phi\hat{i} + \cos\theta\sin\phi\hat{j} - \sin\theta\hat{k}]\dot{\theta} \\ + \sin\theta[-\sin\phi\hat{i} + \cos\phi\hat{j}]\dot{\phi}$$

$$\Rightarrow \frac{d}{dt}(\hat{e}_r) = \hat{e}_\theta\dot{\theta} + \sin\theta\hat{e}_\phi\dot{\phi}$$

Similarly:

$$\frac{d}{dt}(\hat{e}_\theta) = -\hat{e}_r \dot{\theta} + \cos \theta \hat{e}_\phi \dot{\phi}$$

and

$$\frac{d}{dt}(\hat{e}_\phi) = -(\cos \phi \hat{i} + \sin \phi \hat{j}) \dot{\phi}$$

here we replace the expressions
of $\hat{i}$ and $\hat{j}$.

$$\begin{aligned}
\frac{d}{dt}(\hat{e}_\phi) &= -\dot{\phi} \left[\cos\phi \left\{ \sin\theta \cos\phi \hat{e}_r + \cos\theta \cos\phi \hat{e}_\theta - \sin\phi \hat{e}_\phi \right\} \right. \\
&\quad \left. + \sin\phi \left\{ \sin\theta \sin\phi \hat{e}_r + \cos\theta \sin\phi \hat{e}_\theta + \cos\phi \hat{e}_\phi \right\} \right] \\
&= -\dot{\phi} \left[\sin\theta \cos^2\phi \hat{e}_r + \cos\theta \cos^2\phi \hat{e}_\theta - \sin\phi \cos\phi \hat{e}_\phi \right. \\
&\quad \left. + \sin\theta \sin^2\phi \hat{e}_r + \cos\theta \sin^2\phi \hat{e}_\theta + \sin\phi \cos\phi \hat{e}_\phi \right] \\
&= -(\sin\theta \hat{e}_r + \cos\theta \hat{e}_\theta) \dot{\phi}
\end{aligned}$$

Spherical Coordinate System

Example: Express the velocity and acceleration of a particle in spherical coordinates.

Solution: Consider:

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

where $x = r \sin \theta \cos \phi$

$$y = r \sin \theta \sin \phi$$

$$z = r \cos \theta$$

Now express $\hat{i}$, $\hat{j}$ and $\hat{k}$ in terms of $\hat{e}_r$, $\hat{e}_\theta$ and $\hat{e}_\phi$.

$$\hat{j} = \sin\phi \sin\theta \hat{e}_r + \sin\phi \cos\theta \hat{e}_\theta + \cos\phi \hat{e}_\phi$$

$$\hat{i} = \cos\phi \sin\theta \hat{e}_r + \cos\phi \cos\theta \hat{e}_\theta - \sin\phi \hat{e}_\phi$$

and

$$\hat{k} = \cos\theta \hat{e}_r - \sin\theta \hat{e}_\theta$$

Now:

$$\begin{aligned} \vec{r} = & r \sin\theta \cos\phi \{ \cos\phi \sin\theta \hat{e}_r + \cos\phi \cos\theta \hat{e}_\theta - \sin\phi \hat{e}_\phi \} \\ & + r \sin\theta \sin\phi \{ \sin\phi \sin\theta \hat{e}_r + \sin\phi \cos\theta \hat{e}_\theta + \cos\phi \hat{e}_\phi \} \\ & + r \cos\theta \{ \cos\theta \hat{e}_r - \sin\theta \hat{e}_\theta \} \end{aligned}$$

after simplifications:

$$\vec{r} = r \hat{e}_r$$

and

$$\vec{v} = \frac{d}{dt}(\vec{r}) = \frac{d}{dt}(r \hat{e}_r) = \dot{r} \hat{e}_r + r \frac{d}{dt}(\hat{e}_r)$$

$$= \dot{r} \hat{e}_r + r(\dot{\theta} \hat{e}_\theta + \sin \theta \hat{e}_\varphi \dot{\varphi})$$

$$\Rightarrow \vec{v} = \dot{r} \hat{e}_r + r \dot{\theta} \hat{e}_\theta + r \sin \theta \dot{\varphi} \hat{e}_\varphi$$

Now:

$$\vec{a} = \frac{d}{dt}(\vec{v}) = \frac{d}{dt}(\dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta + r\sin\theta\dot{\phi}\hat{e}_\phi)$$

$$\begin{aligned}\vec{a} &= \ddot{r}\hat{e}_r + \dot{r}(\dot{\theta}\hat{e}_\theta + \sin\theta\dot{\phi}\hat{e}_\phi) + \dot{r}\dot{\theta}\hat{e}_\theta + r\ddot{\theta}\hat{e}_\theta \\ &\quad + r\dot{\theta}(-\dot{\theta}\hat{e}_r + \cos\theta\dot{\phi}\hat{e}_\phi) + \dot{r}\sin\theta\dot{\phi}\hat{e}_\phi \\ &\quad + r\dot{\phi}\sin\theta\hat{e}_\phi + r\cos\theta\dot{\theta}\dot{\phi}\hat{e}_\phi + \\ &\quad r\sin\theta\dot{\phi}(-\sin\theta\dot{\phi}\hat{e}_r - \cos\theta\dot{\phi}\hat{e}_\theta) \\ \vec{a} &= (\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\phi}^2)\hat{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta} \\ &\quad - r\sin\theta\cos\theta\dot{\phi}^2)\hat{e}_\theta + (2\dot{r}\dot{\phi}\sin\theta + \\ &\quad 2r\dot{\theta}\dot{\phi}\cos\theta + r\ddot{\phi}\sin\theta)\hat{e}_\phi\end{aligned}$$

Example; Represent the following vector in spherical coordinates.

$$\vec{A} = 2y\hat{i} - z\hat{j} + 3x\hat{k}.$$

Solution: Since

$$\hat{i} = \sin\theta \cos\phi \hat{e}_r + \cos\theta \cos\phi \hat{e}_\theta - \sin\phi \hat{e}_\phi$$

$$\hat{j} = \sin\theta \sin\phi \hat{e}_r + \cos\theta \sin\phi \hat{e}_\theta + \cos\phi \hat{e}_\phi$$

$$\hat{k} = \cos\theta \hat{e}_r - \sin\theta \hat{e}_\theta$$

and

$$x = r \sin\theta \cos\phi$$

$$y = r \sin\theta \sin\phi$$

$$z = r \cos\theta$$

Implies:

$\vec{A} = 2y\hat{i} - z\hat{j} + 3x\hat{k}$, becomes:

$\vec{A} = A_r\hat{e}_r + A_\theta\hat{e}_\theta + A_\phi\hat{e}_\phi$, where

$$A_r = 2r \sin^2\theta \sin\phi \cos\phi - r \sin\theta \cos\theta \sin\phi + 3r \sin\theta \cos\theta \cos\phi$$

$$A_\theta = 2r \sin\theta \cos\theta \sin\phi \cos\phi - r \cos^2\theta \sin\phi - 3r \sin^2\theta \cos\phi$$

$$A_\phi = -2r \sin\theta \sin^2\phi - r \cos\theta \cos\phi$$

Example: Express the following in Spherical coordinates.

a) the sphere $x^2 + y^2 + z^2 = 9$

using transformation equations:

$$(r \sin \theta \cos \phi)^2 + (r \sin \theta \sin \phi)^2 + (r \cos \theta)^2 = 9$$

$$r^2 \sin^2 \theta (\cos^2 \phi + \sin^2 \phi) + r^2 \cos^2 \theta = 9$$

$$\Rightarrow r^2 (\sin^2 \theta + \cos^2 \theta) = 9$$

$$\Rightarrow r^2 = 9 \quad \text{or} \quad r = 3$$

b) the paraboloid: $z = x^2 + y^2$

$$\Rightarrow r \cos \theta = r^2 \sin^2 \theta \cos^2 \phi + r^2 \sin^2 \theta \sin^2 \phi$$

$$r \cos \theta = r^2 \sin^2 \theta$$

$$\Rightarrow \cos \theta = r \sin^2 \theta$$

c) the plane $y = x$

$$\Rightarrow r \sin \theta \sin \phi = r \sin \theta \cos \phi$$

$$\Rightarrow \tan \phi = 1 \Rightarrow \phi = \tan^{-1}(1)$$

$$\Rightarrow \phi = \pi/4 \text{ and } 5\pi/4 \text{ b/c } 0 \leq \phi \leq 2\pi$$

d) the cone $z^2 = 3(x^2 + y^2)$

$$\Rightarrow r^2 \cos^2 \theta = 3 r^2 \sin^2 \theta (\cos^2 \phi + \sin^2 \phi)$$

$$\Rightarrow \tan^2 \theta = 1/3$$

$$\Rightarrow \tan \theta = 1/\sqrt{3}$$

$$\Rightarrow \theta = \tan^{-1}(1/\sqrt{3})$$

$$\Rightarrow \theta = \pi/6 \quad \text{or} \quad \theta = 5\pi/6 \quad 0 \leq \theta \leq \pi$$

e) the plane $Z=0$

$$\Rightarrow r \cos \theta = 0$$

$$r \neq 0 \Rightarrow \cos \theta = 0$$

$$\Rightarrow \theta = \cos^{-1}(0)$$

$$\Rightarrow \theta = \pi/2$$

Example: Express the heat equation
 $\frac{\partial u}{\partial t} = k \nabla^2 u$ in spherical coordinates

if u is independent of

a) ϕ

b) ϕ and θ

c) r and t

d) ϕ , θ and t

a) Consider $\frac{\partial u}{\partial t} = k \nabla^2 u$

first consider ∇^2 in spherical coordinates

$$\nabla^2 u = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial u}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial u}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2}$$

when u is independent of ϕ , we have

$$\frac{\partial u}{\partial t} = k \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial u}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial u}{\partial \theta} \right) \right]$$

b) similarly when u is independent of θ and ϕ :

$$\frac{\partial u}{\partial t} = \frac{k}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial u}{\partial r} \right)$$

c) independent of r and t gives:

$$\sin \theta \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial u}{\partial \theta} \right) + \frac{\partial^2 u}{\partial \phi^2} = 0$$

$$d) \frac{d}{dr} \left(r^2 \frac{du}{dr} \right) = 0 \quad \text{OR} \quad r^2 \frac{du}{dr} = \text{constant}$$

Contravariant Vectors

Important points:

- On transforming a vector the components change and the basis vectors change however, the vector itself does not change its direction and magnitude remains same.
- This actually defines a tensor which does not change with a change of coordinate transformation. i.e., it remains invariant under a change of coordinate systems.

The components of the vector change under a coordinate transformation. These components may change in one of the two ways

- Contravariant
- Covariant

A vector whose components transform in a contravariant fashion is a contravariant vector.

Similarly, a vector whose components transform in a covariant fashion is a covariant vector.

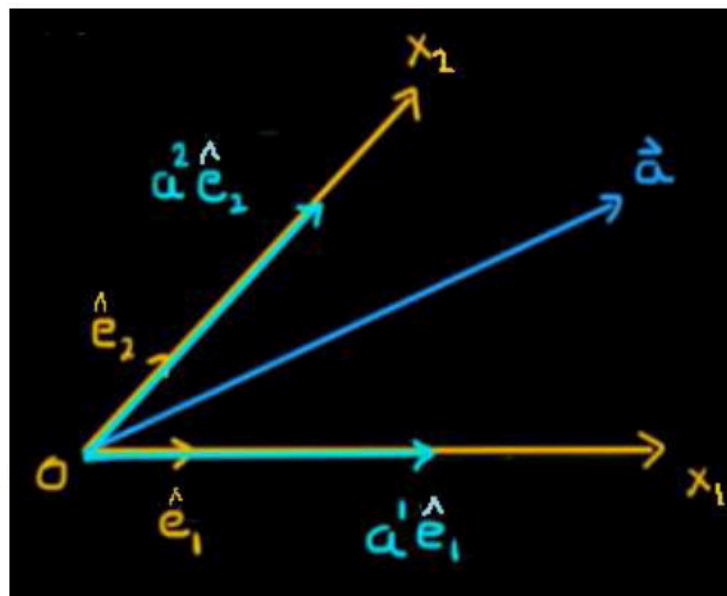
Contravariant vectors:

- A contravariant vector or tangent vector is often abbreviated simply as vector, such as a direction vector or velocity vector.
- The contravariant vector has components that **contra-vary** with a change of basis to compensate.
- That is, the matrix that transforms the vector components must be the inverse of the matrix that transforms the basis vectors.
- Examples of vectors with contravariant components include the position of an object relative to an observer, or any derivative of position with respect to time, including velocity and acceleration.
- In Einstein notation, contravariant components are denoted with upper indices.

Contravariant Vectors:

Consider a coordinate system with x_1 – axis and x_2 – axis and $\hat{e}_1$ and $\hat{e}_2$ are unit vectors along these axes. Further consider a vector $\vec{a}$ in this coordinate system such that in component form it can be written as

$$\vec{a} = a^1 \hat{e}_1 + a^2 \hat{e}_2 \rightarrow (1)$$



Now we will observe change in above vector when the basis vector $\hat{e}_2$ becomes $2\hat{e}_2$. It means we are transforming the vector $\vec{a}$ from the x_1x_2 coordinate system to a new one named as $x'_1x'_2$ system such that the new bases vectors $\hat{e}'_1$ and $\hat{e}'_2$ are related with $\hat{e}_1$ and $\hat{e}_2$ as: $\hat{e}'_1 = \hat{e}_1$ and $\hat{e}'_2 = 2\hat{e}_2$.

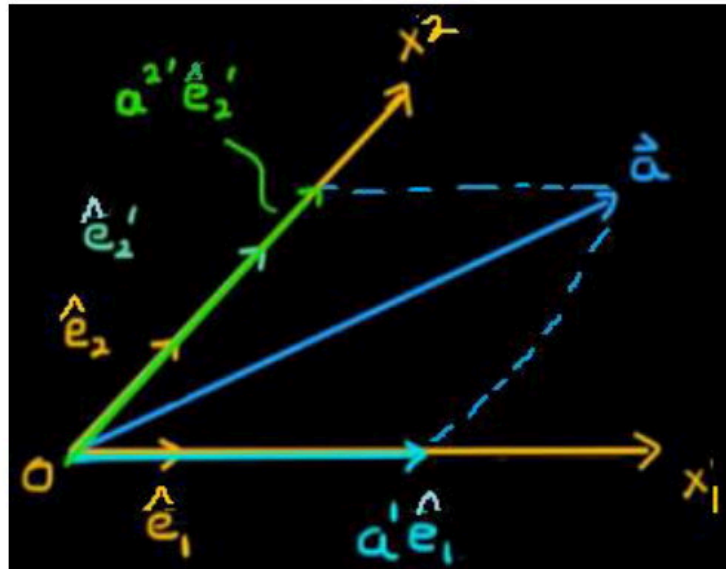
In this new system:

$$\vec{a} = a^1 \hat{e}_1 + a^2 \hat{e}_2 \rightarrow (2)$$

Now comparing (1) and (2), we have

$a^2 \hat{e}_2 = a'^2 \hat{e}'_2$ using $\hat{e}'_2 = 2\hat{e}_2$, we obtain:

$a^2 \hat{e}_2 = 2a'^2 \hat{e}_2$ implies $a^2 = 2a'^2$ or $a'^2 = \frac{a^2}{2}$



Covariant Vectors

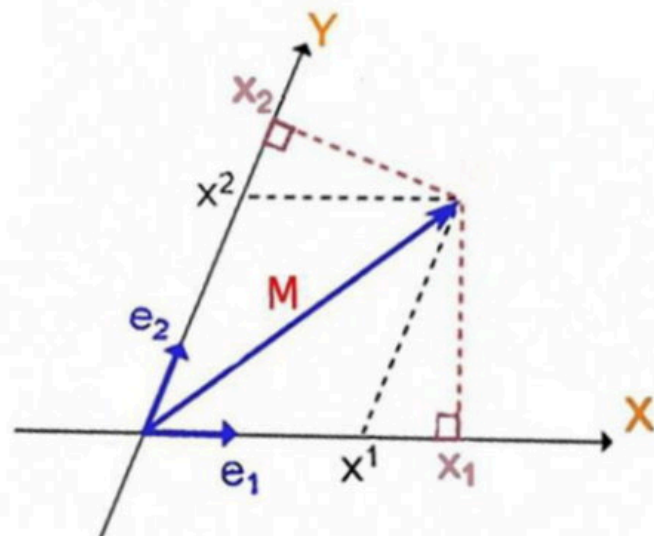
Covariant Vectors:

- In contrast, co-vectors also called dual vectors typically have units of the inverse of distance.
- An example of a co-vector is the gradient, which has units of a spatial derivative, or distance inverse. The components of co-vectors change in the same way as changes to scale of the reference axes and consequently are called covariant.
- A covariant vector often abbreviated as co-vector has components that **co-vary** with a change of basis.
- That is, the components must be transformed by the same matrix as the change of basis matrix.
- In Einstein notation, covariant components are denoted with lower indices.

Relation between contravariant and covariant components of a vector:

Note that:

- Contravariant components of a vector represent its parallel projections onto coordinate axes.
- Covariant components of a vector represent its perpendicular projection.



(x_1, x_2) covariant components.

(x^1, x^2) contravariant components.

Consider the vector $\vec{a}$ in contravariant component form as

$$\vec{a} = a^1 \hat{e}_1 + a^2 \hat{e}_2 \rightarrow (1)$$

Since its covariant components are perpendicular projections of $\vec{a}$ onto the coordinate axes so,

$$a_1 = \vec{a} \cdot \hat{e}_1 \rightarrow (2) ,$$

$$a_2 = \vec{a} \cdot \hat{e}_2 \rightarrow (3)$$

Now replacing the expression of vector $\vec{a}$ from Eq.1 into (2) and (3)

We have:

$$a_1 = (a^1 \hat{e}_1 + a^2 \hat{e}_2) \cdot \hat{e}_1 \quad \text{and} \quad a_2 = (a^1 \hat{e}_1 + a^2 \hat{e}_2) \cdot \hat{e}_2$$

Implies

$$a_1 = a^1(\hat{e}_1 \cdot \hat{e}_1) + a^2(\hat{e}_2 \cdot \hat{e}_1) \rightarrow (2')$$

$$a_2 = a^1(\hat{e}_1 \cdot \hat{e}_2) + a^2(\hat{e}_2 \cdot \hat{e}_2) \rightarrow (3')$$

Covariant vector components transformation:

Now similar to contravariant case, consider a coordinate system with x_1 – axis and x_2 – axis and $\hat{e}_1$ and $\hat{e}_2$ are unit vectors along these axes. Further consider a vector $\vec{a}$ in this coordinate system such that in component form it can be written as

$$\vec{a} = a^1 \hat{e}_1 + a^2 \hat{e}_2 \rightarrow (1)$$

To obtain a relation for covariant components, consider

$$a'_1 = \vec{a} \cdot \hat{e}_1 = a^1 (\hat{e}_1 \cdot \hat{e}_1) + a^2 (\hat{e}_2 \cdot \hat{e}_1)$$

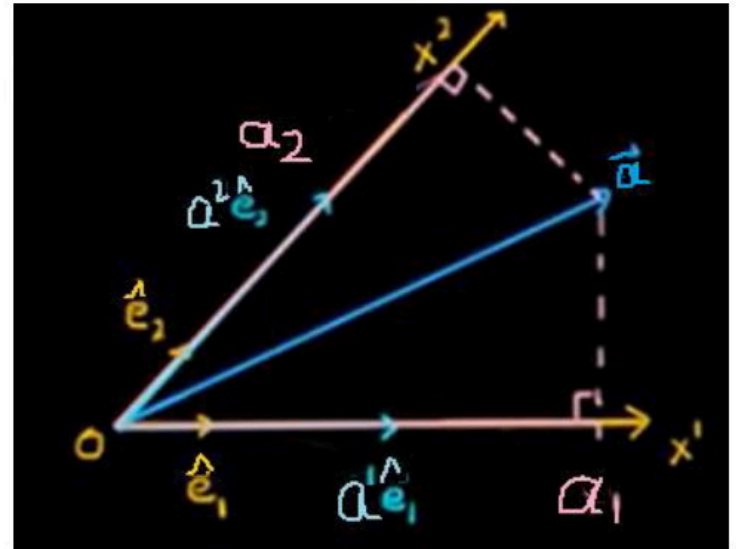
$$a'_2 = \vec{a} \cdot \hat{e}'_2 = a^1 (\hat{e}_1 \cdot \hat{e}'_2) + a^2 (\hat{e}_2 \cdot \hat{e}'_2)$$

Now using $\hat{e}'_2 = 2\hat{e}_2$ and $\hat{a}^2 = \frac{a^2}{2}$, we get

$$a'_1 = a^1 (\hat{e}_1 \cdot \hat{e}_1) + a^2 (\hat{e}_2 \cdot \hat{e}_1) = a_1 \quad \text{and}$$

$$a'_2 = 2(a^1 (\hat{e}_1 \cdot \hat{e}_2) + a^2 (\hat{e}_2 \cdot \hat{e}_2)) = 2a_2$$

Implies that a covariant component transforms in the same manner as the basis vectors.



Note:

If we restrict ourselves to Cartesian co-ordinate systems having orthonormal basis vectors then we do not have to make a distinction between co- and contravariant indices.

First order tensors

Tensor of rank one:

Tensors of order one are actually vectors. Here we study the transformation rules for contravariant and covariant first order tensors.

Transformation Law for Contravariant tensor:

A vector field $\vec{A}$ is a contravariant tensor of order one provided its components A^i and $\tilde{A}^i$ relative to the coordinate systems x^i and $\tilde{x}^i$ obey the transformation

$$\tilde{A}^i = A^r \frac{\partial \tilde{x}^i}{\partial x^r} \quad ; \quad 1 \leq i \leq n$$

$$A_i = a_{ij} \tilde{A}^j$$

Question: Show that the tangent vector of curve is a contravariant vector.

Solution: Let C be curve whose parametric representation in x^i – system is $x^i = x^i(t); a \leq t \leq b$.

The tangent vector $T = T^i$ is defined by the usual differentiation:

$$T^i = \frac{dx^i}{dt}$$

Under a change of coordinates to $\tilde{x}^i$ system, the same curve is given by $\tilde{x}^i = \tilde{x}^i(t); a \leq t \leq b$

And the tangent for this system is

$$\tilde{T}^i = \frac{d\tilde{x}^i}{dt}$$

Then by chain rule:

$$\frac{d\tilde{x}^i}{dt} = \frac{\partial \tilde{x}^i}{\partial x^r} \frac{dx^r}{dt} \quad \text{or} \quad \tilde{T}^i = T^r \frac{\partial \tilde{x}^i}{\partial x^r}$$

Implying that tangent vector $\vec{T}$ is a contravariant vector.

Transformation Law for Covariant tensor:

A vector field $\vec{A}$ is a covariant tensor of order one provided its components A_i and $\tilde{A}_i$ relative to the coordinate systems x^i and $\tilde{x}^i$ obey the transformation

$$\tilde{A}_i = A_r \frac{\partial x^r}{\partial \tilde{x}^i} \quad ; 1 \leq i \leq n$$

Example: Suppose that T^i is a contravariant vector on R^2 and that $T^i = (x^2, x^1)$ in the x^i system. Calculate $\tilde{T}^i$ in the $\tilde{x}^i$ -system, under the change of coordinates.

$$\tilde{x}^1 = (x^2)^2$$

$$\tilde{x}^2 = x^1 x^2$$

By definition of contravariance,

$$\tilde{T}^i = T^r \frac{\partial \tilde{x}^i}{\partial x^r} = T^1 \frac{\partial \tilde{x}^i}{\partial x^1} + T^2 \frac{\partial \tilde{x}^i}{\partial x^2} = x^2 \frac{\partial \tilde{x}^i}{\partial x^1} + x^1 \frac{\partial \tilde{x}^i}{\partial x^2}$$

$$\text{For } i = 1: \tilde{T}^1 = x^2 \frac{\partial \tilde{x}^1}{\partial x^1} + x^1 \frac{\partial \tilde{x}^1}{\partial x^2} = 2x^1 x^2$$

$$\text{For } i = 2: \tilde{T}^2 = x^2 \frac{\partial \tilde{x}^2}{\partial x^1} + x^1 \frac{\partial \tilde{x}^2}{\partial x^2} = (x^2)^2 + (x^1)^2$$

Which in terms of tilted coordinates become:

$$\tilde{T}^1 = 2 \tilde{x}^2, \tilde{T}^2 = \tilde{x}^1 + \frac{(\tilde{x}^2)^2}{\tilde{x}^1}$$

Example: Calculate $\tilde{T}^i$ in the $\tilde{x}^i$ -system if $T^i = (x^2, x^1 + 2x^2)$ is a contravariant vector.

Solution: Given $T^i = (x^2, x^1 + 2x^2)$

we have to find $\tilde{T}_i$.

using the transformation law:

$$\tilde{T}^i = \frac{\partial \tilde{x}^i}{\partial x^j} T^j$$

$$\Rightarrow \tilde{T}^1 = \frac{\partial \tilde{x}^1}{\partial x^1} T^1 + \frac{\partial \tilde{x}^1}{\partial x^2} T^2$$

$$\text{and } \tilde{T}^2 = \frac{\partial \tilde{x}^2}{\partial x^1} T^1 + \frac{\partial \tilde{x}^2}{\partial x^2} T^2$$

using $\tilde{x}' = (x^2)^2$ and $\tilde{x}^2 = x' x^2$

$$\Rightarrow \frac{\partial \tilde{x}'}{\partial x'} = 0, \frac{\partial \tilde{x}'}{\partial x^2} = 2x^2 \quad \left| \quad \frac{\partial \tilde{x}^2}{\partial x'} = x^2 \text{ and } \frac{\partial \tilde{x}^2}{\partial x^2} = x'$$

$$\Rightarrow \tilde{T}' = (0)(x^2) + 2x^2(x' + 2x^2) = 2x'x^2 + 4(x^2)^2$$

$$\frac{\tilde{T}^2}{T} = x^2(x^2) + (x')(x' + 2x^2) = (x^2)^2 + (x')^2 + 2x'x^2$$

$$\Rightarrow \tilde{T}' = 2\tilde{x}^2 + 4\tilde{x}'$$

and $\tilde{T}^2 = \tilde{x}' + 2\tilde{x}^2 + \frac{(\tilde{x}^2)^2}{x'}$

$$\left\{ \begin{array}{l} \sqrt{\tilde{x}'} = x^2 \\ \Rightarrow \tilde{x}^2 = x' \sqrt{\tilde{x}'} \\ \Rightarrow x' = \frac{\tilde{x}^2}{\sqrt{\tilde{x}'}} \end{array} \right.$$

Question: Show that there is no distinction between contravariant and covariant vectors when use the transformation of the type:

$\tilde{x}^i = a^i_m x^m + b^i$, where a 's and b 's are constants such that: $a^i_k a^i_m = \delta^k_m$

Solution: consider, $\tilde{x}^i = a^i_m x^m + b^i$

$$\Rightarrow a^i_m x^m = \tilde{x}^i - b^i$$

multiplying by a^i_k on both sides:

$$a^i_k a^i_m x^m = a^i_k \tilde{x}^i - a^i_k b^i$$

$$\Rightarrow \delta_m^l x^m = a_r^i \tilde{x}^i - b^i a_r^i$$

$$\Rightarrow x^r = a_r^i \tilde{x}^i - b^i a_r^i$$

$$\Rightarrow \frac{\partial x^r}{\partial \tilde{x}^i} = a_r^i$$

Now consider: $\tilde{x}^i = a_s^i x^s + b^i$

$$\Rightarrow \frac{\partial \tilde{x}^i}{\partial x^s} = a_s^i$$

Hence the contravariant transformation law is: $\tilde{A}^i = \frac{\partial \tilde{x}^i}{\partial x^s} A^s = a^i_s A^s$

and from covariant transformation law: $\tilde{A}_i = \frac{\partial x^s}{\partial \tilde{x}^i} A_s = a^s_i A_s$

$\Rightarrow$ There is no distinction between covariant and contravariant vectors under the given transformation.

Second Order Tensors

Higher order tensors:

Tensors of arbitrary order may be defined. We begin with the three types of second order tensors.

- Contravariant tensor
- Covariant tensor
- Mixed tensor

Let $V = [V^{ij}]$ denote a matrix field and $[V^{ij}]$ is an $n \times n$ matrix of scalar fields. These are defined over the same region in R^n . As before, there are two systems of coordinates x^i and $\tilde{x}^i$ with $V = [V^{ij}]$ in x^i -system and $V = [\tilde{V}^{ij}]$ in the tilted coordinate system. Where the coordinates from one system to another can be transformed via some transformation.

Contravariant tensor:

The matrix field V is a contravariant tensor of order two if its components T^{ij} in x^i -system and $\tilde{T}^{ij}$ in $\tilde{x}^i$ -system obey the law of transformation

$$\tilde{T}^{ij} = T^{rs} \frac{\partial \tilde{x}^i}{\partial x^r} \frac{\partial \tilde{x}^j}{\partial x^s}$$

$$\tilde{T}^{ij} = T^{rs} \frac{\partial \tilde{x}^i}{\partial x^r} \frac{\partial \tilde{x}^j}{\partial x^s}$$

Covariant tensor:

The matrix field V is a covariant tensor of order two if its components T_{ij} in x^i -system and $\tilde{T}_{ij}$ in $\tilde{x}^i$ -system obey the law of transformation

$$\tilde{T}_{ij} = T_{rs} \frac{\partial x^r}{\partial \tilde{x}^i} \frac{\partial x^s}{\partial \tilde{x}^j}$$

Mixed tensor:

The matrix field V is a mixed tensor of order two, contravariant of order one and covariant of order one, if its components T_j^i in x^i -system and $\tilde{T}_{\cancel{j}}^{\tilde{i}}$ in $\tilde{x}^i$ -system obey the law of transformation

$$\tilde{T}_j^i = T_s^r \frac{\partial \tilde{x}^i}{\partial x^r} \frac{\partial x^s}{\partial \tilde{x}^j}$$

Question: Suppose that the components of a contravariant tensor T of order 2 in a coordinate system x^i of $\mathbb{R}^2$ are $T^{11} = 1$, $T^{12} = 1$, and $T^{21} = -1$ and $T^{22} = 2$

a) Find the components $\tilde{T}^{ij}$ of T in the $\tilde{x}^i$ -system, connected to the x^i -system via $\tilde{x}^1 = (x^1)^2 \neq 0$
 $\tilde{x}^2 = x^1 x^2$

b) Compute the values of the point which corresponds to $x^1 = 1$, $x^2 = -2$.

Solution: (a) using the transformation

$$\tilde{T}^{ij} = T^{rs} \frac{\partial \tilde{x}^i}{\partial x^r} \frac{\partial \tilde{x}^j}{\partial x^s}; \quad i, j, r, s \text{ have values } 1, 2$$

$$\Rightarrow \tilde{T}^{11} = T^{rs} \frac{\partial \tilde{x}^1}{\partial x^r} \frac{\partial \tilde{x}^1}{\partial x^s} \quad (\because r \text{ and } s \text{ are dummy indices})$$

$$= T^{1s} \frac{\partial \tilde{x}^1}{\partial x^1} \frac{\partial \tilde{x}^1}{\partial x^s} + T^{2s} \frac{\partial \tilde{x}^1}{\partial x^2} \frac{\partial \tilde{x}^1}{\partial x^s}$$

$$= T^{11} \frac{\partial \tilde{x}^1}{\partial x^1} \frac{\partial \tilde{x}^1}{\partial x^1} + T^{12} \frac{\partial \tilde{x}^1}{\partial x^1} \frac{\partial \tilde{x}^1}{\partial x^2}$$

$$+ T^{21} \frac{\partial \tilde{x}^1}{\partial x^2} \frac{\partial \tilde{x}^1}{\partial x^1} + T^{22} \frac{\partial \tilde{x}^1}{\partial x^2} \frac{\partial \tilde{x}^1}{\partial x^2}$$

$$\tilde{x}^1 = (x^1)^2, \quad \tilde{x}^2 = x^1 x^2$$

$$\Rightarrow \frac{\partial \tilde{x}^1}{\partial x^1} = 2x^1 \text{ and } \frac{\partial \tilde{x}^1}{\partial x^2} = 0$$

$$\frac{\partial \tilde{x}^2}{\partial x^1} = x^2 \text{ and } \frac{\partial \tilde{x}^2}{\partial x^2} = x^1$$

$$\Rightarrow \tilde{T}'' = (1)(2x^1)(2x^1) + 0 + 0 + 0$$

$$\tilde{T}'' = 4(x^1)^2$$

Similarly:

$$\tilde{T}_{12} = 2x'x^2 + 2(x')^2$$

$$\tilde{T}_{21} = 2x'x^2 - 2(x')^2$$

$$\tilde{T}_{22} = 2(x')^2 + (x^2)^2$$

(b) At the point $(1, -2)$

$$\tilde{T}'' = 4(1)^2 = 4$$

$$\tilde{T}^{12} = 2(1)(-2) + 2(1)^2 = -2$$

$$\tilde{T}^{21} = 2(1)(-2) - 2(1)^2 = -6$$

$$\tilde{T}^{22} = 2(1)^2 + (-2)^2 = 6$$

Question: Show that if S^i and T^i are contravariant vectors on $\mathbb{R}^n$, the matrix $[U^{ij}] = [S^i T^j]_{nn}$, defined in this manner for all coordinates systems represents a contravariant tensor of order 2.

Solution: Given S^i and T^i are contravariant vectors in $\mathbb{R}^n$, we have

$$\tilde{S}^i = S^r \frac{\partial \tilde{x}^i}{\partial x^r} \quad \text{and} \quad \tilde{T}^j = T^s \frac{\partial \tilde{x}^j}{\partial x^s}$$

By multiplying both we have

$$\tilde{S}^i \tilde{T}^j = S^r T^s \frac{\partial \tilde{x}^i}{\partial x^r} \frac{\partial \tilde{x}^j}{\partial x^s}$$

$$\text{where } S^i T^j = V^{ij}$$

$\Rightarrow \tilde{V}^{ij} = V^{rs} \frac{\partial \tilde{x}^i}{\partial x^r} \frac{\partial \tilde{x}^j}{\partial x^s} \Rightarrow V^{ij}$ is a contravariant tensor of rank 2.

Question: let T_{ij} be a covariant tensor of order 2.

Find the components $\tilde{T}_{ij}$ in $\tilde{x}_i$ -system if $T_{11} = x^2$,

$T_{12} = T_{21} = 0$, $T_{22} = x^1$ are the components in x^i -system.
Also the transformations b/w x^i and $\tilde{x}^i$ system are:

$$\tilde{x}^1 = (x^1)^2$$

$$\tilde{x}^2 = x^1 x^2$$

Solution: First consider

$$\tilde{x}^1 = (x^1)^2 \text{ and } \tilde{x}^2 = x^1 x^2$$
$$\Rightarrow x^1 = \sqrt{\tilde{x}^1}, \quad x^2 = \frac{\tilde{x}^2}{x^1} = \frac{\tilde{x}^2}{\sqrt{\tilde{x}^1}}$$

$$\Rightarrow x^1 = \sqrt{\tilde{x}^1} \text{ and } x^2 = \frac{\tilde{x}^2}{\sqrt{\tilde{x}^1}}$$

Now using the transformation equation:

$$\tilde{T}_{ij} = T_{ks} \frac{\partial x^k}{\partial \tilde{x}^i} \frac{\partial x^s}{\partial \tilde{x}^j}$$

$$\Rightarrow \tilde{T}_{11} = T_{ks} \frac{\partial x^k}{\partial \tilde{x}^1} \frac{\partial x^s}{\partial \tilde{x}^1} = T_{1s} \frac{\partial x^1}{\partial \tilde{x}^1} \frac{\partial x^s}{\partial \tilde{x}^1} + T_{2s} \frac{\partial x^2}{\partial \tilde{x}^1} \frac{\partial x^s}{\partial \tilde{x}^1}$$

$$\Rightarrow \tilde{T}_{11} = T_{11} \frac{\partial x^1}{\partial \tilde{x}^1} \frac{\partial x^1}{\partial \tilde{x}^1} + T_{12} \frac{\partial x^1}{\partial \tilde{x}^1} \frac{\partial x^2}{\partial \tilde{x}^1} + T_{21} \frac{\partial x^2}{\partial \tilde{x}^1} \frac{\partial x^1}{\partial \tilde{x}^1} + T_{22} \frac{\partial x^2}{\partial \tilde{x}^1} \frac{\partial x^2}{\partial \tilde{x}^1}$$

now: $\frac{\partial x^1}{\partial \tilde{x}^1} = \frac{1}{2\sqrt{\tilde{x}_1}}$, $\frac{\partial x^2}{\partial \tilde{x}^1} = \frac{-\tilde{x}_1^2}{2(\tilde{x}^1)^{3/2}}$

$$\Rightarrow \tilde{T}_{11} = x^2 \frac{1}{2\sqrt{\tilde{x}_1}} \cdot \frac{1}{2\sqrt{\tilde{x}_1}} + 0 + 0 + x^1 \left(\frac{-\tilde{x}_1^2}{2(\tilde{x}^1)^{3/2}} \right) \left(\frac{-\tilde{x}_1^2}{2(\tilde{x}^1)^{3/2}} \right)$$

$$\Rightarrow \tilde{T}_{11} = \frac{1}{4} \frac{x^2}{x'} + x' \frac{(\tilde{x}^2)^2}{4(\tilde{x}')^3}$$

$$= \frac{1}{4} \frac{x^2}{(x')^2} + \frac{x' (x' x^2)^2}{4(x')^6}$$

$$= \frac{1}{4} \frac{x^2}{(x')^2} + \frac{(x^2)^2}{4(x')^3} = \frac{1}{4(x')^3} [x' x^2 + (x^2)^2]$$

Similarly:

$$\tilde{T}_{12} = -\frac{x^2}{2(x')^2}$$

$$\tilde{T}_{21} = -\frac{x^2}{2(x')^2}$$

$$\tilde{T}_{22} = \frac{1}{x'}$$

Tensor of arbitrary order:

Vectors and matrix fields are inadequate for higher order tensors.

The generalized vector field V is a tensor of order $m = p+q$,

contravariant of order p and covariant of order q , if its

components $T_{j_1 j_2 j_3 \dots j_q}^{i_1 i_2 i_3 \dots i_p}$ in x^i -system and $\tilde{T}_{j_1 j_2 j_3 \dots j_q}^{i_1 i_2 i_3 \dots i_p}$

in $\tilde{x}^i$ - system obey the law of transformation

$$\tilde{T}_{j_1 j_2 j_3 \dots j_q}^{i_1 i_2 i_3 \dots i_p} = T_{j_1 j_2 j_3 \dots j_q}^{i_1 i_2 i_3 \dots i_p} \frac{\partial \tilde{x}^{i_1}}{\partial x^{r_1}} \frac{\partial \tilde{x}^{i_2}}{\partial x^{r_2}} \dots \frac{\partial \tilde{x}^{i_p}}{\partial x^{r_p}} \frac{\partial x^{s_1}}{\partial \tilde{x}^{j_1}} \frac{\partial x^{s_1}}{\partial \tilde{x}^{j_1}} \dots \frac{\partial x^{s_1}}{\partial \tilde{x}^{j_1}}$$

Question: Display the transformation law for a third order tensor that is contravariant of order 2 and covariant of order 1.

Take $p=2$ and $q=1$ in the definition of general tensors.

$$\Rightarrow \tilde{T}^{ij}_k = T^{rs}_t \frac{\partial \tilde{x}^i}{\partial x^r} \frac{\partial \tilde{x}^j}{\partial x^s} \frac{\partial x^t}{\partial \tilde{x}^k}$$

Question: T^{ij}_{klm} is a mixed tensor of order five. Prove that T^{ij}_{kij} is a covariant tensor of order one.

Solution:

$$\tilde{T}^{ij}_{klm} = T^{rs}_{tuv} \frac{\partial \tilde{x}^i}{\partial x^r} \frac{\partial \tilde{x}^j}{\partial x^s} \frac{\partial x^t}{\partial \tilde{x}^k} \frac{\partial x^u}{\partial \tilde{x}^l} \frac{\partial x^v}{\partial \tilde{x}^m}$$

set $l=i$ and $m=j$:

$$\begin{aligned} \tilde{T}^{ij}_{kij} &= T^{rs}_{tuv} \frac{\partial \tilde{x}^i}{\partial x^r} \frac{\partial \tilde{x}^j}{\partial x^s} \frac{\partial x^t}{\partial \tilde{x}^k} \frac{\partial x^u}{\partial \tilde{x}^i} \frac{\partial x^v}{\partial \tilde{x}^j} \\ &= T^{rs}_{tuv} \left(\frac{\partial \tilde{x}^i}{\partial x^r} \frac{\partial x^u}{\partial \tilde{x}^i} \right) \left(\frac{\partial \tilde{x}^j}{\partial x^s} \frac{\partial x^v}{\partial \tilde{x}^j} \right) \frac{\partial x^t}{\partial \tilde{x}^k} \end{aligned}$$

$$\tilde{T}_{kij}^{ij} = T_{tuv}^{rs} \delta_r^u \delta_s^v \frac{\partial x^t}{\partial \tilde{x}^k}$$

$\downarrow$
 $\tilde{T}_k$

$T_{trs}^{rs} \rightarrow T_t$

$$\frac{\partial x^u}{\partial x^r} = \delta_r^u$$

OR

$$\tilde{T}_k = T_t \frac{\partial x^t}{\partial \tilde{x}^k}$$

✓

Question: Show that Kronecker delta is a mixed tensor of rank two.

Solution: x^i -system
 $\tilde{x}^i$ -system

$$\tilde{\delta}_{\dot{j}}^{\dot{i}} = \begin{cases} 1 & \text{if } \dot{i} = \dot{j} \\ 0 & \text{if } \dot{i} \neq \dot{j} \end{cases}$$

$$\text{So, } \tilde{\delta}_{\dot{j}}^{\dot{i}} = \frac{\partial \tilde{x}^{\dot{i}}}{\partial \tilde{x}^{\dot{j}}} = \frac{\partial \tilde{x}^{\dot{i}}}{\partial x^k} \cdot \frac{\partial x^l}{\partial \tilde{x}^{\dot{j}}} \frac{\partial x^k}{\partial x^l}$$

$$\tilde{\delta}_{\dot{j}}^{\dot{i}} = \frac{\partial \tilde{x}^{\dot{i}}}{\partial x^k} \cdot \frac{\partial x^l}{\partial \tilde{x}^{\dot{j}}} \delta_l^k \} \Rightarrow \delta_{\dot{j}}^{\dot{i}} \text{ is a mixed tensor of rank 2.}$$

Question: Compute the formulas for the tensor components $\tilde{T}^{ij}$ in polar coordinates if the components T^{ij} in rectangular coordinates are given.

Solution: here $x^1 = x$, $x^2 = y$, $\tilde{x}^1 = r$, $\tilde{x}^2 = \theta$

also, $x = r \cos \theta$, $y = r \sin \theta$, $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1} y/x$

using the transformation:

$$\tilde{T}^{ij} = T^{uv} \frac{\partial \tilde{x}^i}{\partial x^u} \frac{\partial \tilde{x}^j}{\partial x^v}$$

So, we require

$$\frac{\partial \tilde{x}'}{\partial x'} = \frac{\partial x}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}} = \frac{x \cos \theta}{r} = \cos \theta$$

$$\frac{\partial \tilde{x}^2}{\partial x'} = \frac{\partial \theta}{\partial x} = \frac{x^2 (-1/x^2)}{x^2 + y^2} = \frac{-y}{x^2 + y^2} = \frac{-x \sin \theta}{r^2} = -\frac{\sin \theta}{r}$$

$$\frac{\partial \tilde{x}'}{\partial x^2} = \frac{\partial x}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}} = \frac{r \sin \theta}{r} = \sin \theta$$

$$\frac{\partial \tilde{x}^2}{\partial x^2} = \frac{\partial \theta}{\partial y} = \frac{x^2 (1/x)}{x^2 + y^2} = \frac{x}{x^2 + y^2} = \frac{x \cos \theta}{r^2} = \frac{\cos \theta}{r}$$

Implies:

$$\begin{aligned}\tilde{T}'' &= T'' \frac{\partial \tilde{x}'}{\partial x^1} \frac{\partial \tilde{x}'}{\partial x^1} + T^{2v} \frac{\partial \tilde{x}'}{\partial x^2} \cdot \frac{\partial \tilde{x}'}{\partial x^v} \\&= T'' \frac{\partial \tilde{x}'}{\partial x^1} \frac{\partial \tilde{x}'}{\partial x^1} + T^{12} \frac{\partial \tilde{x}'}{\partial x^1} \frac{\partial \tilde{x}'}{\partial x^2} + T^{21} \frac{\partial \tilde{x}'}{\partial x^2} \frac{\partial \tilde{x}'}{\partial x^1} + T^{22} \frac{\partial \tilde{x}'}{\partial x^2} \frac{\partial \tilde{x}'}{\partial x^2} \\&= T'' (\cos \theta)(\cos \theta) + T^{12} (\cos \theta)(\sin \theta) + T^{21} (\sin \theta)(\cos \theta) + T^{22} (\sin \theta)(\sin \theta) \\ \tilde{T}'' &= \cos^2 \theta T'' + \frac{1}{2} \sin 2\theta T^{12} + \frac{1}{2} \sin 2\theta T^{21} + \sin^2 \theta T^{22}\end{aligned}$$

Implies:

$$\tilde{T}^{11} = \cos^2 \theta T^{11} + \frac{1}{2} \sin 2\theta T^{21} + \frac{1}{2} \sin 2\theta T^{12} + \sin^2 \theta T^{22}$$

$$\tilde{T}^{12} = \frac{-1}{2\hbar} \sin 2\theta T^{11} - \frac{\sin^2 \theta}{\hbar} T^{21} + \frac{\cos^2 \theta}{\hbar} T^{12} + \frac{1}{2\hbar} \sin 2\theta T^{22}$$

$$\tilde{T}^{21} = -\frac{1}{2\hbar} \sin 2\theta T^{11} + \frac{1}{\hbar} \cos^2 \theta T^{21} - \frac{1}{\hbar} \sin^2 \theta T^{12} + \frac{1}{2\hbar} \sin 2\theta T^{22}$$

$$\tilde{T}^{22} = \frac{\sin^2 \theta}{\hbar^2} T^{11} - \frac{\sin 2\theta}{2\hbar^2} T^{21} - \frac{\sin 2\theta}{2\hbar^2} T^{12} + \frac{\cos^2 \theta}{\hbar^2} T^{22}$$

Question: Compute the formulas for the tensor components $\tilde{T}_{ij}$ in cylindrical coordinates if the components T_{ij} in rectangular coordinates are given.

Solution: $x^1 = x, x^2 = y, x^3 = z$ and $\tilde{x}^1 = r, \tilde{x}^2 = \theta, \tilde{x}^3 = z$

also, $x = r \cos \theta, y = r \sin \theta, r = \sqrt{x^2 + y^2}, \theta = \tan^{-1} y/x$
and $z = z$

using the transformation:

$$\tilde{T}_{ij} = T_{uv} \frac{\partial x^u}{\partial \tilde{x}^i} \frac{\partial x^v}{\partial \tilde{x}^j}$$

$$\frac{\partial x'}{\partial \tilde{x}'} = \frac{\partial x}{\partial r} = \cos \theta$$

$$\frac{\partial x'}{\partial \tilde{x}^2} = \frac{\partial x}{\partial \theta} = -r \sin \theta$$

$$\frac{\partial x'}{\partial \tilde{x}^3} = \frac{\partial x}{\partial z} = 0$$

$$\left| \frac{\partial x^2}{\partial \tilde{x}'} = \frac{\partial y}{\partial r} = \sin \theta \right.$$

$$\frac{\partial x^2}{\partial \tilde{x}^2} = \frac{\partial y}{\partial \theta} = r \cos \theta$$

$$\frac{\partial x^2}{\partial \tilde{x}^3} = \frac{\partial y}{\partial z} = 0$$

$$\frac{\partial x^3}{\partial \tilde{x}^1} = \frac{\partial z}{\partial r} = 0$$

$$\frac{\partial x^3}{\partial \tilde{x}^2} = \frac{\partial z}{\partial \theta} = 0$$

$$\frac{\partial x^3}{\partial \tilde{x}^3} = \frac{\partial z}{\partial z} = 1$$

Implies:

$$\begin{aligned}
 \tilde{T}_{11} &= T_{\mu\nu} \frac{\partial x^\mu}{\partial \tilde{x}^1} \cdot \frac{\partial x^\nu}{\partial \tilde{x}^1} \\
 &= T_{11} \frac{\partial x^1}{\partial \tilde{x}^1} \frac{\partial x^1}{\partial \tilde{x}^1} + T_{21} \frac{\partial x^2}{\partial \tilde{x}^1} \frac{\partial x^1}{\partial \tilde{x}^1} + T_{31} \frac{\partial x^3}{\partial \tilde{x}^1} \frac{\partial x^1}{\partial \tilde{x}^1} \\
 &= T_{11} \frac{\partial x^1}{\partial \tilde{x}^1} \frac{\partial x^1}{\partial \tilde{x}^1} + T_{12} \frac{\partial x^1}{\partial \tilde{x}^1} \frac{\partial x^2}{\partial \tilde{x}^1} + T_{13} \frac{\partial x^1}{\partial \tilde{x}^1} \frac{\partial x^3}{\partial \tilde{x}^1} \rightarrow 0 \\
 &\quad + T_{21} \frac{\partial x^2}{\partial \tilde{x}^1} \frac{\partial x^1}{\partial \tilde{x}^1} + T_{22} \frac{\partial x^2}{\partial \tilde{x}^1} \frac{\partial x^2}{\partial \tilde{x}^1} + T_{23} \frac{\partial x^2}{\partial \tilde{x}^1} \frac{\partial x^3}{\partial \tilde{x}^1} \rightarrow 0 \\
 &\quad + T_{31} \frac{\partial x^3}{\partial \tilde{x}^1} \frac{\partial x^1}{\partial \tilde{x}^1} + T_{32} \frac{\partial x^3}{\partial \tilde{x}^1} \frac{\partial x^2}{\partial \tilde{x}^1} + T_{33} \frac{\partial x^3}{\partial \tilde{x}^1} \frac{\partial x^3}{\partial \tilde{x}^1} \rightarrow 0
 \end{aligned}$$

$$\Rightarrow \tilde{T}_{11} = \cos^2 \theta T_{11} + T_{12} (\cos \theta \sin \theta) + T_{21} (\cos \theta \sin \theta) + \sin^2 \theta T_{22}$$

$$\Rightarrow \tilde{T}_{11} = \cos^2 \theta T_{11} + \sin^2 \theta T_{22} + \sin \theta \cos \theta (T_{12} + T_{21})$$

Alternative Formulas (For tensors of order 2)

In terms of Jacobian matrix:

where

$$J = \frac{\partial(\tilde{x}^1, \tilde{x}^2)}{\partial(x^1, x^2)}, \quad J^{-1} = \frac{\partial(x^1, x^2)}{\partial(\tilde{x}^1, \tilde{x}^2)}$$

The transformation law for contravariant tensor is: $\tilde{T} = J T J^t$

Similarly, for a covariant tensor:

$$\tilde{T} = (J^{-1})^t T J^{-1}$$

and for mixed tensor:

$$\tilde{T} = J T J^{-1}$$

Question: Given the components of tensor T^{ij} in x^i -system: $T^{11} = 1$, $T^{12} = 1$, $T^{21} = -1$ and $T^{22} = 2$. Find the components of T^{ij} in $\tilde{x}^i$ -system given: $\tilde{x}^1 = (x^1)^2$, $\tilde{x}^2 = x^1 x^2$

Solution: Using $T = [T^{ij}]$

$$\text{and } \tilde{T} = J T J^t$$

First we find J matrix.

$$J = \frac{\partial(\tilde{x}^1, \tilde{x}^2)}{\partial(x^1, x^2)} = \begin{bmatrix} \frac{\partial \tilde{x}^1}{\partial x^1} & \frac{\partial \tilde{x}^1}{\partial x^2} \\ \frac{\partial \tilde{x}^2}{\partial x^1} & \frac{\partial \tilde{x}^2}{\partial x^2} \end{bmatrix}$$

$$\tilde{x}^1 = (x^1)^2, \quad \tilde{x}^2 = x^1 x^2$$

$$\Rightarrow J = \begin{bmatrix} 2x^1 & 0 \\ x^2 & x^1 \end{bmatrix} \text{ and } J^t = \begin{bmatrix} 2x^1 & x^2 \\ 0 & x^1 \end{bmatrix}$$

now

$$\tilde{T} = J T J^t$$

gives

$$\tilde{T} = \begin{bmatrix} 2x' & 0 \\ x^2 & x' \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2x' & x^2 \\ 0 & x' \end{bmatrix}$$

$$\Rightarrow \tilde{T} = \begin{bmatrix} 2x' & 2x' \\ x^2 - x' & x^2 + 2x' \end{bmatrix} \begin{bmatrix} 2x' & x^2 \\ 0 & x' \end{bmatrix}$$

$$= \begin{bmatrix} 4(x')^2 & 2x'x^2 + 2(x')^2 \\ 2x'x^2 - 2(x')^2 & (x^2)^2 - \cancel{x}x^2 + \cancel{x}x' + 2(x')^2 \end{bmatrix}$$

$$\tilde{T} = \begin{bmatrix} 4(x')^2 & 2x'x^2 + 2(x')^2 \\ 2x'x^2 - 2(x')^2 & (x^2)^2 + 2(x')^2 \end{bmatrix}$$

Question: Compute the formulas for the tensor components $\tilde{T}^{ij}$ in polar coordinates if the components T^{ij} in rectangular coordinates are given.

Solution: let $T = [T^{ij}]$

$$\text{and } \tilde{T} = J T J^t$$

$$\text{where } J = \frac{\partial(x, y)}{\partial(r, \theta)} \quad \text{and} \quad J^{-1} = \frac{\partial(r, \theta)}{\partial(x, y)}$$

As $x = r \cos \theta$ and $y = r \sin \theta$

$$r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1} y/x$$

$$J^{-1} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{bmatrix}$$

$$\Rightarrow J = \begin{bmatrix} \cos \theta & \sin \theta \\ -\frac{\sin \theta}{r} & \frac{\cos \theta}{r} \end{bmatrix}$$

$$\text{and } J^t = \begin{bmatrix} \cos \theta & -\sin \theta / r \\ \sin \theta & \cos \theta / r \end{bmatrix}$$

Now

$$\tilde{T} = J T J^t$$

$$\Rightarrow \tilde{T} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} \end{bmatrix} \begin{bmatrix} T^{11} & T^{12} \\ T^{21} & T^{22} \end{bmatrix} \begin{bmatrix} \cos\theta & -\frac{\sin\theta}{r} \\ \sin\theta & \frac{\cos\theta}{r} \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta T^{11} + \sin\theta T^{21} & \cos\theta T^{12} + \sin\theta T^{22} \\ -\frac{\sin\theta}{r} T^{11} + \frac{\cos\theta}{r} T^{21} & -\frac{\sin\theta}{r} T^{12} + \frac{\cos\theta}{r} T^{22} \end{bmatrix} \begin{bmatrix} \cos\theta & -\frac{\sin\theta}{r} \\ \sin\theta & \frac{\cos\theta}{r} \end{bmatrix}$$

Implies:

$$\tilde{T}^{11} = \cos^2 \theta T^{11} + \frac{1}{2} \sin 2\theta T^{21} + \frac{1}{2} \sin 2\theta T^{12} + \sin^2 \theta T^{22}$$

$$\tilde{T}^{12} = \frac{-1}{2\hbar} \sin 2\theta T^{11} - \frac{\sin^2 \theta}{\hbar} T^{21} + \frac{\cos^2 \theta}{\hbar} T^{12} + \frac{1}{2\hbar} \sin 2\theta T^{22}$$

$$\tilde{T}^{21} = -\frac{1}{2\hbar} \sin 2\theta T^{11} + \frac{1}{\hbar} \cos^2 \theta T^{21} - \frac{1}{\hbar} \sin^2 \theta T^{12} + \frac{1}{2\hbar} \sin 2\theta T^{22}$$

$$\tilde{T}^{22} = \frac{\sin^2 \theta}{\hbar^2} T^{11} - \frac{\sin 2\theta}{2\hbar^2} T^{21} - \frac{\sin 2\theta}{2\hbar^2} T^{12} + \frac{\cos^2 \theta}{\hbar^2} T^{22}$$

Invariant:

A function $\varphi(x^1, x^2, x^3, \dots, x^n)$ is called scalar or an invariant if its original value does not change upon transformation of coordinates from x^i -system to $\tilde{x}^i$ -system. *i.e.*,

$$\varphi(x^1, x^2, x^3, \dots, x^n) = \tilde{\varphi}(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3, \dots, \tilde{x}^n)$$

- Scalar is a tensor of rank zero.
- Objects, functions, equations or formulas that are independent of the coordinate system used to express them are invariants and are of fundamental significance.
- An invariant of a vector (rank-1 tensor) under coordinate transformations is its magnitude, *i.e.* length (the direction is also invariant but it is not scalar!).

- The main three independent scalar invariants of a rank-2 tensor under change of basis are:

If $A = [A_{ij}]$, then

$$I = \text{tr}(A) = A_{ii}$$

$$II = \text{tr}(A^2) = A_{ij}A_{ji}$$

$$III = \text{tr}(A^3) = A_{ij}A_{jk}A_{ki}$$

$$I_1 = I = A_{ii}$$

$$I_2 = \frac{1}{2}(I^2 - II)$$

$$I_3 = \det(A) = \frac{1}{3!}(I^3 - 3I \cdot II + 2III)$$

Question: Let A^i and A_i are contravariant and covariant components of a vector $\vec{A}$ then $A^i A_i$ is an invariant.

Solution: Using the transformation rules for A^i and A_i :

$$\tilde{A}^i = \frac{\partial \tilde{x}^i}{\partial x^k} A^k \quad \text{and} \quad \tilde{A}_i = \frac{\partial x^j}{\partial \tilde{x}^i} A_j \quad \underline{\underline{\tilde{A}^i \tilde{A}_i = A^i A_i}}$$

$$\Rightarrow \tilde{A}^i \tilde{A}_i = \frac{\partial \tilde{x}^i}{\partial x^k} \cdot \frac{\partial x^j}{\partial \tilde{x}^i} A^k A_j = \delta_k^j A^k A_j$$

$$\tilde{A}^i \tilde{A}_i = A^j A_j = A^i A_i$$

$$\Rightarrow \tilde{A}^i \tilde{A}_i = A^i A_i$$

Question: Prove that if T^i_j is a mixed tensor of order two T^i_i is an invariant.

Solution: $\tilde{T}^i_j = \frac{\partial \tilde{x}^i}{\partial x^k} \frac{\partial x^s}{\partial \tilde{x}^j} T^k_s$

$\tilde{T}^i_i = T^i_i$

$$\Rightarrow \tilde{T}^i_i = \frac{\partial \tilde{x}^i}{\partial x^k} \frac{\partial x^s}{\partial \tilde{x}^i} T^k_s$$

$$\Rightarrow \tilde{T}^i_i = \delta^s_k T^k_s = T^k_k$$

$$\Rightarrow \tilde{T}^i_i = T^i_i \Rightarrow T^i_i \text{ is invariant}$$

Question: If S_{jk}^i and T^i are tensors of the type and order indicated, $S_{jr}^r T^j$ is an invariant.

Solution:

$$\tilde{T}^i = \frac{\partial \tilde{x}^i}{\partial x^l} T^l$$

$$\tilde{S}_{jk}^i = \frac{\partial \tilde{x}^i}{\partial x^m} \frac{\partial x^n}{\partial \tilde{x}^j} \frac{\partial x^q}{\partial \tilde{x}^k} S_{nq}^m$$

Now,

$$\tilde{S}_{jk}^i \tilde{T}^p = \frac{\partial \tilde{x}^i}{\partial x^m} \frac{\partial x^n}{\partial \tilde{x}^j} \frac{\partial x^q}{\partial \tilde{x}^k} \cdot \frac{\partial \tilde{x}^p}{\partial x^l} S_{nq}^m T^l$$

$$\Rightarrow \tilde{S}_{j\ell}^{\ell} \tilde{T}^j = \frac{\partial \cancel{x}^{\ell}}{\partial x^m} \frac{\partial x^n}{\partial \cancel{x}^j} \frac{\partial x^q}{\partial \cancel{x}^{\ell}} \cdot \frac{\partial \tilde{x}^j}{\partial x^l} S_{nq}^m T^l$$

$$\tilde{S}_{j\ell}^{\ell} \tilde{T}^j = \delta_m^q \delta_l^n S_{nq}^m T^l$$

for $q=m$ and $n=l$

$$\Rightarrow \tilde{S}_{j\ell}^{\ell} \tilde{T}^j = S_{nm}^m T^n$$

$$m \rightarrow \ell, \quad n \rightarrow j$$

$$\Rightarrow \tilde{S}_{j\ell}^{\ell} \tilde{T}^j = S_{j\ell}^{\ell} T^j \Rightarrow \text{It is invariant.}$$

Question: Show that δ_j^i is an invariant.

Solution:

$$\tilde{\delta}_j^i = \frac{\partial \tilde{x}^i}{\partial x^k} \left(\frac{\partial x^l}{\partial \tilde{x}^j} \delta_l^k \right)$$

$$= \frac{\partial \tilde{x}^i}{\partial \cancel{x^k}} \cdot \frac{\cancel{\partial x^k}}{\partial \tilde{x}^j}$$

$$= \frac{\partial \tilde{x}^i}{\partial \tilde{x}^j} = \delta_j^i$$

$\Rightarrow$

Kronecker delta has same components in every coordinate system.

Question: Show that $\frac{\partial \phi}{\partial \tilde{x}^i}$ is a covariant vector, where ϕ is a scalar function.

Solution: Consider there are two coordinate systems: x^i -system and $\tilde{x}^i$ -system. ϕ is a scalar function such that:

$$\phi(\tilde{x}^1, \tilde{x}^2, \dots, \tilde{x}^n) = \phi(x^1, x^2, \dots, x^n)$$

$$\frac{\partial \phi}{\partial \tilde{x}^i} = \frac{\partial \phi}{\partial x^j} \frac{\partial x^j}{\partial \tilde{x}^i}$$

$$\Rightarrow \frac{\partial \phi}{\partial \tilde{x}^i} = \frac{\partial x^j}{\partial \tilde{x}^i} \frac{\partial \phi}{\partial x^j}, \quad \begin{matrix} i = 1, 2, \dots, n \\ j = 1, 2, \dots, n \end{matrix}$$

Addition and subtraction of tensors

The sum or difference of two tensors which have same number of covariant and contravariant indices is again a tensor of the rank and type as the given tensors.

Since the transformation law for tensor of any arbitrary order is linear in the tensor components, it is clear that if $T_1, T_2, \dots, T_n$ are tensors of the same type and order and if $\alpha_1, \alpha_2, \dots, \alpha_n$ are invariant scalars, then

$$\alpha_1 T_1 + \alpha_2 T_2 + \dots + \alpha_n T_n$$

Is a tensor of the same type and order as T_i .

Question: Show that if S^i and T^i are components of contravariant vectors then $\lambda S^i + \mu T^i$ is a contravariant vector, where λ and μ are scalars.

Solution: We have to find the transformation law for $\lambda S^i + \mu T^i$, where

$$\begin{aligned} S^i &= \frac{\partial \tilde{x}^i}{\partial x^m} S^m \\ \tilde{T}^i &= \frac{\partial \tilde{x}^i}{\partial x^n} T^n \end{aligned} \quad \left| \begin{array}{l} \text{also} \\ \tilde{\lambda} = \lambda \\ \text{and} \\ \tilde{\mu} = \mu \end{array} \right.$$

So,

$$\begin{aligned}\tilde{\lambda} \tilde{S}^i + \tilde{\mu} \tilde{T}^i &= \lambda \left(\frac{\partial \tilde{x}^i}{\partial x^m} S^m \right) + \mu \left(\frac{\partial \tilde{x}^i}{\partial x^n} T^n \right) \\ &= \lambda \left(\frac{\partial \tilde{x}^i}{\partial x^m} S^m \right) + \mu \left(\frac{\partial \tilde{x}^i}{\partial x^m} T^m \right) \\ &= \left(\lambda S^m + \mu T^m \right) \frac{\partial \tilde{x}^i}{\partial x^m}\end{aligned}$$

Question: If $A_{\underset{k}{k}}^{ij}$ and $B_{\underset{n}{n}}^{lm}$ are tensors then their sum and difference are tensors of the same rank and type.

Solution:

$$\tilde{A}_{\underset{k}{k}}^{ij} = \frac{\partial \tilde{x}^i}{\partial x^p} \frac{\partial \tilde{x}^j}{\partial x^q} \frac{\partial x^r}{\partial \tilde{x}^k} A_{\underset{r}{r}}^{pq}$$

$$\tilde{B}_{\underset{k}{k}}^{ij} = \frac{\partial \tilde{x}^i}{\partial x^p} \frac{\partial \tilde{x}^j}{\partial x^q} \frac{\partial x^r}{\partial \tilde{x}^k} B_{\underset{r}{r}}^{pq}$$

$$\tilde{A}_k^{ij} \pm \tilde{B}_k^{ij} = \frac{\partial \tilde{x}^i}{\partial x^p} \frac{\partial \tilde{x}^j}{\partial x^q} \frac{\partial x^r}{\partial \tilde{x}^k} (A_r^{pq} \pm B_r^{pq})$$

$$\text{If } \tilde{A}_k^{ij} \pm \tilde{B}_k^{ij} = \tilde{C}_k^{ij}$$

$$\text{and } A_r^{pq} \pm B_r^{pq} = C_r^{pq}$$

$$\Rightarrow \tilde{C}_k^{ij} = \frac{\partial \tilde{x}^i}{\partial x^p} \frac{\partial \tilde{x}^j}{\partial x^q} \frac{\partial x^r}{\partial \tilde{x}^k} C_r^{pq}$$

Outer product

The outer product of the tensors

$$S = S_{\substack{\hat{i}_1 \hat{i}_2 \dots \hat{i}_p \\ \hat{j}_1 \hat{j}_2 \dots \hat{j}_q}} \quad \text{and} \quad T = T_{\substack{k_1 k_2 \dots k_r \\ l_1 l_2 \dots l_s}}$$

Is the tensor

$$ST = S_{\substack{\hat{i}_1 \hat{i}_2 \dots \hat{i}_p \\ \hat{j}_1 \hat{j}_2 \dots \hat{j}_q}} \cdot T_{\substack{k_1 k_2 \dots k_r \\ l_1 l_2 \dots l_s}}$$

Which is of order

$$p + q + r + s$$

i.e., the sum of orders of S and T ,
contravariant of order $p + r$ and covariant
of order $q + s$.

Question: Find the outer product of two tensors S^i_j and T_k .

Solution: Let $S^i_j T_k = P^i_{jk}$ ✓

$$\tilde{S}^i_j = \frac{\partial \tilde{x}^i}{\partial x^r} \frac{\partial x^s}{\partial \tilde{x}^j} S^r_s$$

$$\tilde{T}_k = \frac{\partial x^\mu}{\partial \tilde{x}^k} T_\mu$$

$$\tilde{P}^i_{jk} = S^r_s \frac{\partial \tilde{x}^i}{\partial x^r} \frac{\partial x^s}{\partial \tilde{x}^j} \cdot T_\mu \frac{\partial x^\mu}{\partial \tilde{x}^k}$$

$$\tilde{P}^i_{jk} = P^r_{s\mu} \frac{\partial \tilde{x}^i}{\partial x^r} \frac{\partial x^s}{\partial \tilde{x}^j} \frac{\partial x^\mu}{\partial \tilde{x}^k}$$

Question: Show that the product of A^i and B_j is a mixed tensor of rank two.

Solution:

$$\tilde{A}^i = \frac{\partial \tilde{x}^i}{\partial x^k} A^k$$

and $\tilde{B}_j = \frac{\partial x^l}{\partial \tilde{x}^j} B_l$

$$\Rightarrow \tilde{A}^i \tilde{B}_j = \frac{\partial \tilde{x}^i}{\partial x^k} \frac{\partial x^l}{\partial \tilde{x}^j} A^k B_l$$

$$\Rightarrow \tilde{P}^i_j = \frac{\partial \tilde{x}^i}{\partial x^k} \frac{\partial x^l}{\partial \tilde{x}^j} P^k_l$$

Inner product

To take an inner product of two tensors, one equates an upper index of one tensor to a lower index of the other, and then apply sum over the repeated dummy index. In effect, the contravariant and covariant behaviors cancel out, which lowers the total order of the tensor product.

$$ST = \sum_{\substack{i_1, \dots, i_p \\ j_1, j_2, \dots, j_q}} T^{i_1 \dots i_p}_{j_1 j_2 \dots j_q} \sum_{\substack{k_1, k_2, \dots, k_r \\ l_1, \dots, l_s}} T^{k_1 k_2 \dots k_r}_{l_1 \dots l_s}$$

It can be observed that there will exist $p \cdot s + r \cdot q$ inner products, in general, all of these will be distinct. Each will be a tensor of order

$$m = p + q + r + s - 2$$

Question: From the tensors $S = S^{ij}$ and $T = T_{klm}$ form the inner product $U_{km}^j = S^{uj} T_{kum}$ and show that it is contravariant of order 1 and covariant of order 2.

Solution: Consider,

$$U_{km}^j = S^{uj} T_{kum}$$

$$\Rightarrow \tilde{U}_{km}^j = \tilde{S}^{uj} \tilde{T}_{kum}$$

$$\Rightarrow \tilde{U}_{km}^j = S^{Pr} \frac{\partial \tilde{x}^u}{\partial x^p} \frac{\partial \tilde{x}^j}{\partial x^u} T_{sqt} \frac{\partial x^s}{\partial \tilde{x}^k} \frac{\partial x^q}{\partial \tilde{x}^u} \frac{\partial x^t}{\partial \tilde{x}^m}$$

$$= S^{Pr} T_{sqt} \frac{\partial \tilde{x}^j}{\partial x^r} \frac{\partial x^s}{\partial \tilde{x}^k} \frac{\partial x^t}{\partial \tilde{x}^m} \frac{\partial x^q}{\partial x^p}$$

$$\tilde{U}_{km}^j = S^{Pr} T_{sqt} \frac{\partial \tilde{x}^j}{\partial x^r} \frac{\partial x^s}{\partial \tilde{x}^k} \frac{\partial x^t}{\partial \tilde{x}^m} \delta_p^q$$

$$\Rightarrow \tilde{U}_{km}^j = U_{St}^r \frac{\partial \tilde{x}^j}{\partial x^r} \frac{\partial x^s}{\partial \tilde{x}^k} \frac{\partial x^t}{\partial \tilde{x}^m}$$

Question: If A^i and B_i are two tensors then $A^i B_i$ is scalar or invariant.

Solution:

$$\tilde{A}^i = \frac{\partial \tilde{x}^i}{\partial x^p} A^p \quad \text{and} \quad \tilde{B}_i = \frac{\partial x^q}{\partial \tilde{x}^i} B_q$$

$$\begin{aligned} \Rightarrow \tilde{A}^i \tilde{B}_i &= \frac{\partial \tilde{x}^i}{\partial x^p} \frac{\partial x^q}{\partial \tilde{x}^i} A^p B_q \\ &= \delta_p^q A^p B_q \end{aligned}$$

$$\Rightarrow \tilde{A}^i \tilde{B}_i = A^p B_p = A^i B_i \Rightarrow \text{It is invariant.}$$

Question: Prove that if g_{ij} is a covariant tensor of order two and U^i and V^j are contravariant vectors, then the double inner product $g_{ij} U^i V^j$ is an invariant.

Solution: Consider the transformation laws

$$\tilde{g}_{ij} = g_{rs} \frac{\partial x^r}{\partial \tilde{x}^i} \frac{\partial x^s}{\partial \tilde{x}^j}$$

$$\tilde{U}^i = U^t \frac{\partial \tilde{x}^i}{\partial x^t}, \quad V^j = V^\mu \frac{\partial \tilde{x}^j}{\partial x^\mu}$$

Now take the product:

$$\begin{aligned}\tilde{g}_{ij} \tilde{U}^i \tilde{V}^j &= g_{rs} U^t V^\mu \frac{\partial x^r}{\partial \tilde{x}^i} \frac{\partial x^s}{\partial \tilde{x}^j} \frac{\partial \tilde{x}^i}{\partial x^t} \frac{\partial \tilde{x}^j}{\partial x^\mu} \\ &= g_{rs} U^t V^\mu \delta_t^r \delta_\mu^s\end{aligned}$$

$$\begin{aligned}\tilde{g}_{ij} \tilde{U}^i \tilde{V}^j &= g_{rs} U^r V^s \\ &= g_{ij} U^i V^j\end{aligned}$$

$\Rightarrow g_{ij} U^i V^j$ is invariant.

Contraction

This is another order reducing operation like the inner product but is applied to single tensors. In contraction of a tensor we equate its one contravariant index and one covariant index.

For example, in the tensor

$$S = S^{i_1 i_2 \dots i_\alpha \dots i_p}_{j_1 j_2 \dots j_\beta \dots j_q}$$

we equate $i_\alpha = \mu = j_\beta$, then resulting tensor is

$$S' = S^{i_1 i_2 \dots \mu \dots i_p}_{j_1 j_2 \dots \mu \dots j_q} \quad \left. \vphantom{S'} \right\} \text{this is called a contraction of } S \text{ with indices } i_\alpha \text{ and } j_\beta.$$

$\Rightarrow S'$ is contravariant of order $p-1$ and covariant of order $q-1$.

Question: How many tensors may be created by repeated contraction of the tensor $T = T_{kl}^{ij}$?

Solution: Single contraction produces the following four mixed tensors:

- 1) for $i=k=v$: T_{vl}^{vj}
- 2) for $j=k=v$: T_{vl}^{iv}
- 3) for $i=l=v$: T_{kv}^{vj}
- 4) for $j=l=v$: T_{kv}^{iv}

and double contraction produces the two zero-order tensors:

5) In first contractions 1 and 4 equating the indices we have:

$$T_{uv}^{uv}$$

6) and from contractions 2 and 3 we have T_{vu}^{uv}

$\Rightarrow$ there are six distinct tensors are formed by contractions.

Question: Show that any contraction of the tensor $T = T^i_{jk}$ results in a covariant vector.

Solution: consider the transformation law:

$$\tilde{T}^i_{jk} = T^r_{ts} \frac{\partial \tilde{x}^i}{\partial x^r} \frac{\partial x^t}{\partial \tilde{x}^j} \frac{\partial x^s}{\partial \tilde{x}^k}$$

Contracting the indices: $i = j$

$$\tilde{T}^i_{ik} = T^r_{ts} \frac{\partial \tilde{x}^i}{\partial x^r} \frac{\partial x^t}{\partial \tilde{x}^i} \frac{\partial x^s}{\partial \tilde{x}^k}$$

$$\tilde{T}^i_{ik} = T^r_{ts} \frac{\partial x^s}{\partial \tilde{x}^k} \delta^t_r = T^r_{rs} \frac{\partial x^s}{\partial \tilde{x}^k}$$

Similarly contracting: $i = k$

$$\Rightarrow \tilde{T}^i_{ji} = T^r_{ts} \frac{\partial \tilde{x}^i}{\partial x^r} \frac{\partial x^t}{\partial \tilde{x}^i} \frac{\partial x^s}{\partial \tilde{x}^i}$$

$$\tilde{T}^i_{ji} = T^r_{ts} \frac{\partial x^t}{\partial x^i} \delta^s_r = T^r_{tr} \frac{\partial x^t}{\partial x^i}$$

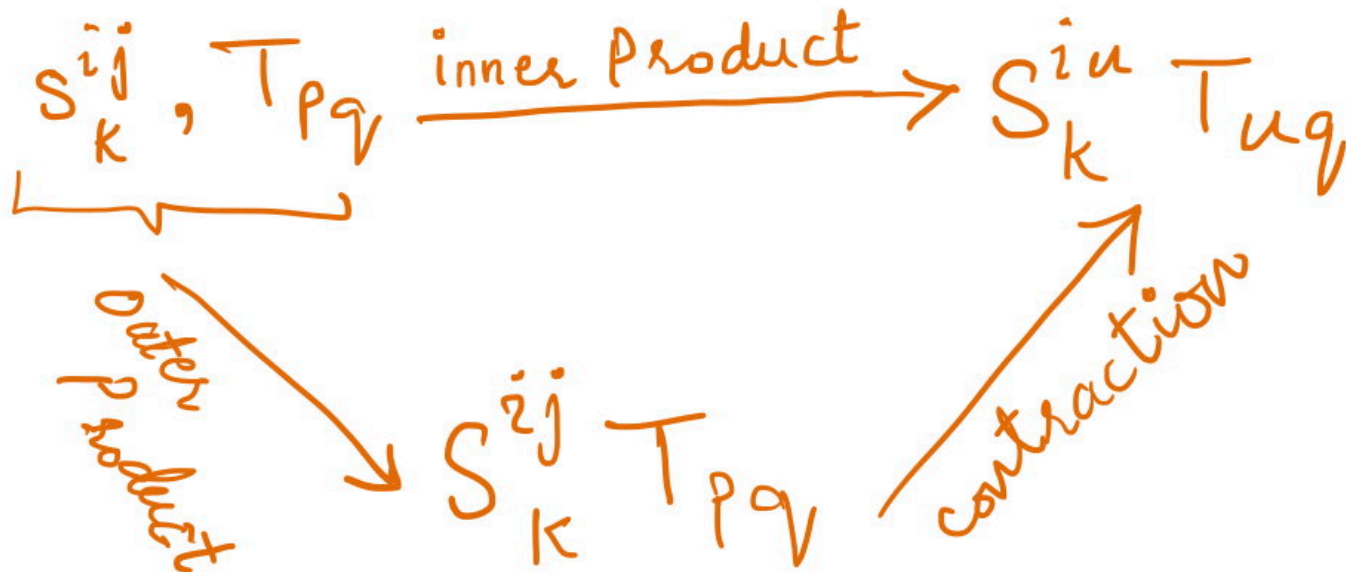
$\Rightarrow$ Both contractions yield a covariant vector.

Combined operations

We can form new tensors from the old ones in a variety of ways by performing different tensor operations discussed above.

One might form the outer product of two tensors, then take an inner product of this with a third tensor, or contract on one or more pair of indices either before or after taking a product.

It is noteworthy to mention that an inner product of two tensors may be characterized as a contraction of their outer product.



Question: Suppose that $S = S^i_j$ and $T = T^i_j$ are tensors from which a contravariant vector V^i is to be constructed using a combination of outer/inner products and contractions.

a) Show that there are six distinct possibilities for V^i .

b) Show that each possible V^i is obtainable as a contraction of an inner product ST .

a) Here $S = S_{\quad k}^{ij}$ and $T = T_m^l$

consider outer product:

$$ST = S_{\quad k}^{ij} T_m^l = U_{\quad km}^{ijl} = U(\text{say})$$

then contravariant vectors can be obtained as the double contractions of $U = U_{\quad km}^{ijl}$:

$$U_{\quad uv}^{uvk}, \quad U_{\quad vu}^{uvk}, \quad U_{\quad uv}^{ujv}, \quad U_{\quad vu}^{ujv},$$

$$U_{\quad uv}^{iuv}, \quad U_{\quad vu}^{iuv}$$

b) The vector

$$U_{uv}^{ak} = S_u^{av} T_v^k$$

may be obtained by first taking the inner product $S_l^{iv} T_v^k$ and then contracting on $i = u = l$.

Similar can be shown for the other vectors of part (a).

Quotient Law

By this law, we can test the given quantity is a tensor or not. Suppose given quantity be A and we do not know that A is a tensor or not. To test A , we take inner product of A with an arbitrary tensor, if this inner product is a tensor then A is also a tensor.

Statement

If the inner product of a set of functions with an arbitrary tensor is a tensor then these set of functions are components of a tensor.

Question: Show that the expression $A(i, j, k)$ is a covariant tensor of rank three if $A(\tilde{i}, \tilde{j}, k) B^k$ is a covariant tensor of rank two and B^k is contravariant vector.

Solution: Given $A(\tilde{i}, \tilde{j}, k) B^k$ is a covariant tensor of rank two then,

$$\tilde{A}(\tilde{i}, \tilde{j}, k) \tilde{B}^k = \frac{\partial x^p}{\partial \tilde{x}^i} \frac{\partial x^q}{\partial \tilde{x}^j} A(p, q, r) B^r$$

Since B^k is contravariant vector

$$\tilde{B}^k = \frac{\partial x^k}{\partial \tilde{x}^l} B^l \quad \text{or} \quad B^l = \frac{\partial x^l}{\partial \tilde{x}^k} \tilde{B}^k$$

$$\Rightarrow \tilde{A}(\tilde{i}, \tilde{j}, k) \tilde{B}^k = \frac{\partial x^p}{\partial \tilde{x}^i} \frac{\partial x^q}{\partial \tilde{x}^j} A(p, q, r) \frac{\partial x^r}{\partial \tilde{x}^k} \tilde{B}^k$$

$$\Rightarrow \tilde{A}(\tilde{i}, \tilde{j}, k) = \frac{\partial x^p}{\partial \tilde{x}^i} \frac{\partial x^q}{\partial \tilde{x}^j} \frac{\partial x^r}{\partial \tilde{x}^k} A(p, q, r)$$

$\Rightarrow A(\tilde{i}, \tilde{j}, k)$ is a covariant tensor of rank three.

Question: If $A(\tilde{i}, j, k) A^{\tilde{i}} B^{\tilde{j}} C_k$ is a scalar for arbitrary vectors $A^{\tilde{i}} B^{\tilde{j}} C_k$ then show that $A(\tilde{i}, j, k)$ is a tensor of type (1, 2).

Solution: As $A(\tilde{i}, j, k) A^{\tilde{i}} B^{\tilde{j}} C_k$ is a scalar: then

$$\tilde{A}(\tilde{i}, j, k) \tilde{A}^{\tilde{i}} \tilde{B}^{\tilde{j}} \tilde{C}_k = A(p, q, r) A^p B^q C_r$$

Also $A^{\tilde{i}}, B^{\tilde{j}}$ and C_k are vectors. Then

$$\tilde{A}_{\tilde{i}} = \frac{\partial \tilde{x}^{\tilde{i}}}{\partial x^p} A^p \quad \text{or} \quad A^p = \frac{\partial x^p}{\partial \tilde{x}^{\tilde{i}}} \tilde{A}_{\tilde{i}}$$

Similarly: $B^v = \frac{\partial x^v}{\partial \tilde{x}^j} \tilde{B}^j$

$$C_r = \frac{\partial \tilde{x}^k}{\partial x^r} \tilde{C}_k$$

$\Rightarrow$

$$\tilde{A}(i, j, k) \tilde{A}^i \tilde{B}^j \tilde{C}_k = A(p, q, r) \frac{\partial x^p}{\partial \tilde{x}^i} \frac{\partial x^q}{\partial \tilde{x}^j} \frac{\partial \tilde{x}^k}{\partial x^r} \tilde{A}^i \tilde{B}^j \tilde{C}_k$$

$$\Rightarrow \tilde{A}(i, j, k) = A(p, q, r) \frac{\partial x^p}{\partial \tilde{x}^i} \frac{\partial x^q}{\partial \tilde{x}^j} \frac{\partial \tilde{x}^k}{\partial x^r}$$

$\Rightarrow A(i, j, k)$ is a tensor of type $(1, 2)$.

Tensor Equations

Much of the importance of tensors in mathematical physics and engineering resides in the fact that if a tensor equation or identity is true in one coordinate system, then it is true in all coordinate systems.

Example: Suppose that in x^i -system the covariant tensor $T = T_{ij}$ vanishes. The components of T in any other system say, $\tilde{x}^i$ -system are:

$$\tilde{T}_{ij} = T_{rs} \frac{\partial x^r}{\partial \tilde{x}^i} \frac{\partial x^s}{\partial \tilde{x}^j} = 0 \quad \text{if } T_{ij} = 0 \Rightarrow \tilde{T}_{ij} = 0$$

Therefore $T = 0$ in every coordinate system.

Example: Consider an equation:

$$\tilde{T}_{ij} = R_{ijk} U^k - A W_i^{kl} M_{jk} I_l = 0 \Rightarrow \tilde{T}_{ij} = 0$$

$$R_{ijk} U^k = A W_i^{kl} M_{jk} I_l$$

which connect six entities that may or may not be tensors. Then if it can be shown that (i) $T_{ij} = R_{ijk} U^k - A W_i^{kl} M_{jk} I_l$ is a tensor and (ii) there exist a coordinate system in which all T_{ij} are zero.

Then the given tensor equation is valid in every coordinate system.

Question: Prove that if T_{jkl}^i is a tensor in x^i -system and $T_{jkl}^i = 3 T_{ljk}^i$. Then $T_{jkl}^i = 3 T_{ljk}^i$ in all coordinate systems.

Solution:

Here we have to prove that

$$\tilde{T}_{jkl}^i = 3 \tilde{T}_{ljk}^i \text{ in } \tilde{x}^i\text{-system.}$$

$$\tilde{T}_{jkl}^i = 3 \tilde{T}_{lik}^i \text{ means: } \tilde{T}_{jkl}^i - 3 \tilde{T}_{lik}^i = 0$$

So consider:

$$\begin{aligned} \tilde{T}_{jkl}^i - 3 \tilde{T}_{lik}^i &= T^p_{rst} \frac{\partial \tilde{x}^i}{\partial x^p} \frac{\partial x^r}{\partial \tilde{x}^j} \frac{\partial x^s}{\partial \tilde{x}^k} \frac{\partial x^t}{\partial \tilde{x}^l} \\ &\quad - 3 T^p_{rst} \frac{\partial \tilde{x}^i}{\partial x^p} \frac{\partial x^r}{\partial \tilde{x}^l} \frac{\partial x^s}{\partial \tilde{x}^j} \frac{\partial x^t}{\partial \tilde{x}^k} \\ &= T^p_{rst} \frac{\partial \tilde{x}^i}{\partial x^p} \frac{\partial x^r}{\partial \tilde{x}^j} \frac{\partial x^s}{\partial \tilde{x}^k} \frac{\partial x^t}{\partial \tilde{x}^l} \begin{cases} r \rightarrow t \\ s \rightarrow r \\ t \rightarrow s \end{cases} \\ &\quad - 3 T^p_{trs} \frac{\partial \tilde{x}^i}{\partial x^p} \frac{\partial x^t}{\partial \tilde{x}^l} \frac{\partial x^r}{\partial \tilde{x}^j} \frac{\partial x^s}{\partial \tilde{x}^k} \end{aligned}$$

$$\Rightarrow \tilde{T}_{jkl}^i - 3\tilde{T}_{ljk}^i = (T_{rst}^p - 3T_{trs}^p) \frac{\partial \tilde{x}^i}{\partial x^p} \frac{\partial x^r}{\partial \tilde{x}^j} \frac{\partial x^s}{\partial \tilde{x}^k} \frac{\partial x^t}{\partial \tilde{x}^l}$$

but $T_{jkl}^i = 3T_{ljk}^i$ in x^i -system

$$\Rightarrow \tilde{T}_{jkl}^i - 3\tilde{T}_{ljk}^i = 0$$

$\Rightarrow \tilde{T}_{jkl}^i = 3\tilde{T}_{ljk}^i$ in every coordinate system.

Question: Prove that if a mixed tensor T^i_j can be expressed as the outer product of contravariant and covariant vectors U^i and V_j in one coordinate system then T^i_j is the outer product of those vectors in general.

Solution: As $T_j^i = U^i V_j$ in x^i -system (say)
and we have to show that

$$\tilde{T}_j^i = \tilde{U}^i \tilde{V}_j \text{ in any } \tilde{x}^i\text{-system.}$$

Consider:

$$\tilde{T}_j^i - \tilde{U}^i \tilde{V}_j = T_s^i \frac{\partial \tilde{x}^i}{\partial x^s} \frac{\partial x^s}{\partial \tilde{x}^j} - U^i \frac{\partial \tilde{x}^i}{\partial x^s} V_s \frac{\partial x^s}{\partial \tilde{x}^j}$$

$$= \underbrace{(T_s^i - U^i V_s)}_{=0} \left(\frac{\partial \tilde{x}^i}{\partial x^s} \frac{\partial x^s}{\partial \tilde{x}^j} \right)$$

$$\Rightarrow \tilde{T}_j^i - \tilde{U}^i \tilde{V}_j = 0 \Rightarrow \tilde{T}_j^i = \tilde{U}^i \tilde{V}_j \text{ as required.}$$

Symmetric Tensors

A tensor is said to be symmetric with respect to two contravariant or two covariant indices if its components remain unchanged on an interchange of the two indices.

For example: (1) The tensor A^{ij} is symmetric if: $A^{ij} = A^{ji}$

(2) The tensor A_{lm}^{ijk} is symmetric w.r.t indices i and j if: $A_{lm}^{ijk} = A_{lm}^{jik}$

and w.r.t indices l and m if:

$$A_{lm}^{ijk} = A_{ml}^{ijk}$$

Question: A symmetric tensor of rank two has only $\frac{1}{2}n(n+1)$ different components in an n -dimensional space.

Solution: let A^{ij} be a symmetric tensor of rank two, i.e., $A^{ij} = A^{ji}$. In matrix form:

$$A = [A^{ij}] = \begin{bmatrix} A^{11} & A^{12} & A^{13} & \dots & A^{1n} \\ A^{21} & A^{22} & A^{23} & \dots & A^{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ A^{n1} & A^{n2} & A^{n3} & \dots & A^{nn} \end{bmatrix}$$

$\Rightarrow A^{ij}$ will have n^2 -components. Out of these n^2 -components n are diagonal, i.e., $A^{11}, A^{22}, \dots, A^{nn}$. In remaining $n^2 - n$

components: $A^{12} = A^{21}, A^{23} = A^{32}$, and so on.

due to symmetry. So, the distinct components are $\frac{1}{2}(n^2 - n)$.

Hence the total number of different components are $= n + \frac{1}{2}(n^2 - n) = \frac{2n - n^2 - n}{2}$
 $= \frac{1}{2}(n + n^2) = \frac{1}{2}n(n+1)$

Skew symmetric tensor

A tensor is said to be skew symmetric with respect to two contravariant or two covariant indices if its components change sign on interchange of the two indices.

For example: (1) The tensor A^{ij} is skew-symmetric if $A^{ij} = -A^{ji}$

(2) The tensor A_{lm}^{ijk} is skew-symmetric w.r.t the indices j and k if:

$$A_{lm}^{ijk} = -A_{lm}^{ikj}$$

Question: A skew symmetric tensor of second order has only $\frac{1}{2}n(n-1)$ different non-zero components.

Solution: let A^{ij} be a skew symmetric tensor. Then $A^{ij} = -A^{ji}$ when $i=j$ (i.e., for diagonal elements)

$$A^{ii} = -A^{ii} \Rightarrow 2A^{ii} = 0 \Rightarrow A^{ii} = 0$$

$\Rightarrow$ diagonal elements of a second order skew symmetric tensor are zero.

Hence

$$A = [A^{ij}] = \begin{bmatrix} 0 & A^{12} & A^{13} & \dots & A^{1n} \\ A^{21} & 0 & A^{23} & \dots & A^{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ A^{n1} & A^{n2} & A^{n3} & \dots & A^{nn} \end{bmatrix}$$

Now A^{ij} has n^2 components out of which n diagonal components are zero. So we are left with $n^2 - n$ components.

Also $A^{12} = -A^{21}$, $A^{13} = -A^{31}$ and so on.

$\Rightarrow$ The different or distinct components of A^{ij} are

$$\frac{1}{2}(n^2 - n)$$

or $\frac{1}{2}n(n-1)$

Question: A contravariant or covariant tensor of rank two say A_{ij} can always be written as the sum of a symmetric and skew symmetric tensor.

Solution: consider a covariant tensor

$$A_{ij} = \frac{1}{2}(A_{ij} + A_{ji}) + \frac{1}{2}(A_{ij} - A_{ji})$$

$$\Rightarrow A_{ij} = S_{ij} + T_{ij}$$

where $S_{ij} = \frac{1}{2} (A_{ij} + A_{ji})$

and $T_{ij} = \frac{1}{2} (A_{ij} - A_{ji})$

Now, $S_{ji} = \frac{1}{2} (A_{ji} + A_{ij}) = S_{ij}$

$\Rightarrow S_{ij}$ is symmetric tensor.

and $T_{ji} = \frac{1}{2} (A_{ji} - A_{ij}) = -\frac{1}{2} (A_{ij} - A_{ji})$

$\Rightarrow T_{ji} = -T_{ij}$

$\Rightarrow T_{ij}$ is skew symmetric tensor.

Question: If $\phi = a_{jk} A^j A^k$. Show that we can always write $\phi = b_{jk} A^j A^k$ where b_{jk} is symmetric.

Solution: As $\phi = a_{jk} A^j A^k$

Interchange the indices j and k

$$\phi = a_{kj} A^k A^j$$

and adding both expressions

$$2\phi = (a_{jk} + a_{kj}) A^j A^k$$

$$\Rightarrow \phi = \frac{1}{2} (a_{jk} + a_{kj}) A^j A^k$$

$$\Rightarrow \phi = b_{jk} A^j A^k$$

$$\text{where } b_{jk} = \frac{1}{2} (a_{jk} + a_{kj})$$

$$\text{and } b_{kj} = \frac{1}{2} (a_{kj} + a_{jk}) = b_{jk}$$

$$\Rightarrow b_{jk} \text{ is symmetric.}$$

Question: If T_i be the component of a covariant vector show that $\left(\frac{\partial T_i}{\partial x^j} - \frac{\partial T_j}{\partial x^i}\right)$ are components of a skew symmetric tensor of rank 2.

Solution: As T_i is a covariant tensor then

$$\tilde{T}_i = \frac{\partial x^k}{\partial \tilde{x}^i} T_k$$

differentiating it w.r.t $\tilde{x}^j$ partially,

$$\frac{\partial \tilde{T}_i}{\partial \tilde{x}^j} = \frac{\partial}{\partial \tilde{x}^j} \left(\frac{\partial x^k}{\partial \tilde{x}^i} T_k \right)$$

$$= \frac{\partial^2 x^k}{\partial \tilde{x}^j \partial \tilde{x}^i} T_k + \frac{\partial x^k}{\partial \tilde{x}^i} \frac{\partial T_k}{\partial \tilde{x}^j}$$

$$\Rightarrow \frac{\partial \tilde{T}_i}{\partial \tilde{x}^j} = \frac{\partial^2 x^k}{\partial \tilde{x}^j \partial \tilde{x}^i} T_k + \frac{\partial x^k}{\partial \tilde{x}^i} \frac{\partial x^l}{\partial \tilde{x}^j} \frac{\partial T_k}{\partial x^l} \rightarrow (1)$$

and similarly:

$$\frac{\partial \tilde{T}_j}{\partial \tilde{x}^i} = \frac{\partial^2 x^k}{\partial \tilde{x}^i \partial \tilde{x}^j} T_k + \frac{\partial x^k}{\partial \tilde{x}^i} \frac{\partial x^l}{\partial \tilde{x}^j} \frac{\partial T_l}{\partial x^k}$$

interchanging the dummy indices k and l in second term:

$$\frac{\partial \tilde{T}_j}{\partial \tilde{x}^i} = \frac{\partial^2 x^k}{\partial \tilde{x}^i \partial \tilde{x}^j} T_k + \frac{\partial x^k}{\partial \tilde{x}^i} \frac{\partial x^l}{\partial \tilde{x}^j} \frac{\partial T_k}{\partial \tilde{x}^l} \rightarrow (2)$$

now subtracting (1) and (2), we have

$$\frac{\partial \tilde{T}_i}{\partial \tilde{x}^j} - \frac{\partial \tilde{T}_j}{\partial \tilde{x}^i} = \frac{\partial x^k}{\partial \tilde{x}^i} \frac{\partial x^l}{\partial \tilde{x}^j} \left(\frac{\partial T_k}{\partial x^l} - \frac{\partial T_l}{\partial x^k} \right)$$

$\Rightarrow \frac{\partial T_i}{\partial x^j} - \frac{\partial T_j}{\partial x^i}$ is a second rank tensor.

Now consider:

$$A_{ij} = \frac{\partial T_i}{\partial x^j} - \frac{\partial T_j}{\partial x^i}$$

$$\Rightarrow A_{ji} = \frac{\partial T_j}{\partial x^i} - \frac{\partial T_i}{\partial x^j}$$

$$= - \left(\frac{\partial T_i}{\partial x^j} - \frac{\partial T_j}{\partial x^i} \right) = - A_{ij}$$

$$\text{or } A_{ij} = - A_{ji}$$

$\Rightarrow A_{ij}$ is Skew Symmetric tensor of rank two.

Question: If the tensors a_{ij} and g_{ij} are symmetric and u^i, v^i are components of contravariant vectors satisfying the equations

$$(a_{ij} - k g_{ij}) u^i = 0, i, j = 1, 2, \dots, n$$

$$(a_{ij} - k' g_{ij}) v^i = 0, k \neq k'$$

Prove that $g_{ij} u^i v^j = 0$

$$a_{ij} u^i v^j = 0$$

Solution: consider the given equations:

$$(a_{ij} - k g_{ij}) u^i = 0 \longrightarrow (1)$$

$$(a_{ij} - k' g_{ij}) v^i = 0 \longrightarrow (2)$$

Multiplying (1) by v^j and (2) by u^j :

$$(a_{ij} - k g_{ij}) u^i v^j = 0 \longrightarrow (1)'$$

$$(a_{ij} - k' g_{ij}) v^i u^j = 0 \longrightarrow (2)'$$

subtracting $(1)'$ and $(2)'$:

$$a_{ij}u^i v^j - a_{ij}v^i u^j - kg_{ij}u^i v^j + k'g_{ij}u^j v^i = 0$$

Interchanging i and j in the second and fourth terms:

$$a_{ij}u^i v^j - a_{ji}u^j v^i - kg_{ij}u^i v^j + k'g_{ji}u^j v^i = 0$$

$$\Rightarrow (a_{ij} - a_{ji})u^i v^j - (kg_{ij} - k'g_{ji})u^i v^j = 0$$

also, $a_{ij} = a_{ji}$, $g_{ij} = g_{ji}$ and $k \neq k'$

$$\Rightarrow 0 - kg_{ij}u^i v^j + k'g_{ij}u^i v^j = 0$$

$$\Rightarrow (k' - k) g_{ij} u^i v^j = 0 \quad k' - k \neq 0$$

$$\Rightarrow g_{ij} u^i v^j = 0$$

Now multiplying (1) by v^j :

$$a_{ij} v^j u^i - k \cancel{g_{ij}} u^i v^j = 0$$

$$\Rightarrow a_{ij} u^i v^j = 0$$

Question: If a_{ij} is symmetric tensor and b_i is a vector and

$$a_{ij}b_k + a_{jk}b_i + a_{ki}b_j = 0$$

then prove that $a_{ij} = 0$ or $b_k = 0$.

Solution: Consider:

$$a_{ij}b_k + a_{jk}b_i + a_{ki}b_j = 0$$

$$\Rightarrow \tilde{a}_{ij}\tilde{b}_k + \tilde{a}_{jk}\tilde{b}_i + \tilde{a}_{ki}\tilde{b}_j = 0$$

using transformation laws we have:

$$a_{pq} \frac{\partial x^p}{\partial \tilde{x}^i} \frac{\partial x^q}{\partial \tilde{x}^j} b_r \frac{\partial x^r}{\partial \tilde{x}^k} + a_{pq} \frac{\partial x^p}{\partial \tilde{x}^j} \frac{\partial x^q}{\partial \tilde{x}^k} b_r \frac{\partial x^r}{\partial \tilde{x}^i}$$

$$+ a_{pq} \frac{\partial x^p}{\partial \tilde{x}^k} \frac{\partial x^q}{\partial \tilde{x}^i} b_r \frac{\partial x^r}{\partial \tilde{x}^j} = 0$$

$$\Rightarrow a_{pq} b_r \left[\frac{\partial x^p}{\partial \tilde{x}^i} \frac{\partial x^q}{\partial \tilde{x}^j} \frac{\partial x^r}{\partial \tilde{x}^k} + \frac{\partial x^p}{\partial \tilde{x}^j} \frac{\partial x^q}{\partial \tilde{x}^k} \frac{\partial x^r}{\partial \tilde{x}^i} + \frac{\partial x^p}{\partial \tilde{x}^k} \frac{\partial x^q}{\partial \tilde{x}^i} \frac{\partial x^r}{\partial \tilde{x}^j} \right] = 0$$

$$\Rightarrow a_{pq} b_r = 0 \Rightarrow a_{pq} = 0 \text{ or } b_r = 0$$

$$\Rightarrow a_{ij} = 0 \text{ or } b_k = 0$$

Question: If A_{ij} is a skew symmetric tensor then $(B_j^i B_l^k + B_l^i B_j^k) A_{ik} = 0$

Solution: As A_{ij} is skew symmetric
 $\Rightarrow A_{ij} = -A_{ji}$

Consider,

$$T = (B_j^i B_l^k + B_l^i B_j^k) A_{ik}$$

$$= B_j^i B_l^k A_{ik} + B_l^i B_j^k A_{ik}$$

$$= B_j^i B_l^k A_{ik} + B_l^k B_j^i A_{ki}$$

$$= \cancel{B_j^i B_l^k A_{ik}} - \cancel{B_l^k B_j^i A_{ik}} = 0 \text{ proved.}$$

Question: If A_{ij} is a skew symmetric tensor
prove that:

$$(\delta_j^i \delta_l^k + \delta_l^i \delta_j^k) A_{ik} = 0$$

Solution: Given A_{ij} is a skew symmetric tensor then $A_{ij} = -A_{ji}$. Consider

$$\begin{aligned} (\delta_j^i \delta_l^k + \delta_l^i \delta_j^k) A_{ik} &= \delta_j^i \delta_l^k A_{ik} + \delta_l^i \delta_j^k A_{ik} \\ &= \delta_j^i A_{il} + \delta_l^i A_{ij} \\ &= A_{jl} + A_{lj} \\ &= A_{jl} - A_{jl} = 0 \end{aligned}$$

Question: If $p_{ij} \neq 0$ are components of a tensor and if the equation $f p_{ij} + g p_{ji} = 0$ holds where f and g are non-zero scalars then prove that either $f = g$ and p_{ij} is skew symmetric or $f = -g$ and p_{ij} is symmetric.

Solution: As $f p_{ij} + g p_{ji} = 0$

$$\Rightarrow f p_{ij} = -g p_{ji}$$

$$\begin{aligned}\Rightarrow f^2 p_{ij} &= -f g p_{ji} = -g (f p_{ji}) \\ &= g (g p_{ij})\end{aligned}$$

$$f^2 p_{ij} = g^2 p_{ij}$$

$$\Rightarrow (f^2 - g^2) p_{ij} = 0$$

$$\Rightarrow p_{ij} = 0 \text{ or } f^2 - g^2 = 0$$

$$\text{but } p_{ij} \neq 0$$

$$\Rightarrow f^2 - g^2 = 0 \Rightarrow f = \pm g$$

Case-I: when $f = g$

$$\text{then } f p_{ij} = -g p_{ji}$$

$$\Rightarrow f p_{ij} = -f p_{ji}$$

$$\Rightarrow p_{ij} = -p_{ji}$$

$\Rightarrow P_{ij}$ is skew symmetric.

Case II: when $f = -g$

$$\Rightarrow f P_{ij} = f P_{ji}$$

$$\Rightarrow P_{ij} = P_{ji}$$

$\Rightarrow P_{ij}$ is symmetric.

Conjugate symmetric tensor:

Consider a covariant symmetric tensor A_{ij} of rank two. Let d denote the determinant of this tensor i.e., $d = |A_{ij}|$ and $d \neq 0$. Then define A^{ij} by

$$A^{ij} = \frac{\text{Cofactor of } A_{ij} \text{ in the determinant } |A_{ij}|}{d}$$

' A^{ij} ' is a contravariant symmetric tensor of rank two which is called conjugate OR reciprocal tensor of A_{ij} .

Question: Find the conjugate or reciprocal of the tensor

$$[A_{ij}] = \begin{bmatrix} 1 & x & y \\ x & 1 & 0 \\ y & 0 & -1 \end{bmatrix}$$

Solution: The conjugate of A_{ij} is:

$$A^{ij} = \frac{\text{Cofactor of } A_{ij} \text{ in } |A_{ij}|}{d}$$

where $d = |A_{ij}|$

First find $|A_{ij}| = d$.

$$d = \begin{vmatrix} 1 & x & y \\ x & 1 & 0 \\ y & 0 & -1 \end{vmatrix}$$

Expanding the determinant by first row:

$$d = 1 \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} - x \begin{vmatrix} x & 0 \\ y & -1 \end{vmatrix} + y \begin{vmatrix} x & 1 \\ y & 0 \end{vmatrix}$$
$$= 1(-1) - x(-x) + y(-y)$$

$$d = -1 + x^2 + y^2$$

$$d = x^2 + y^2 - 1$$

So

$$A'' = \frac{\text{cof. of } A_{11}}{d} = (-1)^2 \frac{\begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}}{d} = \frac{-1}{d}$$

$$A^{12} = \frac{\text{cof. of } A_{12}}{d} = (-1)^3 \frac{\begin{vmatrix} x & 0 \\ y & -1 \end{vmatrix}}{d} = \frac{x}{d}$$

$$A^{13} = \frac{\text{cof. of } A_{13}}{d} = (-1)^4 \frac{\begin{vmatrix} x & 1 \\ y & 0 \end{vmatrix}}{d} = \frac{-y}{d}$$

$$A^{21} = \frac{\text{cof. of } A_{21}}{d} = (-1)^3 \frac{\begin{vmatrix} x & y \\ 0 & -1 \end{vmatrix}}{d} = \frac{x}{d}$$

$$A^{22} = \frac{\text{cof. of } A_{22}}{d} = (-1)^4 \begin{vmatrix} 1 & y \\ y & -1 \end{vmatrix} = \frac{-1 - y^2}{d}$$

$$A^{23} = \frac{\text{cof. of } A_{23}}{d} = (-1)^5 \begin{vmatrix} 1 & x \\ y & 0 \end{vmatrix} = \frac{xy}{d}$$

$$A^{31} = \frac{\text{cof. of } A_{31}}{d} = (-1)^4 \begin{vmatrix} x & y \\ 1 & 0 \end{vmatrix} = \frac{-y}{d}$$

$$A^{32} = \frac{\text{cof. of } A_{32}}{d} = (-1)^5 \begin{vmatrix} 1 & y \\ x & 0 \end{vmatrix} = \frac{xy}{d}$$

$$A^{33} = \frac{\text{cof. of } A_{33}}{d} = (-1)^6 \begin{vmatrix} 1 & x \\ x & 1 \end{vmatrix} = \frac{1 - x^2}{d}$$

$$\Rightarrow [A^{ij}] = \begin{bmatrix} -\frac{1}{d} & \frac{x}{d} & -\frac{y}{d} \\ \frac{x}{d} & -\frac{1-y^2}{d} & \frac{xy}{d} \\ -\frac{y}{d} & \frac{xy}{d} & 1-\frac{x^2}{d} \end{bmatrix} \quad \begin{bmatrix} 1 & x & y \\ x & 1 & 0 \\ y & 0 & -1 \end{bmatrix} \downarrow [A_{ij}]$$

You can check that

$$[A_{ij}][A^{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

Also if A^{ij} is a symmetric tensor.

Note : If B_{ij} is the cofactor of the element A_{ij} in the determinant $d = |A_{ij}| \neq 0$ and A^{ij} defined as:

$$A^{ij} = \frac{B_{ij}}{d}$$

Then :

$$A_{ij} A^{kj} = \delta_i^k$$

Cartesian Tensors:

A cartesian tensor is a tensor in a three dimensional Euclidean space. Unlike general tensors there is no distinction between covariant and contravariant indices for Cartesian tensors. However, tensors in non Euclidean spaces do require this distinction.

In other words, tensors defined in a geometry obeying the coordinate transformation from non tilde system to tilde system as

$$\tilde{x}^i = a_j^i x^j \text{ with } |a_j^i| \neq 0$$

and the transformation matrix a_j^i is orthogonal are called Cartesian tensors. This transformation should also be distance preserving.

Transformation laws for Cartesian tensors:

$$\text{If } J = \left[\frac{\partial \tilde{x}^i}{\partial x^j} \right]_{nn} \text{ and } J^{-1} = \left[\frac{\partial x^i}{\partial \tilde{x}^j} \right]$$

$$\star J^{-1} = J^t$$

$$\underline{\underline{b^i_j = a^j_i}}$$

$$\text{Or } J = [a^i_j]_{nn} \text{ and } J^{-1} = [b^i_j]_{nn}$$

the transformation laws for Cartesian tensors are,

$$\left. \begin{array}{l} \text{Contra: } \tilde{T}^i = a^i_r T^r \\ \text{Cov: } \tilde{T}_i = a^r_i T_r \end{array} \right\} \text{Rank one tensors}$$

For rank two tensors:

$$\tilde{T}^{ij} = a^i_r a^j_s T^{rs}$$

$$\tilde{T}_{ij} = a^i_r a^j_s T_{rs}$$

$$\tilde{T}^i_j = a^i_r a^j_s T^{rs}$$

Here contravariant and covariant behaviors do not distinguishable
So, Consequently Cartesian tensors can be written as:

$$\tilde{x}_i = a_{ij} x_j \text{ or } x_i = a_{ji} \tilde{x}_j$$

coordinate transformations

$$\tilde{T}_i = a_{ik} T_k, \quad \tilde{T}_{ij} = a_{ik} a_{js} T_{ks}$$

Note that $J J^t = I$ implies

$$|J J^t| = |I|$$

$$\Rightarrow |J| |J^t| = |I| = 1$$

$$\Rightarrow |J| |J| = 1$$

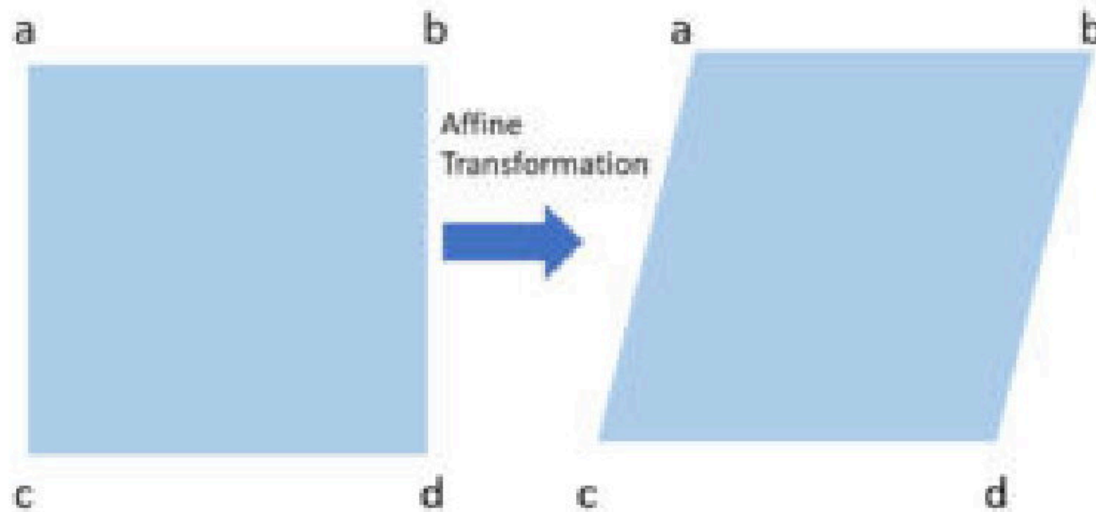
$$\Rightarrow |J|^2 = 1$$

$$\Rightarrow |J| = \pm 1$$

when $|J| = 1$, then the corresponding tensors are called direct tensors.

Affine Tensors

The adjective "affine" indicates everything that is related to the geometry of affine spaces. A coordinate system for the n -dimensional affine space is determined by any basis of n vectors, which are not necessarily orthonormal. Therefore the resulting axes are not mutually perpendicular.



Consider a transformation of the form that takes the points from a rectangular coordinate system into an affine coordinate system. Tensors defined on the class of all such affine coordinate systems are called affine tensors.

$$\text{As } J = \left[\frac{\partial \tilde{x}^i}{\partial x^j} \right]_{nn} = [a^i_j]_{nn} \text{ and}$$

$$J^{-1} = \left[\frac{\partial x^i}{\partial \tilde{x}^j} \right]_{nn} = [b^i_j]_{nn}$$

The transformation laws for affine tensors are:

$$\tilde{T}^{ij} = a^i_k a^j_s T^{ks} \rightarrow \text{contravariant}$$

$$\tilde{T}_{ij} = b^k_i b^s_j T_{ks} \rightarrow \text{covariant}$$

$$\tilde{T}^i_j = a^i_k b^s_j T^k_s \rightarrow \text{Mixed}$$

Metric Tensor:

This is a rank two tensor and is also called the fundamental tensor.

The main purpose of the metric tensor is to generalize the concept of distance to general curvilinear coordinate frames and maintain the invariance of distance in different coordinate systems.

Explanation:

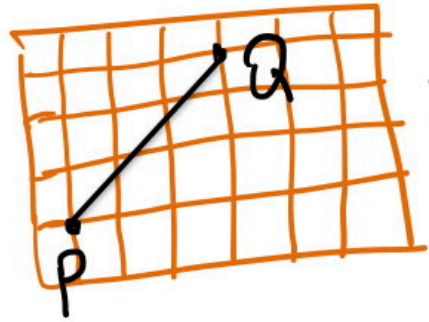
Every point in coordinate system has its own metric tensor. We do not care about this usually because our Euclidean coordinate system has same metric tensor everywhere. In advance Physics and Mathematics, we have to deal with non-Euclidean

Spaces with varying metric tensor at every point.

To understand this, first we discuss distance between two points in Euclidean space.

Distance between 2-points:

Consider a rubber sheet

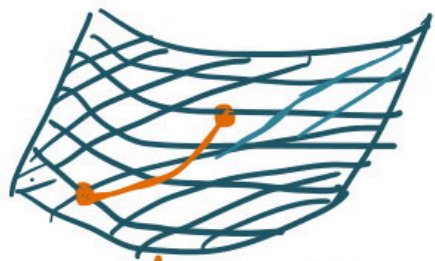


→ unstretched sheet
 $ds^2 = dx^2 + dy^2$
(distance formula)

If this sheet is stretched in both x and y direction 2-times, then

$$ds^2 = (2dx^2 + 2dy^2)$$

But if we stretch it in some random direction. We cannot write the distance b/w two points using distance formula. As the rubber sheet may be like this



→ stretched in random direction.

and the corresponding coordinate system may be non-Euclidean.

In this case the distance will be

$$ds^2 = g_{11} dx^2 + (g_{12} + g_{21}) dx dy + g_{22} dy^2$$

Hence whole stretching information is captured by these coefficients

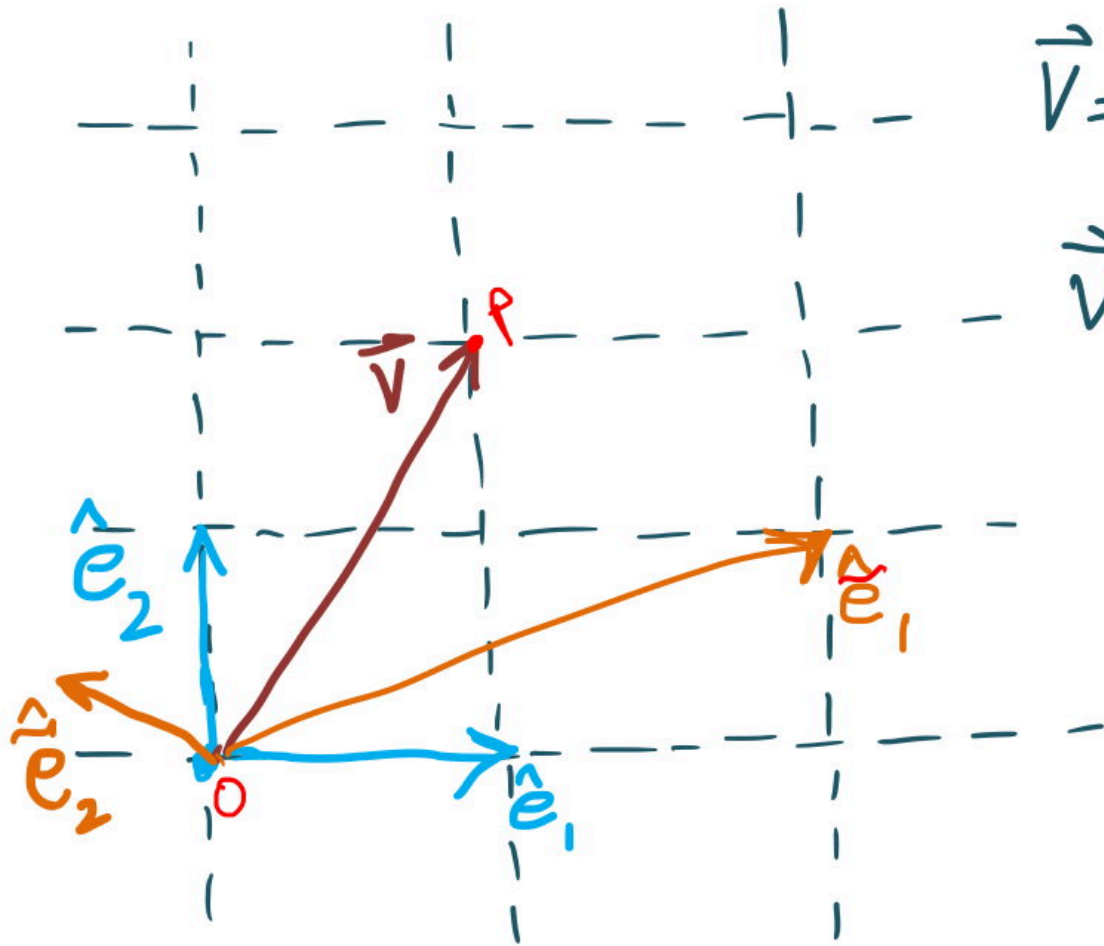
$(g_{11}, g_{12}, g_{21}, g_{22})$. Furthermore, this information is stored in the form of tensor called metric tensor.

Another Explanation:

Consider a vector $\vec{V}$. Such that

$$\vec{V} = v^1 \hat{e}_1 + v^2 \hat{e}_2$$

$$\vec{V} = \tilde{v}^1 \hat{e}_1 + \tilde{v}^2 \hat{e}_2$$



Now:

$$\begin{aligned} |\vec{V}| &= \vec{V} \cdot \vec{V} = (v^1 \hat{e}_1 + v^2 \hat{e}_2) \cdot (v^1 \hat{e}_1 + v^2 \hat{e}_2) \\ &= (v^1)^2 (\hat{e}_1 \cdot \hat{e}_1) + 2v^1 v^2 (\hat{e}_1 \cdot \hat{e}_2) + \\ &\quad (v^2)^2 (\hat{e}_2 \cdot \hat{e}_2) \end{aligned}$$

As system is orthonormal

$$\Rightarrow |\vec{V}| = (v^1)^2 + (v^2)^2$$

corresponding Metric: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Similarly in tild system, we have

$$|\vec{V}| = (\tilde{V}^1)^2 (\hat{\tilde{e}}_1 \cdot \hat{\tilde{e}}_1) + 2\tilde{V}^1 \tilde{V}^2 (\hat{\tilde{e}}_1 \cdot \hat{\tilde{e}}_2) + (\tilde{V}^2)^2 (\hat{\tilde{e}}_2 \cdot \hat{\tilde{e}}_2)$$

consider

$$\hat{\tilde{e}}_1 = c_1 \hat{e}_1 + c_2 \hat{e}_2$$

$$\hat{\tilde{e}}_2 = c_3 \hat{e}_1 + c_4 \hat{e}_2$$

Implies:

$$\hat{\tilde{e}}_1 \cdot \hat{\tilde{e}}_1 = c_1^2 + c_2^2 \rightarrow g_{11}$$

$$\hat{\tilde{e}}_1 \cdot \hat{\tilde{e}}_2 = c_1 c_3 + c_2 c_4 \rightarrow g_{12}, g_{21}$$

$$\hat{\tilde{e}}_2 \cdot \hat{\tilde{e}}_2 = c_3^2 + c_4^2 \rightarrow g_{22}$$

$$\Rightarrow |\vec{v}| = (v^1)^2 (c_1^2 + c_2^2) + 2v^1 v^2 (c_1 c_3 + c_2 c_4) + (v^2)^2 (c_3^2 + c_4^2)$$

Implies in tild system the metric tensor is:

$$[g_{ij}] = \begin{bmatrix} c_1^2 + c_2^2 & c_1 c_3 + c_2 c_4 \\ c_1 c_3 + c_2 c_4 & c_3^2 + c_4^2 \end{bmatrix}$$

$$\begin{aligned} |\vec{V}| &= (\vec{V}) [g_{ij}] (\vec{V})^t \\ &= \begin{bmatrix} v^1 & v^2 \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \end{aligned}$$

Hence:

$$[g_{ij}] = \begin{bmatrix} \hat{e}_1 \cdot \hat{e}_1 & \hat{e}_1 \cdot \hat{e}_2 \\ \hat{e}_2 \cdot \hat{e}_1 & \hat{e}_2 \cdot \hat{e}_2 \end{bmatrix}$$

$$[\tilde{g}_{ij}] = \begin{bmatrix} \tilde{e}_1 \cdot \tilde{e}_1 & \tilde{e}_1 \cdot \tilde{e}_2 \\ \tilde{e}_2 \cdot \tilde{e}_1 & \tilde{e}_2 \cdot \tilde{e}_2 \end{bmatrix}$$

$$\Rightarrow g_{ij} = \hat{e}_i \cdot \hat{e}_j \quad \text{and} \quad \tilde{g}_{ij} = \tilde{e}_i \cdot \tilde{e}_j$$

$$\text{As } \tilde{g}_{ij} = \hat{e}_i \cdot \hat{e}_j$$

$$\tilde{g}_{ij} = \frac{\partial x^k}{\partial \tilde{x}^i} \hat{e}_k \cdot \frac{\partial x^l}{\partial \tilde{x}^j} \hat{e}_l$$

$$= \frac{\partial x^k}{\partial \tilde{x}^i} \frac{\partial x^l}{\partial \tilde{x}^j} (\hat{e}_k \cdot \hat{e}_l)$$

$\Rightarrow g_{ij}$ is a covariant tensor
of rank 2.

The metric tensor ' g_{ij} ' is a function which tells how to compute the distance between any two points in a given space. Generally

$$ds^2 = g_{11}(dx^1)^2 + g_{12} dx^1 dx^2 + g_{22}(dx^2)^2 + \dots$$

or

$$ds^2 = g_{ij} dx^i dx^j$$

Examples:

1) For Euclidean space in a rectangular coordinate system

$$g_{ij} = \delta_{ij}$$

$\Rightarrow$ In 3D

$$ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2$$

$$ds^2 = g_{ij} dx^i dx^j$$

2) In polar coordinates: $(x^1, x^2) = (r, \theta)$

$$ds^2 = (dx^1)^2 + (x^1)^2 (dx^2)^2$$

or $ds^2 = dr^2 + r^2 d\theta^2 \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix}$

3) In cylindrical coordinates:
 $(x^1, x^2, x^3) = (r, \theta, z)$

$$ds^2 = dr^2 + r^2 d\theta^2 + dz^2 \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & (x^1)^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$ds^2 = (dx^1)^2 + (x^1)^2 (dx^2)^2 + (dx^3)^2$$

4) Spherical coordinates:

$$(x^1, x^2, x^3) = (r, \theta, \phi)$$

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$$

$$ds^2 = (dx^1)^2 + (x^1)^2 (dx^2)^2 + (x^1 \sin x^2)^2 (dx^3)^2$$

Properties of Metric Tensor:

Assume that a matrix $g = [g_{ij}]$ exists satisfying in all (admissible) coordinate systems x^i :

- 1) g is of differentiability class C^2 .
- 2) g is symmetric i.e., $g_{ij} = g_{ji}$
- 3) g is non-singular i.e., $|g_{ij}| \neq 0$.

4) $ds^2 = g_{ij} dx^i dx^j$ and hence the distance concept generated by g is invariant with respect to a change of coordinates.

$$\underline{d\tilde{s}^2 = ds^2}$$

Example: Suppose that on $\mathbb{R}^3$ a matrix field is given in x^i -system by:

$$[g_{ij}] = \begin{bmatrix} (x^1)^2 - 1 & 1 & 0 \\ 1 & (x^2)^2 & 0 \\ 0 & 0 & 64/9 \end{bmatrix}$$

where $[(x^1)^2 - 1](x^2)^2 \neq 1$

Show that this matrix field is a metric.

Solution:

To prove that $[g_{ij}]$ is a metric tensor, we show that it satisfies the properties of metric tensor.

(1) Since g_{ij} is a polynomial in x^1 and x^2 for each i and j . Implies it is of differentiability class C^2 .

$$(2) \quad [g_{ji}] = \begin{bmatrix} (x^1)^2 - 1 & 1 & 0 \\ 1 & (x^2)^2 & 0 \\ 0 & 0 & 64/9 \end{bmatrix} = [g_{ij}]$$

$\Rightarrow g = [g_{ij}]$ is symmetric tensor.

$$\begin{aligned}
 3) \text{ Also } |g_{ij}| &= \frac{64}{9} \begin{vmatrix} (x^1)^2 - 1 & 1 \\ 1 & (x^2)^2 \end{vmatrix} \\
 &= \frac{64}{9} \{ (x^2)^2 [(x^1)^2 - 1] - 1 \} \neq 0
 \end{aligned}$$

$$4) \text{ As } \tilde{g}_{ij} = g_{rs} \frac{\partial x^r}{\partial \tilde{x}^i} \frac{\partial x^s}{\partial \tilde{x}^j}$$

$$\text{let } dx^i = v^i \text{ and } dx^j = v^j$$

$$\text{then } ds^2 = g_{ij} v^i v^j$$

$$\tilde{v}^i = v^t \frac{\partial \tilde{x}^i}{\partial x^t} \text{ and } \tilde{v}^j = v^l \frac{\partial \tilde{x}^j}{\partial x^l}$$

Now:

$$(d\tilde{s})^2 = \tilde{g}_{ij} \tilde{U}^i \tilde{V}^j = g_{rs} U^t V^l \frac{\partial x^r}{\partial \tilde{x}^i} \frac{\partial x^s}{\partial \tilde{x}^j} \frac{\partial \tilde{x}^i}{\partial x^t} \frac{\partial \tilde{x}^j}{\partial x^l}$$

$$= g_{rs} U^t V^l \delta_t^r \delta_l^s$$

$$= g_{rs} U^r V^s$$

$$= g_{ij} U^i V^j$$

$$(d\tilde{s})^2 = (ds)^2$$

$\Rightarrow ds^2$ is invariant.

Important Result:

If the Jacobian matrix of the transformation from a given coordinate system x^i to a rectangular coordinate system $\tilde{x}^i$ is

$$J = \frac{\partial \tilde{x}^i}{\partial x^j},$$

then the matrix $g = [g_{ij}]$ of the Euclidean metric tensor in the x^i -system is given by

$$g = J^T J$$

Note:

Only Euclidean metric admits a representation of the form $g = J^t J$.

Because if g is non-Euclidean, there exists no coordinate system $\tilde{x}^i$ in which

$$\tilde{g} = I$$

This can be checked as:

consider
$$g_{ij} = \frac{\partial \tilde{x}^r}{\partial x^i} \frac{\partial \tilde{x}^s}{\partial x^j} \tilde{g}_{rs}$$

Now just consider:

$$g_{||} = \frac{\partial \tilde{x}^r}{\partial x'} \frac{\partial \tilde{x}^s}{\partial x'} \tilde{g}_{rs}$$

$$\Rightarrow g_{||} = \frac{\partial \tilde{x}^1}{\partial x'} \frac{\partial \tilde{x}^1}{\partial x'} + \frac{\partial \tilde{x}^2}{\partial x'} \frac{\partial \tilde{x}^2}{\partial x'} + \frac{\partial \tilde{x}^3}{\partial x'} \frac{\partial \tilde{x}^3}{\partial x'}$$

if we consider $[\tilde{g}_{rs}] = [\delta_{rs}] = I$

similarly from $g = J^t J$

$$g_{||} = \frac{\partial \tilde{x}^1}{\partial x'} \frac{\partial \tilde{x}^1}{\partial x'} + \frac{\partial \tilde{x}^2}{\partial x'} \frac{\partial \tilde{x}^2}{\partial x'} + \frac{\partial \tilde{x}^3}{\partial x'} \frac{\partial \tilde{x}^3}{\partial x'}$$

hence $g = [g_{ij}] = J^t J = \left[\frac{\partial \tilde{x}^i}{\partial x^j} \right] \left[\frac{\partial \tilde{x}^j}{\partial x^i} \right]$

if $[\tilde{g}_{ij}] = [\delta_{ij}] = I$

Question: Find metric tensor for cylindrical coordinates x^i , connected to rectangular coordinates $\tilde{x}^i$ through the relations:

$$\tilde{x}^1 = x^1 \cos x^2, \quad \tilde{x}^2 = x^1 \sin x^2, \quad \tilde{x}^3 = x^3$$

Solution: First we find Jacobian matrix:

$$J = \left[\frac{\partial \tilde{x}^i}{\partial x^j} \right] = \begin{bmatrix} \cos x^2 & -x^1 \sin x^2 & 0 \\ \sin x^2 & x^1 \cos x^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

hence the Euclidean metric for cylindrical coordinates is given by:

$$g = J^t J = \begin{bmatrix} \cos x^2 & \sin x^2 & 0 \\ -x^1 \sin x^2 & x^1 \cos x^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos x^2 & -x^1 \sin x^2 & 0 \\ \sin x^2 & x^1 \cos x^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & (x^1)^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Question: Find the components g_{ij} of the Euclidean metric tensor in the special coordinate system x^i , related to rectangular coordinates $\tilde{x}^i$ by:

$$x^1 = \tilde{x}^1, \quad x^2 = e^{\tilde{x}^2 - \tilde{x}^1}$$

$$J = \begin{bmatrix} \frac{\partial \tilde{x}^1}{\partial x^1} & \frac{\partial \tilde{x}^1}{\partial x^2} \\ \frac{\partial \tilde{x}^2}{\partial x^1} & \frac{\partial \tilde{x}^2}{\partial x^2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1/x^2 \end{bmatrix}$$

$$\star \quad \tilde{x}^1 = x^1 \quad \text{and} \quad \tilde{x}^2 = x^1 + \ln x^2$$

Implies:

$$g = J^T J = \begin{bmatrix} 1 & 1 \\ 0 & 1/x^2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1/x^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1 & 1/x^2 \\ 1/x^2 & (1/x^2)^2 \end{bmatrix} = \begin{bmatrix} 2 & 1/x^2 \\ 1/x^2 & (1/x^2)^2 \end{bmatrix}$$

Metric Tensor and Arc Lengths:

Arc length of a curve $\vec{x} = \vec{x}(t)$ is generally defined as:

$$L = \int_a^b \left| \frac{d\vec{x}}{dt} \right| dt$$

$$ds^2 = g_{ij} dx^i dx^j$$

Now $\vec{x} = x^1 \hat{e}_1 + x^2 \hat{e}_2$

$$\frac{d\vec{r}}{dt} = \frac{\partial \vec{r}}{\partial x^1} \frac{dx^1}{dt} + \frac{\partial \vec{r}}{\partial x^2} \frac{dx^2}{dt}$$

$$\left| \frac{d\vec{r}}{dt} \right|^2 = \frac{d\vec{r}}{dt} \cdot \frac{d\vec{r}}{dt}$$

$$= \left(\frac{dx^1}{dt} \right)^2 \left(\frac{\partial \vec{r}}{\partial x^1} \cdot \frac{\partial \vec{r}}{\partial x^1} \right) + 2 \frac{dx^1}{dt} \frac{dx^2}{dt} \left(\frac{\partial \vec{r}}{\partial x^1} \cdot \frac{\partial \vec{r}}{\partial x^2} \right)$$

$$+ \left(\frac{dx^2}{dt} \right)^2 \left(\frac{\partial \vec{r}}{\partial x^2} \cdot \frac{\partial \vec{r}}{\partial x^2} \right)$$

$$\Rightarrow \left| \frac{d\vec{r}}{dt} \right|^2 = \frac{dx^i}{dt} \frac{\partial \vec{r}}{\partial x^i} \cdot \frac{dx^j}{dt} \frac{\partial \vec{r}}{\partial x^j}$$

$$= \frac{dx^i}{dt} \frac{dx^j}{dt} \left(\frac{\partial \vec{r}}{\partial x^i} \cdot \frac{\partial \vec{r}}{\partial x^j} \right)$$

$$= \frac{dx^i}{dt} \frac{dx^j}{dt} (\hat{e}_i \cdot \hat{e}_j)$$

$$(\because \hat{e}_i \cdot \hat{e}_j = g_{ij})$$

$$\left| \frac{d\vec{r}}{dt} \right| = \sqrt{g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt}}$$

$$\Rightarrow L = \int_a^b \sqrt{g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt}} dt$$

Example: Suppose that on $\mathbb{R}^3$ a matrix field is given in x^i -system by:

$$[g_{ij}] = \begin{bmatrix} (x^1)^2 - 1 & 1 & 0 \\ 1 & (x^2)^2 & 0 \\ 0 & 0 & 64/9 \end{bmatrix}$$

where $[(x^1)^2 - 1](x^2)^2 \neq 1$

For this metric, compute the arc-length parameter and length of the given curve.

Solution:

Using this metric tensor we have to find the arc length and length of the curve

$$C: \begin{cases} x^1 = 2t - 1 \\ x^2 = 2t^2 \\ x^3 = t^3 \end{cases} ; 0 \leq t \leq 1$$

$$dx^1 = 2dt, \quad dx^2 = 4t dt, \quad dx^3 = 3t^2 dt$$

Now:

$$\begin{aligned} ds^2 &= g_{ij} dx^i dx^j \\ &= g_{11} dx^1 dx^1 + g_{12} dx^1 dx^2 + g_{21} dx^2 dx^1 + g_{22} dx^2 dx^2 \\ &\quad + g_{33} dx^3 dx^3 \end{aligned}$$

$$\Rightarrow ds^2 = [(x')^2 - 1] 4dt^2 + 2(2dt)(4tdt) + (x^2)^2 (4tdt)^2 + \frac{64}{9} (3t^2 dt)^2$$

$$= \left\{ 4[(x')^2 - 1] + 16t + (x^2)^2 (16t^2) + 9t^4 \cdot \frac{64}{9} \right\} dt^2$$

$$ds^2 = 4 \left[(x')^2 - 1 + 4t + 4t^2 (x^2)^2 + 16t^4 \right] dt^2$$

$$\Rightarrow ds^2 = 4 \left[(2t-1)^2 - 1 + 4t + 4t^2 (4t^4) + 16t^4 \right] dt^2$$

$$ds^2 = 4 \left[4t^2 + 16t^4 + 16t^6 \right] dt^2$$

$$ds^2 = (8t^3 + 4t)^2 dt^2$$

$$\Rightarrow ds = (8t^3 + 4t) dt$$

Arc length:

$$\Rightarrow S = \int_0^t (8u^3 + 4u) du$$

$$= 2t^4 + 2t^2$$

and Length of the curve:

$$L = \int_0^1 (8t^3 + 4t) dt = 4$$

$$L = \int_a^b \sqrt{g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt}} dt$$

$$L = \int_a^b \sqrt{g_{ij} dx^i dx^j}$$

$$= \int_a^b \sqrt{ds^2}$$

$$L = \int_a^b ds$$

L

Question: A curve is given in spherical coordinates x^i by:

$$x^1 = t, \quad x^2 = \sin^{-1} \frac{1}{t}, \quad x^3 = \sqrt{t^2 - 1}$$

Find the length of the arc for $1 \leq t \leq 2$.

Solution:

$$L = \int_a^b \sqrt{g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt}} dt$$

where g_{ij} for spherical coordinates is:

$$[g_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (x^1)^2 & 0 \\ 0 & 0 & (x^1 \sin x^2)^2 \end{bmatrix}$$

$$L = \int_1^2 \left[g_{11} \left(\frac{dx^1}{dt} \right)^2 + g_{22} \left(\frac{dx^2}{dt} \right)^2 + g_{33} \left(\frac{dx^3}{dt} \right)^2 \right]^{1/2} dt$$

where

$$\frac{dx^1}{dt} = 1, \quad \frac{dx^2}{dt} = \frac{-1/t^2}{\sqrt{1 - (1/t)^2}}, \quad \frac{dx^3}{dt} = \frac{t}{\sqrt{t^2 - 1}}$$

$$\left(\frac{dx^1}{dt}\right)^2 = 1, \left(\frac{dx^2}{dt}\right)^2 = \frac{1}{t^2(t^2-1)}, \left(\frac{dx^3}{dt}\right)^2 = \frac{t^2}{t^2-1}$$

$$\Rightarrow \mathcal{L} = \int_1^2 \left[1 + \frac{t^2}{t^2(t^2-1)} + t^2 \left(\frac{1}{t^2} \right) \frac{t^2}{t^2-1} \right]^{1/2} dt$$

$$= \int_1^2 \left[1 + \frac{1}{t^2-1} + \frac{t^2}{t^2-1} \right]^{1/2} dt$$

$$= \int_1^2 \left(\frac{2t^2}{t^2-1} \right)^{1/2} dt$$

$$\Rightarrow \mathcal{L} = \int_1^2 \frac{\sqrt{2}t}{\sqrt{t^2-1}} dt = \frac{\sqrt{2}}{2} \int_1^2 \frac{2t}{\sqrt{t^2-1}} dt$$

$$= \frac{2}{\sqrt{2}} \left| (t^2-1)^{1/2} \right|_1^2 = \sqrt{2} (\sqrt{3}) = \sqrt{6}$$

Question: Find the length of the curve defined
by $x^1 = 1, x^2 = t; 1 \leq t \leq 2$

if the metric is that of the hyperbolic
plane ($x^2 > 0$):

$$g_{11} = g_{22} = \frac{1}{(x^2)^2}, \quad g_{12} = g_{21} = 0$$

Solution:

consider the arc length formula:

$$L = \int_1^2 \sqrt{g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt}} dt$$

$$\Rightarrow \mathcal{L}_1 = \int_1^2 \left[g_{11} \left(\frac{dx^1}{dt} \right)^2 + g_{22} \left(\frac{dx^2}{dt} \right)^2 \right]^{1/2} dt$$

$$= \int_1^2 \left[\frac{1}{t^2} (0) + \frac{1}{t^2} (1)^2 \right]^{1/2} dt$$

$$= \int_1^2 t^{-1} dt = \ln t \Big|_1^2 = \ln 2 - \ln 1^0$$

$$\mathcal{L} = \ln 2$$

Question: Using Euclidean metric for cylindrical coordinates, calculate the length of arc along the circular helix

$$\vec{x}^1 = a \cos t, \quad \vec{x}^2 = a \sin t, \quad \vec{x}^3 = bt$$

with a and b positive constants,
from $t=0$ to $t=C>0$.

Solution: $\tilde{x}^i \rightarrow$ for rectangular coordinates
 $x^i \rightarrow$ for cylindrical coordinates

also

$$\tilde{x}^1 = x^1 \cos x^2, \quad \tilde{x}^2 = x^1 \sin x^2, \quad \tilde{x}^3 = x^3$$

$$[g_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (x^1)^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Given curve has parametric Eqs:

$$\vec{x}^1 = a \cos t, \quad \vec{x}^2 = a \sin t, \quad \vec{x}^3 = bt$$

$$\Rightarrow x^1 = a, \quad x^2 = t \quad \text{and} \quad x^3 = bt$$

are the parametric Eqs of curve in cylindrical coordinates.

$$\text{Now consider, } L = \int_a^b \sqrt{g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt}} dt$$

$$\Rightarrow \mathcal{L} = \int_0^c \left[g_{11} \frac{dx^1}{dt} \frac{dx^1}{dt} + g_{22} \frac{dx^2}{dt} \frac{dx^2}{dt} + g_{33} \frac{dx^3}{dt} \frac{dx^3}{dt} \right]^{1/2} dt$$

$$= \int_0^c \left[0 + (x')^2 (1)(1) + (1)(b)(b) \right]^{1/2} dt$$

$$= \int_0^c \underbrace{(a^2 + b^2)}^{1/2} dt \rightarrow \int_0^c dt$$

$$= c (a^2 + b^2)^{1/2}$$

Question: Using the metric

$$[g_{ij}] = \begin{bmatrix} 12 & 4 & 0 \\ 4 & 1 & 1 \\ 0 & 1 & (x')^2 \end{bmatrix}$$

calculate the length of the curve
given by

$$x^1 = 3 - t, x^2 = 6t + 3, x^3 = \ln t$$

where $1 \leq t \leq e$.

Solution:

Consider

$$L = \int_a^b \left[g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt} \right]^{1/2} dt$$

$$= \int_a^b \left[g_{11} \left(\frac{dx^1}{dt} \right)^2 + 2g_{12} \frac{dx^1}{dt} \frac{dx^2}{dt} + g_{22} \left(\frac{dx^2}{dt} \right)^2 + 2g_{23} \frac{dx^2}{dt} \frac{dx^3}{dt} + g_{33} \left(\frac{dx^3}{dt} \right)^2 \right]^{1/2} dt$$

$$\Rightarrow e$$

$$\mathcal{L} = \int_1^e \left[12(-1)^2 + 2(4)(-1)(6) + (1)(6)^2 + 2(1)(6)\left(\frac{1}{t}\right) + (3-t)^2\left(\frac{1}{t^2}\right) \right]^{1/2} dt$$

$$= \int_1^e \left[\cancel{12} + \cancel{36} - \cancel{48} + \frac{12}{t} + \frac{9+t^2-6t}{t^2} \right]^{1/2} dt$$

$$= \int_1^e \left(\frac{9+t^2+6t}{t^2} \right)^{1/2} dt$$

$$\Rightarrow \mathcal{L} = \int_1^e \left(\frac{3+t}{t} \right)^{2 \times 1/2} dt$$

$$= \int_1^e \left(\frac{3}{t} + 1 \right) dt = \left| 3 \ln t + t \right|_1^e$$

$$= 3 \overset{1}{\ln e} - 3 \overset{0}{\ln 1} + e - 1$$

$$= 3 + e - 1$$

$$\mathcal{L} = 2 + e$$

Conjugate Metric Tensor:

The inverse of the fundamental tensor
or metric tensor

$$[g^{ij}]_{nn} = [g_{ij}]_{nn}^{-1}$$

is called the conjugate metric tensor.

* $g_{ij} = \hat{e}_i \cdot \hat{e}_j$
then $g^{ij} = \hat{e}'_i \cdot \hat{e}'_j$ } where $\hat{e}_i$ and $\hat{e}'_i$ are the
two sets of basis vectors
associated normal to the
coordinate surfaces and
tangent to the coordinate
curves.

The covariant and contravariant metric tensors are inverses of each other that is

$$[g_{ij}] = [g^{ij}]^{-1}$$

$$\text{and } [g^{ij}] = [g_{ij}]^{-1}$$

$$\Rightarrow g_{ik} g^{kj} = \delta_i^j$$

$$\text{and } g^{ik} g_{kj} = \delta^i_j$$

★ It is common to call covariant form as metric tensor and contravariant form as inverse metric, associate metric, conjugate metric or reciprocal metric tensor.

$$g_{ij} = \begin{bmatrix} \hat{e}_1 \cdot \hat{e}_1 & \hat{e}_1 \cdot \hat{e}_2 & \dots & \hat{e}_1 \cdot \hat{e}_n \\ \vdots & \hat{e}_2 \cdot \hat{e}_2 & \dots & \vdots \\ \hat{e}_n \cdot \hat{e}_1 & \dots & \dots & \hat{e}_n \cdot \hat{e}_n \end{bmatrix}$$

★ For orthogonal coordinate system the metric tensor is diagonal, i.e.,

$$g_{ij} = g^{ij} = 0 \quad \text{for } i \neq j$$

Raising and Lowering Indices:

The contravariant metric tensor is used for raising indices of covariant tensors and the covariant metric tensor is used for lowering indices of contravariant tensors

Example:

Consider a mixed tensor T^{ij}_k and convert it into a covariant and contravariant tensor.

Solution:

Contravariant form: $T^{ijk} = T^{ij}_r g^{rk}$

Covariant form: $T_{ijk} = g_{is} g_{jr} T^{sr}_k$

Question: Given that V^i is a contravariant vector in $\mathbb{R}^3$, find its associated covariant vector V_i in cylindrical coordinates x^i under the Euclidean metric.

Solution: Here the metric is

$$[g_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (x^1)^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and $V_i = g_{is} V^s \rightarrow [V_i] = [g_{ij}] [V^i]$

⇒ In matrix form:

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (\chi')^2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix}$$

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v^1 \\ (\chi')^2 v^2 \\ v^3 \end{bmatrix}$$

$$\Rightarrow v_1 = v^1$$

$$v_2 = (\chi')^2 v^2 = \chi^2 v^2$$

$$v_3 = v^3$$

Question: In Spherical coordinates

$$x^\alpha = (1, r, \theta) \text{ and}$$

$$y_\alpha = (0, -r^2, \cos^2 \theta)$$

Find x_α and y^α .

Solution: Here the metric is:

$$[g_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \theta \end{bmatrix}$$

Now

$$X_{\alpha} = g_{\alpha\beta} X^{\beta} \quad [X_{\alpha}] = [g_{\alpha\beta}] [X^{\beta}]$$

$$\Rightarrow X_{\alpha} = g_{\alpha 1} X^1 + g_{\alpha 2} X^2 + g_{\alpha 3} X^3$$

$$\Rightarrow X_1 = g_{11} X^1 = 1$$

$$X_2 = g_{22} X^2 = r^2(r) = r^3$$

$$X_3 = g_{33} X^3 = r^2 \sin^2 \theta(0) = 0$$

$$\Rightarrow X_{\alpha} = (1, r^3, 0)$$

Now

$$y^\alpha = g^{\alpha\beta} y_\beta$$

where

$$g^{\alpha\beta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1/r^2 \sin^2 \theta \end{bmatrix}$$

$$\Rightarrow y^1 = g^{11} y_1 = (1)(0) = 0$$

$$y^2 = g^{22} y_2 = \frac{1}{r^2} (-r^2) = -1$$

$$y^3 = g^{33} y_3 = \frac{1}{r^2 \sin^2 \theta} (\cos^2 \theta) = \frac{\cot^2 \theta}{r^2}$$

Question: Given the two dimensional
line element $(g_{12} + g_{21}) \rightarrow \begin{bmatrix} x^2 & 1 \\ 1 & -1 \end{bmatrix}$
 $ds^2 = x^2 dx^2 + 2dx dy - dy^2$

Write down g_{ij} , g^{ij} and then

raise and lower indices of

$$V_i = (1, -1)$$

$$W^i = (0, 1)$$

Solution: From the given line element
we have

$$g_{ij} = \begin{bmatrix} x^2 & 1 \\ 1 & -1 \end{bmatrix}$$

and its inverse is

$$\rightarrow |g_{ij}| = -(x^2 + 1)$$
$$g^{ij} = \frac{1}{|g_{ij}|} \text{adj } g_{ij}$$

$$g^{ij} = \frac{1}{-(x^2 + 1)} \begin{bmatrix} -1 & -1 \\ -1 & x^2 \end{bmatrix} = \frac{1}{x^2 + 1} \begin{bmatrix} 1 & 1 \\ 1 & -x^2 \end{bmatrix}$$

Now:

$$v^i = g^{ij} v_j = g^{i1} v_1 + g^{i2} v_2$$

$$\Rightarrow v^1 = g^{11} v_1 + g^{12} v_2$$

$$v^1 = \frac{1}{1+x^2} (1) + \frac{1}{1+x^2} (-1) = 0$$

$$v^2 = g^{21} v_1 + g^{22} v_2$$

$$= \frac{1}{1+x^2} (1) - \frac{x^2}{1+x^2} (-1) = \frac{1+x^2}{1+x^2} = 1$$

$$\Rightarrow v^i = (0, 1)$$

$$W_i = g_{ij} W^j$$

$$\Rightarrow W_1 = g_{11} W^1 + g_{12} W^2$$

$$= x^2(0) + (1)(1) = 1$$

$$W_2 = g_{21} W^1 + g_{22} W^2$$

$$= (1)(0) + (-1)(1) = -1$$

$$\Rightarrow W_i = (1, -1)$$

Question: In curvilinear coordinates x^i , two contravariant vectors are:

$$u^i = \left(-\frac{x^1}{x^2}, 1, 0 \right), \quad v^i = \left(\frac{1}{x^2}, 0, 0 \right)$$

where

$$\tilde{x}^1 = x^2, \quad \tilde{x}^2 = x^3 \quad \text{and} \quad \tilde{x}^3 = x^1 x^2; \quad x^2 \neq 0.$$

Find u_i and v_i .

Solution: First we find the metric tensor by

$$[g_{ij}] = J^t J$$

$$J = \frac{\partial \tilde{x}^i}{\partial x^i} \begin{matrix} \rightarrow \text{rectang.} \\ \rightarrow \text{curvi} \end{matrix}$$

$$J = \begin{bmatrix} \partial \tilde{x}^1 / \partial x^1 & \partial \tilde{x}^1 / \partial x^2 & \partial \tilde{x}^1 / \partial x^3 \\ \partial \tilde{x}^2 / \partial x^1 & \partial \tilde{x}^2 / \partial x^2 & \partial \tilde{x}^2 / \partial x^3 \\ \partial \tilde{x}^3 / \partial x^1 & \partial \tilde{x}^3 / \partial x^2 & \partial \tilde{x}^3 / \partial x^3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ x^2 & x^1 & 0 \end{bmatrix}$$

$$\Rightarrow [g_{ij}] = J^t J = \begin{bmatrix} (x^2)^2 & x^1 x^2 & 0 \\ x^1 x^2 & 1 + (x^1)^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now Given

$$u^i = (-x'/x^2, 1, 0)$$

$$u_i = g_{ij} u^j = g_{i1} u^1 + g_{i2} u^2 + g_{i3} u^3$$

$$\Rightarrow u_1 = (x^2)^x \left(-\frac{x'}{x^2} \right) + x' x^2 (1) + (0)$$
$$= -x' x^2 + x' x^2 = 0$$

$$u_2 = x' x^2 \left(-\frac{x'}{x^2} \right) + (1 + (x')^2) (1) + 0$$
$$= -(\cancel{x'})^2 + 1 + (\cancel{x'})^2 = 1$$

$$u_3 = 0 + 0 + 0 = 0$$

$$\Rightarrow u_i = (0, 1, 0)$$

Given: $V^i = \left(\frac{1}{x^2}, 0, 0 \right)$

$$V_i = g_{ij} V^j$$

$$\Rightarrow V_1 = g_{11} V^1 + g_{12} V^2 + g_{13} V^3 = (x^2)^2 \left(\frac{1}{x^2} \right) = x^2$$

$$V_2 = g_{21} V^1 + g_{22} V^2 + g_{23} V^3 = x^1 x^2 \left(\frac{1}{x^2} \right) = x^1$$

$$V_3 = g_{31} V^1 + g_{32} V^2 + g_{33} V^3 = 0$$

$$\Rightarrow V_i = (x^2, x^1, 0).$$

Generalized inner product:

$$ds^2 = g_{ij} \underline{dx^i dx^j}$$

Suppose that g_{ij} is a metric on R^n and u^i, v^i are two vectors in R^n , then the generalized inner product of u^i and v^i is defined

as

$$I = u^i v_i = u_i v^i = g_{ij} u^i v^j \\ = g^{ij} u_i v_j$$

$$L = \int_a^b \sqrt{g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt}} dt$$

Concept of Length and Angle:

The norm (or length) of an arbitrary vector $\vec{v}^i$ or v_i is the non-negative real number.

$$\|\vec{v}\| = \sqrt{v^2} = \sqrt{v_i v^i}$$

The angle θ between two non-null contravariant vectors $\vec{U}$ and $\vec{V}$ is defined by:

$$\cos \theta = \frac{\vec{U} \cdot \vec{V}}{\|\vec{U}\| \|\vec{V}\|} = \frac{g_{ij} U^i V^j}{\sqrt{g_{pq} U^p U^q} \sqrt{g_{rs} V^r V^s}}$$

Question: Show that the length of any contravariant vector v^i equals the length of its associated covariant vector v_i .

Solution: By definition:

$$\|v^i\| = \sqrt{g_{ij} v^i v^j} \quad \text{and} \quad \|v_i\| = \sqrt{g^{ij} v_i v_j}$$

Consider:

$$\begin{aligned} g_{ij} v^i v^j &= g_{ij} (g^{ir} v_r) (g^{js} v_s) = g_{ji} g^{ir} g^{js} v_r v_s \\ &= \delta_j^r g^{is} v_r v_s = g^{rs} v_r v_s \end{aligned}$$

Question: Show that under the Euclidean metric for polar coordinates, the vectors

$$\vec{u} = u^i = \left(\frac{3}{5}, \frac{4}{5} \hat{x}^i \right) \text{ and } \vec{v} = v^i = \left(-\frac{4}{5}, \frac{3}{5} \hat{x}^i \right)$$

are orthonormal.

Solution:

To show that these vectors are orthonormal, we have to show that

$$\|\vec{u}\| = 1, \quad \|\vec{v}\| = 1$$

and $\vec{u} \cdot \vec{v} = 0$

here $[g_{ij}] = \begin{bmatrix} 1 & 0 \\ 0 & (x')^2 \end{bmatrix}$, $u^i = (\frac{3}{5}, \frac{4}{5x'})$

$$\|\vec{u}\| = u^i u_i = g_{ij} u^j u^i = g_{11} u^1 u^1 + g_{22} u^2 u^2$$

$$= (1) \left(\frac{3}{5}\right)^2 + (x')^2 \left(\frac{4}{5x'}\right)^2$$

$$= \frac{9}{25} + \frac{16}{25} = 1$$

Similarly:

$$\begin{aligned}\|\vec{V}\| &= V_i V^i = g_{ij} V^j V^i \\ &= g_{11} V^1 V^1 + g_{22} V^2 V^2 \\ &= (1) \left(-\frac{4}{5}\right)^2 + \left(\frac{3}{5x'}\right)^2 (x')^2 \\ &= \frac{16}{25} + \frac{9}{25} = 1\end{aligned}$$

$$\text{Now: } \vec{u} \cdot \vec{v} = g_{ij} u^i v^j$$

$$\Rightarrow = g_{11} u^1 v^1 + g_{22} u^2 v^2$$

$$= (1) \left(\frac{3}{5} \right) \left(-\frac{4}{5} \right) + (x')^2 \left(-\frac{4}{5x'} \right) \left(\frac{3}{5x'} \right)$$

$$= -\frac{12}{25} + \frac{12}{25}$$

$$= 0$$

$\Rightarrow \vec{u}$ and $\vec{v}$ are orthonormal.

Question: Find the condition for two curves on a sphere a to be orthogonal, if the curves are represented in spherical coordinates by:

$$C_1: = f(\theta) \text{ and } C_2: = g(\theta)$$

Solution: The two curves can be parameterized in spherical coordinates (r, θ, ϕ) :

$$C_1: \begin{cases} r = a \\ \theta = t \\ \phi = f(t) \end{cases} \quad \text{and} \quad C_2: \begin{cases} r = a \\ \theta = u \\ \phi = f(u) \end{cases}$$

Also let (a, θ_0, ϕ_0) is the point of intersection.

The tangents to the curves are:

For C_1 : $u^i = \left(\frac{dx}{dt}, \frac{d\theta}{dt}, \frac{d\phi}{dt} \right) = (0, 1, f'(\theta))$

at (a, θ_0, ϕ_0) :

$$u^i = (0, 1, f'(\theta_0))$$

$$u^i = \frac{dx^i}{dt}$$

For C_2 : $v^i = \left(\frac{dx}{du}, \frac{d\theta}{du}, \frac{d\phi}{du} \right) = (0, 1, g'(\theta))$

and at (a, θ_0, ϕ_0) $v^i(a, \theta_0, \phi_0) = (0, 1, g'(\theta_0))$

These are orthogonal if and only if

$$g_{ij} u^i v^j = 0$$

where

$$[g_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & a^2 \sin^2 \theta_0 \end{bmatrix} \text{ at } (a, \theta_0, \phi_0)$$

$$g_{ij} u^i v^j = 0$$

$$\Rightarrow g_{11} u^1 v^1 + g_{22} u^2 v^2 + g_{33} u^3 v^3 = 0$$

$$(1)(0) + a^2 (1) + a^2 \sin^2 \theta_0 f'(\theta_0) g'(\theta_0) = 0$$

$$0 + a^2 + a^2 \sin^2 \theta_0 f'(\theta_0) g'(\theta_0) = 0$$

$$\Rightarrow \sec^2 \theta_0 + f'(\theta_0) g'(\theta_0) = 0$$

$$\Rightarrow f'(\theta_0) g'(\theta_0) = -\sec^2 \theta_0$$

at any point of intersection (a, θ_0, ϕ_0) .

Question: In curvilinear coordinates x^i , Show that

the vectors

$$u^i = \left(-\frac{x^1}{x^2}, 1, 0\right) \text{ and } v^i = \left(\frac{1}{x^2}, 0, 0\right)$$

are orthonormal if x^i are related to rectangular coordinates $\tilde{x}^i$ by:

$$\tilde{x}^1 = x^2, \tilde{x}^2 = x^3 \text{ and } \tilde{x}^3 = x^1 x^2$$

Solution: We are to prove

$$\|u^i\| = 1, \|v^i\| = 1 \text{ and}$$

$$u_i v^i = 0$$

$$\begin{aligned} \vec{u} \cdot \vec{v} &= u_i v^i \\ &= u^i v_i \\ &= g_{ij} u^i v^j \end{aligned}$$

here the metric is:

$$[g_{ij}] = J^t J = \begin{bmatrix} (x^2)^2 & x' x^2 & 0 \\ x' x^2 & 1 + (x')^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} \Rightarrow \|u^i\| &= \sqrt{g_{ij} u^i u^j} = \left(g_{11} u^1 u^1 + 2g_{12} u^1 u^2 + g_{22} u^2 u^2 \right. \\ &\quad \left. + g_{33} u^3 u^3 \right)^{1/2} \\ &= \left((x^2)^2 \left(-\frac{x'}{x^2} \right)^2 + 2 x' \cancel{x^2} \left(-\frac{x'}{\cancel{x^2}} \right) (1) + (1 + (x')^2) (1)^2 \right)^{1/2} \\ &= \left(\cancel{(x^2)^2} - 2 \cancel{(x^2)^2} + 1 + \cancel{(x')^2} \right)^{1/2} = 1 \end{aligned}$$

Similarly:

$$\|\vec{V}\| = (g_{ij} v^i v^j)^{1/2}$$

$$= \left((x^2)^2 \frac{1}{(x^2)^2} + 2x'x^2(0) + 0 + 0 \right)^{1/2}$$

$$\begin{aligned} u_i v^i &= g_{ij} u^j v^i = (x^2)^2 \left(\frac{-x'}{x^2} \right) \left(\frac{1}{x^2} \right) + x'x^2(1) \frac{1}{x^2} \\ &= -x' + x' = 0 \end{aligned}$$

$\Rightarrow u^i$ and v^i are orthonormal.

Question: Under the metric

$$[g_{ij}] = \begin{bmatrix} 1 & \cos 2x^2 \\ \cos 2x^2 & 1 \end{bmatrix}$$

Compute the norm of $\vec{V} = \frac{dx^i}{dt}$, along the

curve $x^1 = -\sin 2t$,

$$x^2 = t$$

and use it to find the arc length

between $t = 0$ and $t = \frac{\pi}{2}$.

Solution:

$$\dot{x}^i = \frac{dx^i}{dt} = \left(\frac{dx^1}{dt}, \frac{dx^2}{dt} \right) = (-2\cos 2t, 1)$$

now

$$\| \dot{x}^i \| = \sqrt{g_{ij} \dot{x}^i \dot{x}^j}$$

$$= [(1)(4\cos^2 2t) + 2\cos 2t(-2\cos 2t)(1) + (1)]^{1/2}$$

$$= (1)^{1/2} = 1$$

$$L = \int_a^b (g_{ij} \dot{x}^i \dot{x}^j)^{1/2} dt$$

$$\mathcal{L} = \int_0^{\pi/2} dt$$

$$= \left| t \right|_0^{\pi/2} = \pi/2$$

The del Operator: '∇' is a partial differential operator. In Cartesian coordinate system, it is defined by:

$$\nabla = \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k}$$

$$\text{OR } \nabla = \frac{\partial}{\partial x_1} \hat{e}_1 + \frac{\partial}{\partial x_2} \hat{e}_2 + \frac{\partial}{\partial x_3} \hat{e}_3$$

Implies in tensor notation, we can write it: $\nabla = \sum_i \frac{\partial}{\partial x_i} \hat{e}_i = \sum_i \nabla_i \hat{e}_i$

$$\text{where } \nabla_i = \frac{\partial}{\partial x_i}$$

Gradient:

The gradient of a differentiable scalar function $\phi(x_i)$ is a vector given by:

$$\nabla \phi = \sum_i \frac{\partial \phi}{\partial x_i} \hat{e}_i$$

and in component form:

$$[\nabla \phi]_i = \nabla_i \phi = \frac{\partial \phi}{\partial x_i} = \phi_{,i}$$

Divergence:

The divergence of a differentiable vector $\vec{A}$ is a scalar given by:

$$\nabla \cdot \vec{A} = \sum_i \frac{\partial A_i}{\partial x_i}$$

$$\begin{aligned} \nabla \cdot \vec{A} &= \delta_{ij} \frac{\partial A_i}{\partial x_j} = \frac{\partial A_i}{\partial x_i} = \nabla_i A_i \\ &= \partial_i A_i \\ &= A_{i,i} \end{aligned}$$

curl:

The curl of a differentiable vector $\vec{A}$ is a vector given by:

(In component form)

$$\begin{aligned} [\nabla \times \vec{A}]_i &= \epsilon_{ijk} \frac{\partial A_k}{\partial x_j} = \epsilon_{ijk} \nabla_j A_k \\ &= \epsilon_{ijk} \partial_j A_k \\ &= \epsilon_{ijk} A_{k,j} \\ &\quad \swarrow \\ &\quad (i\text{th component}) \end{aligned}$$

Laplacian:

The Laplacian operator, also called the harmonic operator, acting on a differentiable scalar function $\phi(x_i)$ is given by:

$$\begin{aligned}\Delta\phi = \nabla^2\phi &= \delta_{ij} \frac{\partial^2\phi}{\partial x_i \partial x_j} = \frac{\partial^2\phi}{\partial x_i \partial x_i} = \nabla_{ii}\phi \\ &= \partial_i \partial_i \phi \\ &= \phi_{,ii}\end{aligned}$$

Tensor Techniques to prove vector Identities:

Examples:

(1) $\nabla \cdot \vec{r} = n$

Consider $\nabla \cdot \vec{r} = \partial_i x_i$
 $= \frac{\partial x_i}{\partial x_i}$
 $= \delta_{ii}$
 $= n$

(i is dummy,
 Σ is not mentioned
due to Einstein
Summation Convention)

$\vec{r}$ is position vector
$$\begin{aligned} \vec{r} &= x\hat{i} + y\hat{j} + z\hat{k} \\ &= x_1\hat{e}_1 + x_2\hat{e}_2 + x_3\hat{e}_3 \\ &= x_i\hat{e}_i \end{aligned}$$

$$2) \nabla \times \vec{x} = \vec{0}$$

Proof:

$$[\nabla \times \vec{x}]_i = \varepsilon_{ijk} \partial_j x_k$$

$$= \varepsilon_{ijk} \delta_{kj}$$

$$= \varepsilon_{iji}$$

$$= 0 \quad (\text{by the def. of } \varepsilon_{ijk})$$

(definition of curl)

$$(\because \partial_j x_k = \frac{\partial x_k}{\partial x_j})$$

(using def. of δ_{ij})

3) $\nabla(\vec{a} \cdot \vec{r}) = \vec{a}$, where $\vec{a}$ is any constant vector.

Proof: consider L.H.S, where

$$\vec{a} \cdot \vec{r} = a_j x_j \text{ and } [\nabla\phi]_i = \partial_i \phi$$

$$\begin{aligned} \Rightarrow [\nabla(\vec{a} \cdot \vec{r})]_i &= \partial_i(a_j x_j) \\ &= a_j \partial_i x_j + x_j \underbrace{\partial_i a_j}_0 \quad (\text{product rule}) \\ &= a_j \partial_i x_j \\ &= a_j \delta_{ij} \\ &= a_i \end{aligned}$$

$$\Rightarrow \nabla(\vec{a} \cdot \vec{r}) = \vec{a}$$

$$4) \nabla \cdot \nabla \phi = \nabla^2 \phi$$

$$\nabla \cdot \vec{A} = \partial_i A_i$$

$\downarrow$
 $\nabla \phi$

Proof: consider,

$$\nabla \cdot \nabla \phi = \partial_i [\nabla \phi]_i$$

$$= \partial_i (\partial_i \phi)$$

$$= \partial_i \partial_i \phi$$

$$= \partial_{ii} \phi$$

$$= \nabla^2 \phi$$

$$\left(\begin{array}{l} \because \nabla_i = \partial_i \\ [\nabla \phi]_i = \partial_i \phi \end{array} \right)$$

$$5) \nabla \cdot (\nabla \times \vec{A}) = 0$$

Proof: $[\nabla \times \vec{A}]_i = \epsilon_{ijk} \partial_j A_k$

$$\nabla \cdot \vec{B} = \partial_i B_i$$

replace $\nabla \times \vec{A}$ in place of $\vec{B}$:

$$\begin{aligned} \Rightarrow \nabla \cdot (\nabla \times \vec{A}) &= \partial_i (\epsilon_{ijk} \partial_j A_k) \\ &= \epsilon_{ijk} \partial_i \partial_j A_k = \epsilon_{ijk} \partial_j \partial_i A_k \\ &= -\epsilon_{jik} \partial_j \partial_i A_k \\ &= -\epsilon_{ijk} \partial_i \partial_j A_k \end{aligned}$$

by interchanging the dummy indices i and j . we have:

$$\begin{aligned}\nabla \cdot (\nabla \times \vec{A}) &= \varepsilon_{ijk} \partial_i \partial_j A_k \\ &= -\varepsilon_{ijk} \partial_i \partial_j A_k\end{aligned}$$

and for any quantity x

$$\text{if } x = -x \Rightarrow \partial x = 0 \Rightarrow x = 0$$

$$\Rightarrow \varepsilon_{ijk} \partial_i \partial_j A_k = 0$$

$$\Rightarrow \nabla \cdot (\nabla \times \vec{A}) = 0.$$

$$6) \nabla \times \nabla \phi = \vec{0}$$

Proof: We consider i^{th} component of the cross product on L.H.S.

$$\begin{aligned} (\nabla \times \nabla \phi)_i &= \epsilon_{ijk} \partial_j (\nabla \phi)_k \\ &= \epsilon_{ijk} \partial_j (\partial_k \phi) \\ &= \epsilon_{ijk} \partial_j \partial_k \phi \end{aligned}$$

$$\Rightarrow (\nabla \times \nabla \Phi)_i = \epsilon_{ijk} \partial_k \partial_j \Phi$$

$$= -\epsilon_{ikj} \partial_k \partial_j \Phi$$

$$= -\epsilon_{ijk} \partial_j \partial_k \Phi$$

(Interchanging dummy indices
j and k)

$$\Rightarrow (\nabla \times \nabla \Phi)_i = \epsilon_{ijk} \partial_j \partial_k \Phi = -\epsilon_{ijk} \partial_j \partial_k \Phi$$

$$\Rightarrow (\nabla \times \nabla \Phi)_i = 0 \Rightarrow \nabla \times \nabla \Phi = \vec{0}$$

$$\Rightarrow (\nabla \times \nabla \Phi)_i = \epsilon_{ijk} \partial_k \partial_j \Phi$$

$$= -\epsilon_{ikj} \partial_k \partial_j \Phi$$

$$= -\epsilon_{ijk} \partial_j \partial_k \Phi$$

(Interchanging dummy indices
j and k)

$$\Rightarrow (\nabla \times \nabla \Phi)_i = \epsilon_{ijk} \partial_j \partial_k \Phi = -\epsilon_{ijk} \partial_j \partial_k \Phi$$

$$\Rightarrow (\nabla \times \nabla \Phi)_i = 0 \Rightarrow \nabla \times \nabla \Phi = \vec{0}$$

$$7) \vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{C} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{C} \times \vec{A})$$

Proof: Consider,

$$\vec{D} = \vec{B} \times \vec{C}, \text{ then}$$

$$D_i = (\vec{B} \times \vec{C})_i = \epsilon_{ijk} B_j C_k$$

and

$$\begin{aligned} \vec{A} \cdot (\vec{B} \times \vec{C}) &= \vec{A} \cdot \vec{D} = A_i D_i \\ &= A_i \epsilon_{ijk} B_j C_k \\ &= \epsilon_{ijk} A_i B_j C_k \end{aligned}$$

$$\begin{aligned}
 \Rightarrow \vec{A} \cdot (\vec{B} \times \vec{C}) &= \epsilon_{kij} A_i B_j C_k \\
 &= \epsilon_{kij} C_k A_i B_j \\
 &= C_k \epsilon_{kij} A_i B_j
 \end{aligned}$$

Similarly, we can prove

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A})$$

$$8) \nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$$

Proof:

$$\begin{aligned} \nabla \cdot (\vec{A} \times \vec{B}) &= \partial_i (\vec{A} \times \vec{B})_i \\ &= \partial_i (\epsilon_{ijk} A_j B_k) \\ &= \epsilon_{ijk} \partial_i (A_j B_k) \\ &= \epsilon_{ijk} (B_k \partial_i A_j + A_j \partial_i B_k) \end{aligned}$$

$$\Rightarrow \nabla \cdot (\vec{A} \times \vec{B}) = \varepsilon_{ijk} B_k \partial_i A_j + \varepsilon_{ijk} A_j \partial_i B_k$$

$$= \varepsilon_{kij} B_k \partial_i A_j - \varepsilon_{jik} A_j \partial_i B_k$$

$$= B_k (\varepsilon_{kij} \partial_i A_j) - A_j (\varepsilon_{jik} \partial_i B_k)$$

$$= B_k (\nabla \times \vec{A})_k - A_j (\nabla \times \vec{B})_j$$

$$= \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$$

$$9) \quad \nabla \times (\nabla \times \vec{A}) = \nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$$

Proof: Consider L.H.S:

$$[\nabla \times (\nabla \times \vec{A})]_i = \epsilon_{ijk} \partial_j (\nabla \times \vec{A})_k$$

$$= \epsilon_{ijk} \partial_j (\epsilon_{klm} \partial_l A_m)$$

$$= \epsilon_{ijk} \epsilon_{klm} \partial_j \partial_l A_m$$

$$\Rightarrow [\nabla \times (\nabla \times \vec{A})]_i = \epsilon_{ijk} \epsilon_{klm} \partial_j \partial_l A_m$$

using $\epsilon_{ijk} \epsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}$

$$\Rightarrow [\nabla \times (\nabla \times \vec{A})]_i = (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) \partial_j \partial_l A_m$$

$$= \delta_{il} \delta_{jm} \partial_j \partial_l A_m - \delta_{im} \delta_{jl} \partial_j \partial_l A_m$$

$$= \partial_m \partial_i A_m - \partial_l \partial_l A_i$$

$$\begin{aligned}
\Rightarrow &= \partial_m \partial_i A_m - \partial_l \partial_l A_i \\
&= \partial_i (\partial_m A_m) - \partial_{ll} A_i \\
&= (\nabla (\nabla \cdot \vec{A}))_i - (\nabla^2 A)_i
\end{aligned}$$

$$\Rightarrow \nabla \times (\nabla \times \vec{A}) = \nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$$

$$10) \vec{A} \times (\nabla \times \vec{B}) = \nabla \vec{B} \cdot \vec{A} - \vec{A} \cdot \nabla \vec{B}$$

Proof: consider,

$$[\vec{A} \times (\nabla \times \vec{B})]_i = \epsilon_{ijk} A_j (\nabla \times \vec{B})_k$$

$$= \epsilon_{ijk} A_j \epsilon_{klm} \partial_l B_m$$

$$= \epsilon_{ijk} \epsilon_{klm} A_j \partial_l B_m$$

$$\Rightarrow = (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) A_j \partial_l B_m$$

$$= \delta_{il} \delta_{jm} A_j \partial_l B_m - \delta_{im} \delta_{jl} A_j \partial_l B_m$$

$$= A_m \partial_i B_m - A_l \partial_l B_i$$

$$= (\partial_i B_m) A_m - A_l (\partial_l B_i)$$

$$= (\nabla \vec{B} \cdot \vec{A})_i - (\vec{A} \cdot \nabla \vec{B})_i$$

Since i is arbitrary index so the identity is true for all components.

Inadequacy of Ordinary Differentiation:

Consider a contravariant tensor

$$T = T^i(x^i)$$

defined on a curve

$$C: x^i = x^i(t)$$

consider the transformation law:

$$\tilde{T}^i = T^r \frac{\partial \tilde{x}^i}{\partial x^r}$$

Differentiating w.r.t 't' gives:

$$\frac{d\tilde{T}^i}{dt} = \frac{d}{dt} \left(T^r \frac{\partial \tilde{x}^i}{\partial x^r} \right)$$

$$= \frac{dT^r}{dt} \frac{\partial \tilde{x}^i}{\partial x^r} + T^r \frac{d}{dt} \left(\frac{\partial \tilde{x}^i}{\partial x^r} \right)$$

$$\frac{d\tilde{T}^i}{dt} = \frac{dT^r}{dt} \frac{\partial \tilde{x}^i}{\partial x^r} + \left[T^r \frac{\partial^2 \tilde{x}^i}{\partial x^s \partial x^r} \cdot \frac{dx^s}{dt} \right]$$

$$\frac{d}{dt} = \frac{\partial}{\partial x^s} \cdot \frac{dx^s}{dt}$$

which shows that the ordinary derivative of tensor T along the curve is a contravariant tensor when and only when the $\tilde{x}^i$ are the linear functions of the x^{λ} .

Theorem:

The derivative of a tensor is a tensor if and only if coordinate changes are restricted to linear transformations.

Example: Let C is a curve defined by
 $C: x^i = x^i(t)$ then

$\vec{T} = \frac{d\vec{x}}{dt}$ is tangent field along Curve C ,

and $k = \left\| \frac{d\vec{T}}{dt} \right\|$ is the curvature.

but 'k' is not an invariant in
curvilinear coordinates since $\frac{d\vec{T}}{dt}$ is
not a general tensor. Clearly, we
require a general concept of tensor
differentiation.

Covariant Derivative :

The covariant derivative of an arbitrary tensor is a tensor of which the covariant order exceeds that of the original tensor by exactly one.

Rank of a tensor derivative:

The derivative of a zero order tensor becomes a first order tensor.

Generally, the derivative of an m -order tensor forms an $m+1$ order tensor.

However, if the derivative index is a dummy index, then the rank of the derivative will be lower than that of the original tensor. For example

$A_{ij,j} \rightarrow$ rank is one.

b/c there is only one free index i .

The Covariant Derivative

- One of the (many) reasons why tensor calculus is so useful is that it allows a proper description of physically meaningful derivatives of vectors.
- We have not mentioned this before, but taking a derivative of a vector (or of any tensor) is a non-trivial thing.
- As usual, in a rectangular orthonormal coordinate system this is usually not an issue. But when the coordinates are curved (like in polar coordinates, for example) then this becomes a big issue.

$$\vec{A} = A_1 \hat{i} + A_2 \hat{j} + A_3 \hat{k}$$

$$\frac{d\vec{A}}{dt} = \frac{dA_1}{dt} \hat{i} + \frac{dA_2}{dt} \hat{j} + \frac{dA_3}{dt} \hat{k}$$

but in curvilinear coordinates

$\hat{e}_i(\vec{x}_i) \rightarrow$ so their derivatives exist

- It is even more of an issue when the manifold is curved (like in general relativity), the techniques we will discuss here will be equally well applicable for curved manifolds.
- We want to construct a derivative of a vector in such a way that it makes physical sense even in the case of curved coordinates.

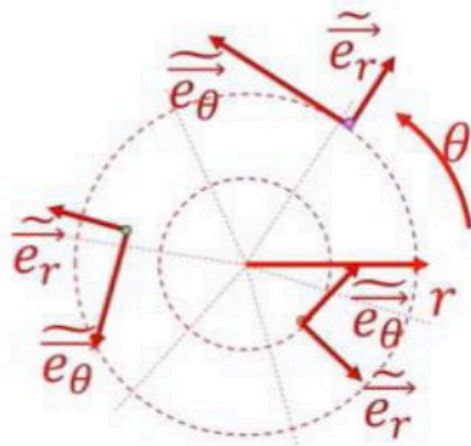
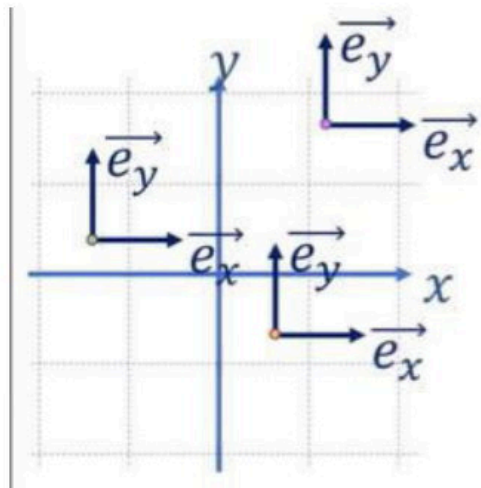
The Covariant Derivative:

Consider the problem of taking the derivative of a vector. In ordinary Cartesian coordinates, this is not complicated because the basis vectors are constant. So we can get the derivative of a vector by differentiating

its components. However, in general things are not so straightforward.

In curved spaces the basis vectors themselves may vary from point to point. This means that when we take derivative of a vector, we will have to

differentiate the basis vectors as well. For this purpose consider a 2-dimensional rectangular coordinate system and polar coordinate system as shown below.



where $\vec{e}_1 = \frac{\partial \vec{r}}{\partial x^1}$, $\vec{e}_2 = \frac{\partial \vec{r}}{\partial x^2}$

$\vec{e}_1 = \frac{\partial \vec{r}}{\partial \tilde{x}^1}$ and $\vec{e}_2 = \frac{\partial \vec{r}}{\partial \tilde{x}^2}$ $\begin{cases} x^1, x^2 \rightarrow r, \theta \\ \tilde{x}^1, \tilde{x}^2 \rightarrow r, \theta \end{cases}$

Now consider the rate of change of some vector field $\vec{V}$ in both coordinate system.

and $\vec{e}_1 = \partial \vec{r} / \partial u$

where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ $\vec{e}_1 = \frac{\partial \vec{r} / \partial u'}{|\partial \vec{r} / \partial u'|}$

In rectangular coordinate system

$$\vec{v} = v^1 \vec{e}_1 + v^2 \vec{e}_2 = \sum_i v^i \vec{e}_i$$

$$\Rightarrow \vec{v} = v^i \vec{e}_i = v^i \frac{\partial \vec{r}}{\partial x^i}$$

$$\Rightarrow \frac{\partial}{\partial x^i} (\vec{v}) = \frac{\partial}{\partial x^i} (v^i \vec{e}_i)$$

$$= \frac{\partial v^i}{\partial x^i} \vec{e}_i + v^i \frac{\partial \vec{e}_i}{\partial x^i}$$

But here

$$\vec{e}_1 = \hat{i} \text{ and } \vec{e}_2 = \hat{j}$$

means constant vectors, Implies

$$\frac{\partial \vec{v}}{\partial x^1} = \frac{\partial v^i}{\partial x^1} \vec{e}_i$$

and similarly:

$$\frac{\partial \vec{v}}{\partial x^2} = \frac{\partial v^i}{\partial x^2} \vec{e}_i$$

Now consider the vector $\vec{V}$ in polar coordinates as:

$$\vec{V} = \tilde{V}^1 \vec{e}_1 + \tilde{V}^2 \vec{e}_2$$

$$\Rightarrow \vec{V} = \tilde{V}^i \vec{e}_i$$

$$\Rightarrow \frac{\partial \vec{V}}{\partial \tilde{x}^1} = \frac{\partial}{\partial \tilde{x}^1} (\tilde{V}^i \vec{e}_i)$$

$$= \frac{\partial \tilde{V}^i}{\partial \tilde{x}^1} \vec{e}_i + \tilde{V}^i \frac{\partial}{\partial \tilde{x}^1} \vec{e}_i$$

and similarly:

$$\frac{\partial \vec{v}}{\partial \vec{x}^2} = \frac{\partial \vec{v}^i}{\partial \vec{x}^2} \vec{e}_i + \vec{v}^i \frac{\partial \vec{e}_i}{\partial \vec{x}^2}$$

Now recall that

$$x = r \cos \theta, \quad y = r \sin \theta$$

$$\vec{e}_r = \cos \theta \vec{e}_1 + \sin \theta \vec{e}_2 = \vec{e}_1$$

$$\vec{e}_\theta = -r \sin \theta \vec{e}_1 + r \cos \theta \vec{e}_2 = \vec{e}_2$$

$$\Rightarrow \frac{\partial \vec{e}_1}{\partial \tilde{x}^1} = 0 ,$$

$$\frac{\partial \vec{e}_2}{\partial \tilde{x}^1} = \frac{\partial \vec{e}_1}{\partial \tilde{x}^2} = \frac{1}{r} \vec{e}_2$$

$$\text{and } \frac{\partial \vec{e}_2}{\partial \tilde{x}^2} = -r \vec{e}_1$$

$$\Rightarrow \frac{\partial \vec{v}}{\partial \tilde{x}'} = \frac{\partial \tilde{v}'}{\partial \tilde{x}'} \vec{e}_1 + \left[\frac{\partial \tilde{v}^2}{\partial \tilde{x}'} + \frac{\tilde{v}^2}{\tilde{x}'} \right] \vec{e}_2$$

and $\frac{\partial \vec{v}}{\partial r} = \frac{\partial \tilde{v}^r}{\partial r} \vec{e}_r + \left[\frac{\partial \tilde{v}^\theta}{\partial r} + \frac{\tilde{v}^\theta}{r} \right] \vec{e}_\theta$

$$\frac{\partial \vec{v}}{\partial \theta} = \left(\frac{\partial \tilde{v}'}{\partial \tilde{x}^2} - \tilde{x}' \tilde{v}^2 \right) \vec{e}_1 + \left(\frac{\partial \tilde{v}^2}{\partial \tilde{x}^2} + \frac{\tilde{v}'}{\tilde{x}'} \right) \vec{e}_2$$

$$= \left[\frac{\partial \tilde{v}^r}{\partial \theta} - r \tilde{v}^\theta \right] \vec{e}_r + \left[\frac{\partial \tilde{v}^\theta}{\partial \theta} + \frac{\tilde{v}^r}{r} \right] \vec{e}_\theta$$

So, in general

$$\frac{\partial \vec{v}}{\partial x^i} = \frac{\partial v^j}{\partial x^i} \tilde{\vec{e}}_j + v^j \frac{\partial \tilde{\vec{e}}_j}{\partial x^i}$$

where

$$\begin{aligned} \frac{\partial \tilde{\vec{e}}_j}{\partial x^i} &= \Gamma_{ij}^k \tilde{\vec{e}}_k \\ &= \Gamma_{ij}^1 \tilde{\vec{e}}_1 + \Gamma_{ij}^2 \tilde{\vec{e}}_2 \end{aligned}$$

$$\Rightarrow \frac{\partial \vec{v}}{\partial x^i} = \frac{\partial v^j}{\partial x^i} \vec{e}_j + v^j \Gamma_{ij}^k \vec{e}_k$$

These Γ_{ij}^k are the christoffel symbols and the derivative on R.H.S is called covariant derivative.

Christoffel Symbols of the First Kind :

In 1869 Christoffel introduced certain functions involving derivatives of the components of the fundamental tensor ' g_{ij} '. These functions are called Christoffel symbols or an affine connection.

Basically, these quantities represent correction terms, which a derivative operator needs to differentiate a tensor and give a result that is another tensor.

Definition:

The quantities or functions defined by

$$\Gamma_{ijk} = \frac{1}{2} \left[\frac{\partial}{\partial x^i} (g_{jk}) + \frac{\partial}{\partial x^j} (g_{ki}) - \frac{\partial}{\partial x^k} (g_{ij}) \right]$$

are the Christoffel Symbols of the first kind.

Transformation Law:

The transformation law for the Γ_{ijk} can be derived from that of g_{ij} . Since g_{ij} is a second rank tensor:

$$\tilde{g}_{ij} = \frac{\partial x^r}{\partial \tilde{x}^i} \frac{\partial x^s}{\partial \tilde{x}^j} g_{rs}$$

differentiating both sides:

$$\tilde{g}_{ij,k} = \frac{\partial}{\partial \tilde{x}^k} \left[\left(\frac{\partial x^r}{\partial \tilde{x}^i} \frac{\partial x^s}{\partial \tilde{x}^j} \right) g_{rs} \right]$$

$\rightarrow \partial \tilde{g}_{ij} / \partial x^k$

$$\Rightarrow \tilde{g}_{ij,k} = \frac{\partial g_{rs}}{\partial \tilde{x}^k} \frac{\partial x^r}{\partial \tilde{x}^i} \frac{\partial x^s}{\partial \tilde{x}^j} + g_{rs} \frac{\partial^2 x^r}{\partial \tilde{x}^k \partial \tilde{x}^i} \frac{\partial x^s}{\partial \tilde{x}^j} + g_{rs} \frac{\partial x^r}{\partial \tilde{x}^i} \frac{\partial^2 x^s}{\partial \tilde{x}^k \partial \tilde{x}^j}$$

using chain rule for:

$$\frac{\partial g_{rs}}{\partial \tilde{x}^k} = \frac{\partial g_{rs}}{\partial x^t} \frac{\partial x^t}{\partial \tilde{x}^k} = g_{rs,t} \frac{\partial x^t}{\partial \tilde{x}^k}$$

$$\Rightarrow \tilde{g}_{ij,k} = g_{rs,t} \frac{\partial x^r}{\partial \tilde{x}^i} \frac{\partial x^s}{\partial \tilde{x}^j} \frac{\partial x^t}{\partial \tilde{x}^k} + g_{rs} \left[\frac{\partial^2 x^r}{\partial \tilde{x}^k \partial \tilde{x}^i} \frac{\partial x^s}{\partial \tilde{x}^j} \right. \\ \left. + \frac{\partial^2 x^s}{\partial \tilde{x}^k \partial \tilde{x}^j} \frac{\partial x^r}{\partial \tilde{x}^i} \right]$$

Implies: $(\tilde{g}_{ij,k}, g_{jk,i}, \tilde{g}_{ki,j})$
 (By interchanging $i \rightarrow j, j \rightarrow k, k \rightarrow i$)

$$\tilde{g}_{jk,i} = g_{st,n} \frac{\partial x^s}{\partial \tilde{x}^j} \frac{\partial x^t}{\partial \tilde{x}^k} \frac{\partial x^u}{\partial \tilde{x}^i} + g_{sn} \left[\frac{\partial^2 x^s}{\partial \tilde{x}^i \partial \tilde{x}^j} \frac{\partial x^u}{\partial \tilde{x}^k} + \right. \\
\left. (1) \left[\frac{\partial^2 x^u}{\partial \tilde{x}^i \partial \tilde{x}^k} \frac{\partial x^s}{\partial \tilde{x}^j} \right] \right]$$

Similarly:

$$\tilde{g}_{ki,j} = g_{tr,s} \frac{\partial x^t}{\partial \tilde{x}^k} \frac{\partial x^r}{\partial \tilde{x}^i} \frac{\partial x^s}{\partial \tilde{x}^j} + g_{sn} \left[\frac{\partial^2 x^s}{\partial \tilde{x}^j \partial \tilde{x}^k} \frac{\partial x^r}{\partial \tilde{x}^i} + \right. \\
\left. (2) \left[\frac{\partial^2 x^r}{\partial \tilde{x}^j \partial \tilde{x}^i} \frac{\partial x^s}{\partial \tilde{x}^k} \right] \right]$$

Consider:

$$\tilde{\Gamma}_{ijk} = \frac{1}{2} (-\tilde{g}_{ij,k} + \tilde{g}_{jk,i} + \tilde{g}_{ki,j})$$

now replacing all values of the metric tensor derivative, we have:

$$\tilde{\Gamma}_{ijk} = \frac{1}{2} [-g_{rs,t} + g_{st,r} + g_{tr,s}] \frac{\partial x^r}{\partial \tilde{x}^i} \frac{\partial x^s}{\partial \tilde{x}^j} \frac{\partial x^t}{\partial \tilde{x}^k}$$

$$+ \frac{1}{2} g_{rs} \left[\frac{\partial^2 x^s}{\partial \tilde{x}^i \partial \tilde{x}^j} \frac{\partial x^r}{\partial \tilde{x}^k} + \frac{\partial^2 x^r}{\partial \tilde{x}^j \partial \tilde{x}^i} \frac{\partial x^s}{\partial \tilde{x}^k} \right]$$

$$\Rightarrow \tilde{\Gamma}_{ijk} = \Gamma_{rst} \frac{\partial x^r}{\partial \tilde{x}^i} \frac{\partial x^s}{\partial \tilde{x}^j} \frac{\partial x^t}{\partial \tilde{x}^k} + g_{rs} \frac{\partial^2 x^r}{\partial \tilde{x}^i \partial \tilde{x}^j} \frac{\partial x^s}{\partial \tilde{x}^k}$$

(by interchanging the dummy indices r and s in last term.)

$\Rightarrow$ Christoffel Symbol is not a general tensor.

Question: Compute the christoffel symbols corresponding to the Euclidean metric for spherical coordinates.

$$[g_{\mu\nu}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (x')^2 & 0 \\ 0 & 0 & (x')^2 \sin^2 x^2 \end{bmatrix}$$

Solution: The formula for christoffel symbols is:

$$\Gamma_{ijk} = \frac{1}{2} \left[-g_{ij,k} + g_{jk,i} + g_{ki,j} \right]$$

$$\begin{aligned}\Gamma_{221} &= \frac{1}{2} \left[-g_{22,1} + g_{21,2} + g_{12,2} \right] \\ &= \frac{1}{2} (-2x' + 0 + 0) = -x' \quad \text{or} \quad \boxed{\Gamma_{221} = -x'}\end{aligned}$$

$$\begin{aligned}\Gamma_{212} &= \frac{1}{2} \left(-\overset{\rightarrow 0}{g_{21,2}} + g_{22,1} + \overset{\rightarrow 0}{g_{12,2}} \right) \\ &= \frac{1}{2} (2x') = x' \quad \text{or} \quad \boxed{\Gamma_{212} = x'}$$

$$\begin{aligned}\Gamma_{122} &= \frac{1}{2} \left(-\overset{\rightarrow 0}{g_{12,2}} + g_{22,1} + \overset{\rightarrow 0}{g_{21,2}} \right) \\ &= \frac{1}{2} (x') = x'\end{aligned}$$

$$\Gamma_{223} = \frac{1}{2} (-g_{22,3} + g_{23,2} + g_{32,2}) = 0$$

Similarly $\Gamma_{232} = \Gamma_{322} = 0$

$$\begin{aligned} \Gamma_{331} &= \frac{1}{2} (-g_{33,1} + g_{31,3}^{\rightarrow 0} + g_{13,3}^{\rightarrow 0}) \\ &= \frac{1}{2} (-2x' \sin^2(x^2)) = -x' \sin^2 x^2 \, dx = x \sin^2 \theta \end{aligned}$$

Similarly:

$$\Gamma_{133} = x' \sin^2 x^2 = \Gamma_{313}$$

$$\Gamma_{332} = \frac{1}{2} (-g_{33,2} + g_{32,3} + g_{23,3})$$

$$= \frac{1}{2} (-(x')^2 2 \sin x^2 \cos x^2)$$

$$= -(x')^2 \sin x^2 \cos x^2 \text{ or } \Gamma_{332} = -x \sin \theta \cos \theta$$

Similarly:

$$\Gamma_{323} = (x')^2 \sin x^2 \cos x^2$$

and

$$\Gamma_{233} = (x')^2 \sin x^2 \cos x^2$$

Properties of Christoffel symbol of first kind:

i) $\Gamma_{ijk} = \Gamma_{jik}$ (symmetric in first two indices)

ii) All Γ_{ijk} vanish if all g_{ij} are constant.

Question:

verify that $\Gamma_{ijk} = \Gamma_{jik}$.

Solution:

By definition,

$$\Gamma_{ijk} = \frac{1}{2} (-g_{ij,k} + g_{jk,i} + g_{ki,j})$$

Interchanging i and j :

$$\Gamma_{jik} = \frac{1}{2} (-g_{ji,k} + g_{ik,j} + g_{ki,i})$$

Since metric tensor is symmetric.
This implies:

$$g_{ij,k} = g_{ji,k} \quad , \quad g_{jk,i} = g_{kj,i} \quad , \quad g_{ki,j} = g_{ik,j}$$

$$\Rightarrow \Gamma_{ijk} = \Gamma_{jik}.$$

Question: Show that if $[g_{ij}]$ is a diagonal matrix, then for all fixed subscripts α and $\beta \neq \alpha$ in the range $1, 2, \dots, n$,

1) $\Gamma_{\alpha\alpha\alpha} = \frac{1}{2} g_{\alpha\alpha, \alpha}$ (not summed on α)

2) $-\Gamma_{\alpha\alpha\beta} = \Gamma_{\alpha\beta\alpha} = \Gamma_{\beta\alpha\alpha} = \frac{1}{2} g_{\alpha\alpha\beta}$ (not summed on α)

3) All remaining Christoffel symbols Γ_{ijk} are zero.

Solution:

$$(1) \Gamma_{\alpha\alpha\alpha} = \frac{1}{2} \left[-g_{\alpha\alpha,\alpha} + g_{\alpha\alpha,\alpha} + g_{\alpha\alpha,\alpha} \right]$$

$$\Gamma_{\alpha\alpha\alpha} = \frac{1}{2} g_{\alpha\alpha,\alpha}$$

$$(2) -\Gamma_{\alpha\alpha\beta} = -\frac{1}{2} \left(-g_{\alpha\alpha,\beta} + g_{\alpha\beta,\alpha}^{\rightarrow\circ} + g_{\beta\alpha,\alpha}^{\rightarrow\circ} \right)$$

$$= \frac{1}{2} g_{\alpha\alpha,\beta}$$

$$\Gamma_{\alpha\beta\alpha} = \Gamma_{\beta\alpha\alpha} = \frac{1}{2} \left(-g_{\alpha\beta,\alpha}^{\rightarrow\circ} + g_{\beta\alpha,\alpha}^{\rightarrow\circ} + g_{\alpha\alpha,\beta} \right)$$
$$= \frac{1}{2} g_{\alpha\alpha,\beta}$$

(3) Let i, j, k be distinct subscripts.

Then $g_{ij} = 0$ and $g_{ij,k} = 0$

implying that

$$\Gamma_{ijk} = 0$$

$$\Gamma_{ijk} = \frac{1}{2} [-g_{ij,k}^{\circ} + g_{jk,i}^{\circ} + g_{ki,j}^{\circ}]$$

$$i \neq j \neq k$$

Question: If all Γ_{ijk} vanish in some coordinate system, then do the metric tensor has constant components in every coordinate system?

Solution: As $\Gamma_{ijk} = \frac{1}{2} (-g_{ij,k} + g_{ik,j} + g_{kj,i})$
 $\Gamma_{jki} = \frac{1}{2} (-g_{jk,i} + g_{ki,j} + g_{ij,k})$

Adding both:

$$\Gamma_{ijk} + \Gamma_{jki} = \frac{1}{2} (2g_{ik,j}) = g_{ik,j}$$

$$\Rightarrow g_{ik,j} = \Gamma_{ijk} + \Gamma_{jki}$$

Now, the conclusion would be valid if the Γ_{ijk} vanished in every coordinate system.

But Γ_{ijk} is not a tensor and the conclusion is false. For instance, all $\Gamma_{ijk} = 0$ for the Euclidean metric in rectangular coordinates, but $g_{22} = (x')^2$, $g_{33} = (x')^2 \sin^2(x')$ in spherical coordinates.

Christoffel Symbols of the Second Kind:

The functions

$$\Gamma_{jk}^i = g^{ik} \Gamma_{jkh}$$

are the Christoffel symbols of the second kind. It is simply the result of raising the third subscript of the Christoffel symbol of the first kind.

Properties:

The basic properties of Γ_{ijk} carry over to Γ^i_{jk} .

1) $\Gamma^i_{jk} = \Gamma^i_{kj}$ (Symmetric in the lower indices.)

2) All Γ^i_{jk} vanish if all g_{ij} are constant.

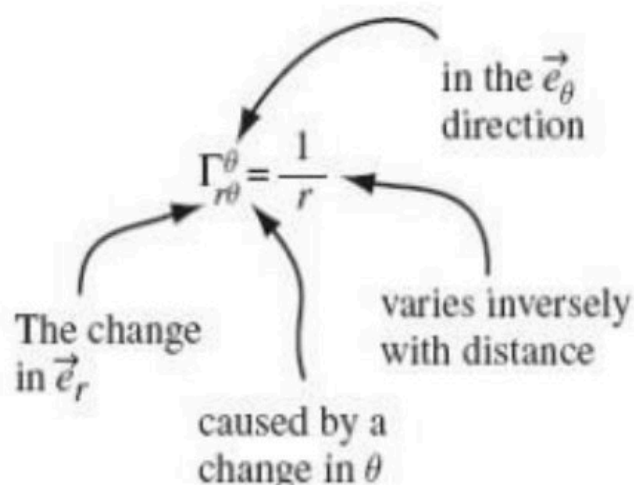
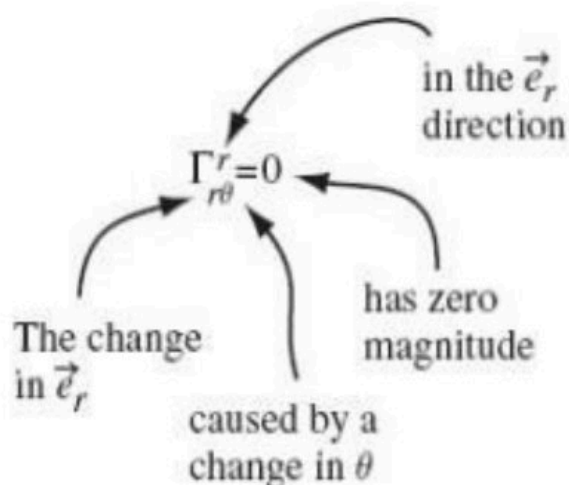
This Christoffel symbol gives you the magnitude of one component of the derivative vector

Tells you which basis vector points in the direction of this component of the derivative vector

$$\Gamma_{ij}^k$$

Tells you which basis vector's change is being considered

Tells you which coordinate is being varied to cause a change in the basis vector



Transformation law:

Starting with:

$$\tilde{\Gamma}_{jk}^i = \tilde{g}^{ik} \tilde{\Gamma}_{jk\alpha} = \left(g^{st} \frac{\partial \tilde{x}^i}{\partial x^s} \frac{\partial \tilde{x}^\alpha}{\partial x^t} \right) \tilde{\Gamma}_{jk\alpha}$$

Substitute for $\tilde{\Gamma}_{jk\alpha}$ from the transformation law of Christoffel symbol of first kind, we have

$$\tilde{\Gamma}_{jk}^i = \left(g^{st} \frac{\partial \tilde{x}^i}{\partial x^s} \frac{\partial \tilde{x}^k}{\partial x^t} \right) \left(\Gamma_{uvw} \frac{\partial x^u}{\partial \tilde{x}^j} \frac{\partial x^v}{\partial \tilde{x}^k} \frac{\partial x^w}{\partial \tilde{x}^l} \right)$$

$$+ \left(g^{st} \frac{\partial \tilde{x}^i}{\partial x^s} \frac{\partial \tilde{x}^k}{\partial x^t} \right) \left(g_{uv} \frac{\partial^2 x^u}{\partial \tilde{x}^j \partial \tilde{x}^k} \frac{\partial x^v}{\partial \tilde{x}^l} \right)$$

$$= g^{st} \Gamma_{uvw} \delta_t^w \frac{\partial \tilde{x}^i}{\partial x^s} \frac{\partial x^u}{\partial \tilde{x}^j} \frac{\partial x^v}{\partial \tilde{x}^k}$$

$$+ g^{st} g_{uv} \delta_t^v \frac{\partial \tilde{x}^i}{\partial x^s} \frac{\partial^2 x^u}{\partial \tilde{x}^j \partial \tilde{x}^k}$$

$$\Rightarrow \tilde{\Gamma}_{jk}^i = g^{st} \Gamma_{unt} \frac{\partial \tilde{x}^i}{\partial x^s} \frac{\partial x^u}{\partial \tilde{x}^j} \frac{\partial x^v}{\partial \tilde{x}^k} + g^{st} g_{ut} \frac{\partial \tilde{x}^i}{\partial x^s} \frac{\partial^2 x^u}{\partial \tilde{x}^j \partial \tilde{x}^k}$$

Since $g^{st} \Gamma_{unt} = \Gamma_{uv}^s$ and $g^{st} g_{ut} = \delta_u^s$,

Implies:

$$\Rightarrow \tilde{\Gamma}_{jk}^i = \Gamma_{st}^r \frac{\partial \tilde{x}^i}{\partial x^r} \frac{\partial x^s}{\partial \tilde{x}^j} \frac{\partial x^t}{\partial \tilde{x}^k} + \frac{\partial^2 x^r}{\partial \tilde{x}^j \partial \tilde{x}^k} \frac{\partial \tilde{x}^i}{\partial x^r}$$

$\Rightarrow \Gamma_{jk}^i$ is not a general tensor

and is merely an affine tensor.

Question: Calculate the Christoffel Symbols of the second kind for the Euclidean metric in polar coordinates.

$$[g_{\mu\nu}] = \begin{bmatrix} 1 & 0 \\ 0 & (x^1)^2 \end{bmatrix} \quad \begin{pmatrix} x^1 \rightarrow r \\ x^2 \rightarrow \theta \end{pmatrix}$$

Solution:

Using the formula:

$$\Gamma_{jk}^{ir} = \frac{g^{ir}}{2} \left[-g_{jk,r} + g_{kr,j} + g_{rj,k} \right]$$

$$\Gamma_{11}^1 = \frac{g^{1k}}{2} [-g_{11,1} + g_{1k,1} + g_{k1,1}]$$

$$= \frac{g^{11}}{2} [-g_{11,1} + g_{11,1} + g_{11,1}]$$

$$= \frac{1}{2} [0 + 0 + 0] = 0$$

$$\Gamma_{21}^1 = \frac{g^{1k}}{2} [-g_{21,k} + g_{1k,2} + g_{k2,1}]$$

$$= \frac{g^{11}}{2} [-g_{21,1}^{\rightarrow 0} + g_{11,2}^{\rightarrow 0} + g_{12,1}^{\rightarrow 0}] = 0$$

Similarly $\Gamma_{12}^1 = 0$

$$\Gamma'_{22} = \frac{g^{12}}{2} \left[-g_{22,1} + g_{2x,2} + g_{x2,2} \right]$$

$$= \frac{g''}{2} \left[-g_{22,1} + g_{22,2} + g_{22,2} \right]$$

$$= \frac{1}{2} \left[-2(x^1) + 0 + \right] = -x^1$$

$$\begin{aligned}
 \Gamma_{21}^{12} &= \frac{g^{22}}{2} \left[-g_{21,1} + g_{12,2} + g_{22,1} \right] \\
 &= \frac{g^{22}}{2} \left[-\overset{\rightarrow 0}{g_{21,2}} + \overset{\rightarrow 0}{g_{12,2}} + g_{22,1} \right] \\
 &= \frac{g^{22}}{2} (2x^1) = (x^1)^{-2} (x^1) = \frac{1}{x^1} \text{ or } \frac{1}{x}
 \end{aligned}$$

and $\Gamma_{12}^{12} = \frac{1}{x^1}$

$$\begin{aligned}
 \Gamma_{11}^{12} &= \frac{g^{22}}{2} \left[-g_{11,1} + g_{12,2} + g_{21,1} \right] \\
 &= \frac{g^{22}}{2} \left[-\overset{\rightarrow 0}{g_{11,2}} + \overset{\rightarrow 0}{g_{12,2}} + \overset{\rightarrow 0}{g_{21,1}} \right] = 0
 \end{aligned}$$

$$\begin{aligned}
\Gamma_{22}^2 &= \frac{g^{22}}{2} \left[-g_{22,2} + g_{22,2} + g_{22,2} \right] \\
&= \frac{g^{22}}{2} \left[-g_{22,2}^{\rightarrow 0} + g_{22,2}^{\rightarrow 0} + g_{22,2}^{\rightarrow 0} \right] \\
&= 0
\end{aligned}$$

Question: If $[g_{ij}]$ is a diagonal matrix,
 Show that for all fixed indices in the
 range $1, 2, \dots, n$.

$$a) \quad \Gamma_{\alpha\beta}^{\alpha} = \Gamma_{\beta\alpha}^{\alpha} = \frac{\partial}{\partial x^{\beta}} \left(\frac{1}{2} \ln |g_{\alpha\alpha}| \right)$$

consider

$$\begin{aligned} \Gamma_{\alpha\beta}^{\alpha} &= \frac{g^{\alpha\alpha}}{2} \left[-g_{\alpha\beta, \alpha}^{\rightarrow 0} + g_{\alpha\alpha, \beta} + g_{\beta\alpha, \alpha} \right] \\ &= \frac{g^{\alpha\alpha}}{2} \left[g_{\alpha\alpha, \beta} + g_{\beta\alpha, \alpha}^{\rightarrow 0} \right] = \frac{1}{2} g^{\alpha\alpha} g_{\alpha\alpha, \beta} \\ &= \frac{1}{2} \frac{g_{\alpha\alpha, \beta}}{g_{\alpha\alpha}} = \frac{\partial}{\partial x^{\beta}} \left(\frac{1}{2} \ln g_{\alpha\alpha} \right) \end{aligned}$$

$$b) \Gamma_{\beta\beta}^{\alpha} = -\frac{1}{2 g_{\alpha\alpha}} g_{\beta\beta,\alpha}$$

$$\begin{aligned} \Gamma_{\beta\beta}^{\alpha} &= \frac{g^{\alpha\lambda}}{2} \left[-g_{\beta\beta,\lambda} + g_{\beta\lambda,\beta} + g_{\beta\lambda,\beta} \right] \\ &= \frac{g^{\alpha\alpha}}{2} \left[-g_{\beta\beta,\alpha} + g_{\beta\alpha,\beta}^{\rightarrow 0} + g_{\beta\alpha,\beta}^{\rightarrow 0} \right] \\ &= -\frac{g^{\alpha\alpha}}{2} g_{\beta\beta,\alpha} \\ &= -\frac{1}{2 g_{\alpha\alpha}} g_{\beta\beta,\alpha} \end{aligned}$$

Question: Christoffel symbols of the second kind for the Euclidean metric in spherical coordinates.

here, $[g_{\alpha\beta}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (x')^2 & 0 \\ 0 & 0 & (x')^2 \sin^2 x^2 \end{bmatrix}$

Then the non zero Christoffel symbols of second kind are:

$$\Gamma_{21}^2 = \Gamma_{12}^2 = \frac{\partial}{\partial x'} \left(\frac{1}{2} \ln (x')^2 \right) = \frac{1}{x'}$$

$$\Gamma_{31}^3 = \Gamma_{13}^3 = \frac{\partial}{\partial x^1} \left(\frac{1}{2} \ln((x')^2 \sin^2 x^2) \right) = \frac{1}{x'}$$

$$\Gamma_{32}^3 = \Gamma_{23}^3 = \frac{\partial}{\partial x^2} \left(\frac{1}{2} \ln((x')^2 \sin^2 x^2) \right) = \cot x^2$$

Similarly:

$$\Gamma_{22}^1 = -\frac{1}{2(1)} \frac{\partial}{\partial x^1} (x')^2 = -x^1$$

$$\Gamma_{33}^1 = -\frac{1}{2(1)} \frac{\partial}{\partial x^1} ((x')^2 \sin^2 x^2) = -x' \sin^2 x^2$$

$$\Gamma_{33}^2 = -\frac{1}{2(x')^2} \frac{\partial}{\partial x^2} ((x')^2 \sin^2 x^2) = -\sin x^2 \cos x^2$$

Question: Use the equation

$$\frac{\partial^2 x^r}{\partial \tilde{x}^i \partial \tilde{x}^j} = \tilde{\Gamma}_{ij}^s \frac{\partial x^r}{\partial \tilde{x}^s} - \Gamma_{st}^r \frac{\partial x^s}{\partial \tilde{x}^i} \frac{\partial x^t}{\partial \tilde{x}^j} \rightarrow (*)$$

to find the most general 3-dimensional transformation of coordinates such that x^i correspond to rectangular and $\tilde{x}^i$ correspond to any other coordinate system for which the Christoffel symbols are:

$$\tilde{\Gamma}_{11}^1 = 1, \quad \tilde{\Gamma}_{22}^2 = 2 \quad \text{and} \quad \tilde{\Gamma}_{33}^3 = 3$$

all other are zero.

Solution: $x^i \rightarrow$ rectangular coordinates

$$\Rightarrow [\sigma_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \Gamma_{st}^r = 0$$

so Eq. ★ implies:

$$\frac{\partial x^r}{\partial \tilde{x}^i \partial \tilde{x}^j} = \tilde{\Gamma}_{ij}^s \frac{\partial x^r}{\partial \tilde{x}^s}$$

$$\text{let } \frac{\partial x^r}{\partial \tilde{x}^s} = \tilde{u}_s^r$$

$$\Rightarrow \frac{\partial \tilde{u}_j^r}{\partial \tilde{x}^i} = \tilde{\Gamma}_{ij}^s \tilde{u}_s^r$$

Moreover, replace $\tilde{u}_j = \tilde{u}_j$

$$\Rightarrow \frac{\partial \tilde{u}_j}{\partial \tilde{x}^i} = \tilde{\Gamma}_{ij}^s \tilde{u}_s$$

$$\Rightarrow \frac{\partial \tilde{u}_1}{\partial \tilde{x}^1} = \tilde{\Gamma}_{11}^1 \tilde{u}_1 \Rightarrow \frac{\partial \tilde{u}_1}{\partial \tilde{x}^1} = \tilde{u}_1$$

$$\Rightarrow \frac{\partial \tilde{u}_1}{\tilde{u}_1} = \partial \tilde{x}^1 \Rightarrow \ln \tilde{u}_1 = \tilde{x}^1 + c_1$$

$$\Rightarrow \tilde{u}_1 = b_1 e^{\tilde{x}^1}$$

and similarly:

$$\tilde{u}_2 = b_2 e^{2\tilde{x}^2} \quad \text{and} \quad \tilde{u}_3 = b_3 e^{3\tilde{x}^3}$$

$$\text{now, } \frac{\partial x^1}{\partial \tilde{x}^1} = b_1 e^{\tilde{x}^1}$$

$$\Rightarrow \partial x^1 = b_1 e^{\tilde{x}^1} \partial \tilde{x}^1$$

$$\Rightarrow x^1 = b_1 e^{\tilde{x}^1} + \phi(\tilde{x}^2, \tilde{x}^3)$$

and similarly:

$$\frac{\partial x^2}{\partial \tilde{x}^2} = b_2 e^{2\tilde{x}^2} \quad \text{and} \quad \frac{\partial x^2}{\partial \tilde{x}^3} = b_3 e^{3\tilde{x}^3}$$

$$\Rightarrow x^2 = b_2 e^{2\tilde{x}^2} + \phi(\tilde{x}^1, \tilde{x}^3)$$

$$\text{and } x^3 = b_3 e^{3\tilde{x}^3} + \phi(\tilde{x}^1, \tilde{x}^2)$$

$$\Rightarrow x = b_1 e^{\tilde{x}^1} + b_2 e^{\tilde{x}^2} + b_3 e^{\tilde{x}^3} + b_4$$

Question: If

$$\tilde{x}^1 = -e^{d_1 x^1} + e^{d_2 x^2} + e^{d_3 x^3}$$

$$\tilde{x}^2 = 2e^{d_1 x^1} - e^{d_2 x^2} + e^{d_3 x^3}$$

$$\tilde{x}^3 = e^{d_1 x^1} - 2e^{d_2 x^2} + 3e^{d_3 x^3}$$

and if all $\tilde{\Gamma}_{jk}^i = 0$, find the Γ_{jk}^i .

Use the formula:

$$\Gamma_{jk}^i = \frac{\partial^2 \tilde{x}^i}{\partial x^j \partial x^k} \frac{\partial x^i}{\partial \tilde{x}^i}$$

Solution: Given

$$\Gamma_{jk}^i = \frac{\partial^2 \tilde{x}^i}{\partial x^j \partial x^k} \frac{\partial x^i}{\partial \tilde{x}^i}$$

$$\frac{\partial \tilde{x}^i}{\partial x^i} = -e^{d, x^i} \cdot \frac{\partial}{\partial x^i} (d, x^i) = \phi_i(x^i)$$

$$\Rightarrow \frac{\partial \tilde{x}^i}{\partial x^2 \partial x^i} = 0 \quad \text{and} \quad \frac{\partial \tilde{x}^i}{\partial x^3 \partial x^i} = 0$$

Similarly

$$\frac{\partial \tilde{x}^i}{\partial x^i \partial x^2} = 0 \quad \text{and} \quad \frac{\partial \tilde{x}^i}{\partial x^i \partial x^3} = 0$$

$$\Rightarrow \frac{\partial^2 \tilde{x}^k}{\partial x^j \partial x^k} \neq 0 \quad \text{for } j=k$$

$$= 0 \quad \text{for } j \neq k$$

$$\Rightarrow \Gamma_{jj}^i = \frac{\partial}{\partial x^j} \left(\frac{\partial \tilde{x}^k}{\partial x^j} \right) \cdot \frac{\partial x^j}{\partial \tilde{x}^k}$$

$$\text{and } \Gamma_{jk}^i = 0 \quad \text{when } j \neq k.$$

Question: let the coordinate system x^i be connected to a system of rectangular coordinates $\tilde{x}^i$ by:

$$\tilde{x}^1 = e^{(x^1 + x^2)}$$

$$\text{and } \tilde{x}^2 = e^{(x^1 - x^2)}$$

Then using $\Gamma_{jk}^i = \frac{\partial^2 \tilde{x}^i}{\partial x^j \partial x^k} \frac{\partial x^i}{\partial \tilde{x}^i}$

Find the non-zero Christoffel symbols.

Solution: Given

$$\Gamma_{jk}^i = \frac{\partial^2 \tilde{x}^i}{\partial x^j \partial x^k} \frac{\partial x^i}{\partial \tilde{x}^i}$$

$$\frac{\partial \tilde{x}^1}{\partial x^1} = e^{x^1 + x^2} \quad (1) = e^{x^1 + x^2}$$

$$\frac{\partial \tilde{x}^1}{\partial x^2} = e^{x^1 + x^2} \quad (1) = e^{x^1 + x^2}$$

$$\frac{\partial^2 \tilde{x}^1}{(\partial x^1)^2} = e^{x^1 + x^2}, \quad \frac{\partial^2 \tilde{x}^1}{\partial x^1 \partial x^2} = e^{x^1 + x^2}$$

$$\text{and } \frac{\partial^2 \tilde{x}^1}{(\partial x^2)^2} = e^{x^1 + x^2}$$

$$\frac{\partial \tilde{x}^2}{\partial x^1} = e^{(x^1 - x^2)} (1) = e^{x^1 - x^2}$$

$$\frac{\partial \tilde{x}^2}{\partial x^2} = -e^{x^1 - x^2}$$

$$\frac{\partial^2 \tilde{x}^2}{(\partial x^1)^2} = e^{x^1 - x^2}, \quad \frac{\partial^2 \tilde{x}^2}{(\partial x^2)^2} = e^{x^1 - x^2}$$

$$\frac{\partial^2 \tilde{x}^2}{\partial x^1 \partial x^2} = -e^{x^1 - x^2} = \frac{\partial^2 \tilde{x}^2}{\partial x^2 \partial x^1}$$

We also require inverse transformations:

$$\tilde{x}^1 = e^{x^1 + x^2} \quad \text{and} \quad \tilde{x}^2 = e^{x^1 - x^2}$$

$$\Rightarrow x^1 + x^2 = \ln \tilde{x}^1 \quad \text{and} \quad x^1 - x^2 = \ln \tilde{x}^2$$

$$\Rightarrow 2x^1 = \ln \tilde{x}^1 + \ln \tilde{x}^2$$

$$\Rightarrow x^1 = \frac{1}{2} (\ln \tilde{x}^1 + \ln \tilde{x}^2)$$

$$\text{and} \quad 2x^2 = \ln \tilde{x}^1 - \ln \tilde{x}^2$$

$$\Rightarrow x^2 = \frac{1}{2} (\ln \tilde{x}^1 - \ln \tilde{x}^2)$$

$$\Rightarrow \frac{\partial x^1}{\partial \tilde{x}^1} = \frac{1}{2\tilde{x}^1} = \frac{1}{2e^{x^1+x^2}}$$

$$\frac{\partial x^1}{\partial \tilde{x}^2} = \frac{1}{2\tilde{x}^2} = \frac{1}{2e^{x^1-x^2}}$$

$$\text{and } \frac{\partial x^2}{\partial \tilde{x}^1} = \frac{1}{2\tilde{x}^1} = \frac{1}{2e^{x^1+x^2}}$$

$$\frac{\partial x^2}{\partial \tilde{x}^2} = -\frac{1}{2\tilde{x}^2} = -\frac{1}{2e^{x^1-x^2}}$$

$$\Rightarrow \Gamma'_{11} = \frac{\partial^2 \tilde{x}^1}{\partial x'^1 \partial x'^1} \cdot \frac{\partial x'^1}{\partial \tilde{x}^1}$$

$$= e^{x'^1 + x^2} \left(\frac{1}{2e^{x'^1 + x^2}} \right) + (e^{x'^1 - x^2}) \left(\frac{1}{2e^{x'^1 - x^2}} \right)$$

$$\Gamma'_{11} = \frac{1}{2} + \frac{1}{2} = 1$$

$$\Gamma'_{12} = \frac{\partial^2 \tilde{x}^2}{\partial x'^1 \partial x'^2} \cdot \frac{\partial x'^1}{\partial \tilde{x}^2}$$

$$= (e^{x'^1 + x^2}) \left(\frac{1}{2e^{x'^1 + x^2}} \right) + (-e^{x'^1 - x^2}) \left(\frac{1}{2e^{x'^1 - x^2}} \right)$$

$$= \frac{1}{2} - \frac{1}{2} = 0 = \Gamma'_{21}$$

$$\Gamma'_{22} = \frac{\partial^2 \tilde{x}^2}{\partial x^2 \partial x^2} \cdot \frac{\partial x^1}{\partial \tilde{x}^2}$$

$$= e^{x^1+x^2} \cdot \frac{1}{2} e^{-(x^1+x^2)} + e^{x^1-x^2} \cdot \frac{1}{2} e^{-(x^1-x^2)}$$

$$= \frac{1}{2} + \frac{1}{2} = 1$$

and similarly:

$$\Gamma^2_{11} = 0, \quad \Gamma^2_{12} = 1 = \Gamma^2_{21}$$

$$\Gamma^2_{22} = 0$$

Covariant Differentiation:

Here we will derive the formula for covariant derivative of a vector. Consider the transformation law:

$$\tilde{T}_i = T_r \frac{\partial x^r}{\partial \tilde{x}^i}$$

Its partial differentiation yields:

$$\frac{\partial \tilde{T}_i}{\partial \tilde{x}^k} = \frac{\partial T_r}{\partial \tilde{x}^k} \frac{\partial x^r}{\partial \tilde{x}^i} + T_r \frac{\partial^2 x^r}{\partial \tilde{x}^k \partial \tilde{x}^i} \rightarrow (\star)$$

now we find an expression for

$\frac{\partial^2 x^h}{\partial \tilde{x}^k \partial \tilde{x}^i}$ in terms of Christoffel symbols of second kind.

Recall the transformation law for Christoffel symbols of second kind:

$$\tilde{\Gamma}_{ik}^s = \Gamma_{ut}^h \frac{\partial \tilde{x}^s}{\partial x^u} \frac{\partial x^u}{\partial \tilde{x}^i} \frac{\partial x^t}{\partial \tilde{x}^k} + \frac{\partial^2 x^h}{\partial \tilde{x}^k \partial \tilde{x}^i} \frac{\partial \tilde{x}^s}{\partial x^h}$$

$$\Rightarrow \frac{\partial^2 x^h}{\partial \tilde{x}^k \partial \tilde{x}^i} = \tilde{\Gamma}_{ik}^s \frac{\partial x^h}{\partial \tilde{x}^s} - \Gamma_{ut}^v \frac{\partial \tilde{x}^s}{\partial x^v} \cdot \frac{\partial x^h}{\partial \tilde{x}^s} \frac{\partial x^u}{\partial \tilde{x}^i} \frac{\partial x^t}{\partial \tilde{x}^k}$$

$$\Rightarrow \frac{\partial^2 x^h}{\partial \tilde{x}^k \partial \tilde{x}^i} = \tilde{\Gamma}_{ik}^s \frac{\partial x^h}{\partial \tilde{x}^s} - \Gamma_{ut}^v \delta_v^h \frac{\partial x^u}{\partial \tilde{x}^i} \frac{\partial x^t}{\partial \tilde{x}^k}$$

$$\Rightarrow \frac{\partial^2 x^\lambda}{\partial \tilde{x}^k \partial \tilde{x}^i} = \tilde{\Gamma}_{ik}^s \frac{\partial x^\lambda}{\partial \tilde{x}^s} - \Gamma_{st}^\lambda \frac{\partial x^s}{\partial \tilde{x}^i} \frac{\partial x^t}{\partial \tilde{x}^k} \quad \checkmark$$

$$\Rightarrow \frac{\partial \tilde{T}_i}{\partial \tilde{x}^k} = \frac{\partial T_\lambda}{\partial \tilde{x}^k} \frac{\partial x^\lambda}{\partial \tilde{x}^i} + T_\lambda \left[\tilde{\Gamma}_{ik}^s \frac{\partial x^\lambda}{\partial \tilde{x}^s} - \Gamma_{st}^\lambda \frac{\partial x^s}{\partial \tilde{x}^i} \frac{\partial x^t}{\partial \tilde{x}^k} \right]$$

$$\Rightarrow \frac{\partial \tilde{T}_i}{\partial \tilde{x}^k} - \tilde{\Gamma}_{ik}^s \left(T_\lambda \frac{\partial x^\lambda}{\partial \tilde{x}^s} \right) = \frac{\partial T_\lambda}{\partial \tilde{x}^s} \frac{\partial x^\lambda}{\partial \tilde{x}^i} \frac{\partial x^s}{\partial \tilde{x}^k} - \Gamma_{st}^\lambda T_\lambda \frac{\partial x^s}{\partial \tilde{x}^i} \frac{\partial x^t}{\partial \tilde{x}^k}$$

$$\Rightarrow \frac{\partial \tilde{T}_i}{\partial \tilde{x}^k} - \tilde{\Gamma}_{ik}^s \tilde{T}_s = \frac{\partial T_r}{\partial x^s} \frac{\partial x^r}{\partial \tilde{x}^i} \frac{\partial x^s}{\partial \tilde{x}^k} - \Gamma_{rs}^t T_t \frac{\partial x^r}{\partial \tilde{x}^i} \frac{\partial x^s}{\partial \tilde{x}^k} \quad \begin{cases} r \rightarrow t \\ s \rightarrow r \\ t \rightarrow s \end{cases}$$

$$\Rightarrow \underbrace{\frac{\partial \tilde{T}_i}{\partial \tilde{x}^k} - \tilde{\Gamma}_{ik}^s \tilde{T}_s}_{\text{corrected}} = \underbrace{\left(\frac{\partial T_r}{\partial x^s} - \Gamma_{rs}^t T_t \right)}_{\text{linear combinations of } T_i} \frac{\partial x^r}{\partial \tilde{x}^i} \frac{\partial x^s}{\partial \tilde{x}^k}$$

$\Rightarrow$ When the components of $\frac{\partial T_i}{\partial x^k}$ are corrected by subtracting certain linear combinations of T_i itself, the result is a tensor.

Definition:

In any coordinate system x^i , the covariant derivative with respect to x^k of a covariant vector T_i is the tensor

$$T_{i;k} = \frac{\partial T_i}{\partial x^k} - \Gamma_{ik}^t T_t$$

* here the second index 'k' emphasize that the derivative is w.r.t kth coordinate.

* The covariant derivative and partial derivative coincide when g_{ij} are constants.

Definition:

In any coordinate system x^i , the covariant derivative with respect to x^k of a contravariant vector T^i is the tensor

$$T^i{}_{;k} = \frac{\partial T^i}{\partial x^k} + \Gamma^i_{jk} T^j$$

Covariant Derivative of any tensor:

In any coordinate system x^i , the covariant derivative with respect to x^k of a tensor $T^{i_1 i_2 \dots i_p}_{j_1 j_2 \dots j_q}$ is defined as:

$$T^{i_1 i_2 \dots i_p}_{j_1 j_2 \dots j_q ; k} = \frac{\partial T^{i_1 i_2 \dots i_p}_{j_1 j_2 \dots j_q}}{\partial x^k} + \Gamma^{i_1}_{t k} T^{t i_2 i_3 \dots i_p}_{j_1 j_2 \dots j_q} + \dots$$

$$- \Gamma^{t}_{j_1 k} T^{i_1 i_2 \dots i_p}_{t j_2 \dots j_q} - \dots - \Gamma^{t}_{j_q k} T^{i_1 i_2 \dots i_p}_{j_1 j_2 \dots t j_q}$$

$$\star T^\alpha_{\beta;\gamma} = T^\alpha_{\beta,\gamma} + \Gamma^\alpha_{\gamma\delta} T^\delta_\beta - \Gamma^\delta_{\beta\gamma} T^\alpha_\delta$$

$$T_{\alpha\beta;\gamma} = T_{\alpha\beta,\gamma} - \Gamma^\delta_{\alpha\gamma} T_{\delta\beta} - \Gamma^\delta_{\beta\gamma} T_{\alpha\delta}$$

$$T^{\alpha\beta}_{;\gamma} = T^{\alpha\beta}_{,\gamma} + \Gamma^\alpha_{\delta\gamma} T^{\delta\beta} + \Gamma^\beta_{\gamma\delta} T^{\alpha\delta}$$

★ The covariant derivative of a Scalar function is just the partial derivative.

$$\phi_{;\alpha} = \phi_{,\alpha}$$

Question: Consider polar coordinates and find the covariant derivative $V^\alpha_{;\alpha}$ of $V^\alpha = r^2 \cos \theta \hat{e}_r - \sin \theta \hat{e}_\theta$.

Solution: Using the formula we have:

$$V^\alpha_{;\alpha} = V^1_{;1} + V^2_{;2}$$

$$V^\alpha_{;\alpha} = V^\alpha_{,\alpha} + \Gamma^\alpha_{\beta\alpha} V^\beta$$

$$V^1_{;1} = V^1_{,1} + \Gamma^1_{\beta 1} V^\beta$$

$$V^2_{;2} = V^2_{,2} + \Gamma^2_{\beta 2} V^\beta$$

Consider,

$$V^1_{;1} = V^1_{,1} + \Gamma^1_{\beta 1} V^\beta$$

$$= (r^2 \cos \theta)_{,r} + \Gamma^1_{11} V^1 + \Gamma^1_{21} V^2$$

now using the Christoffel symbols
given below:

$$\Gamma^1_{11} = 0, \quad \Gamma^1_{21} = 0$$

$$\Gamma^1_{12} = \Gamma^1_{21} = \frac{1}{r} \quad \text{and} \quad \Gamma^1_{22} = -r$$

$$\Rightarrow V^1_{;1} = 2r \cos \theta$$

$$\begin{aligned}
 V_{j2}^2 &= V_{,2}^2 + \Gamma_{\beta 2}^2 V^\beta \\
 &= (-\sin\theta)_{,2} + \Gamma_{12}^2 V^1 + \Gamma_{22}^2 V^2 \\
 &= -\cos\theta + \frac{1}{r}(r^2 \cos\theta) \\
 &= -\cos\theta + r \cos\theta
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow V_{j\alpha}^\alpha &= 2r \cos\theta + r \cos\theta - \cos\theta \\
 V_{j\alpha}^\alpha &= 3r \cos\theta - \cos\theta \\
 &= (3r - 1) \cos\theta
 \end{aligned}$$

Question: Using the spherical polar coordinates $(r, \theta, \phi) \rightarrow (x^1, x^2, x^3)$ and the christoffel symbols

$$\Gamma_{21}^2 = \Gamma_{12}^2 = \frac{1}{r}, \quad \Gamma_{31}^3 = \Gamma_{13}^3 = \frac{1}{r}$$

$$\Gamma_{32}^3 = \Gamma_{23}^3 = \cot \theta, \quad \Gamma_{22}^1 = -r,$$

$$\Gamma_{33}^1 = -r \sin^2 \theta, \quad \Gamma_{33}^2 = -\sin \theta \cos \theta$$

find the covariant derivative $V^\alpha_{;\beta}$ for $\beta=1,2,3$.
 $\alpha=1,2,3$

$$\text{where } V^\alpha = (r, r^2 \cos \theta, \sin \phi)$$

Solution:

$$V^\alpha_{; \beta} = V^\alpha_{, \beta} + \Gamma^\alpha_{t\beta} V^t$$

$$V'^1_{; 1} = V'^1_{, 1} + \Gamma'^1_{t1} V^t = V'^1_{, 1} + 0$$

$$= V'^1_{, 1} = (r)_{, 1} = 1$$

$$V'^1_{; 2} = V'^1_{, 2} + \Gamma'^1_{t2} V^t$$

$$V'^1_{; 2} = V'^1_{, 2} + \Gamma'^1_{22} V^2 = 0 + (-r)(r^2 \cos \theta) = -r^3 \cos \theta$$

$$V'^1_{; 3} = V'^1_{, 3} + \Gamma'^1_{t3} V^t = \overset{\rightarrow 0}{V'^1_{, 3}} + \Gamma'^1_{33} V^3$$

$$V'^1_{; 3} = (-r \sin^2 \theta) (\sin \phi) = -r \sin^2 \theta \sin \phi$$

$$V_{j1}^2 = V_{,1}^2 + \Gamma_{t1}^2 V^t = V_{,1}^2 + \Gamma_{21}^2 V^2$$

$$V_{j1}^2 = (r^2 \cos \theta)_{,1} + \left(\frac{1}{r}\right) (r^2 \cos \theta)$$

$$V_{j1}^2 = 2r \cos \theta + r \cos \theta = 3r \cos \theta$$

$$V_{j2}^2 = V_{,2}^2 + \Gamma_{t2}^2 V^t = V_{,2}^2 + \Gamma_{12}^2 V^1$$

$$= (r^2 \cos \theta)_{,2} + \left(\frac{1}{r}\right) (r)$$

$$V_{j2}^2 = -r^2 \sin \theta + 1$$

$$V_{j3}^2 = V_{,3}^2 + \Gamma_{3t}^2 V^t = V_{,3}^2 + \overset{0}{\Gamma_{33}^2} V^3$$

$$= (-\sin \theta \cos \theta) \sin \phi$$

$$\begin{aligned}
 V_{j1}^3 &= V_{j1}^3 + \Gamma_{1t}^3 V^t \\
 &= V_{j1}^3 \rightarrow 0 + \Gamma_{13}^3 V^3 = \frac{1}{\lambda} \sin \phi
 \end{aligned}$$

$$\begin{aligned}
 V_{j2}^3 &= V_{j2}^3 + \Gamma_{2t}^3 V^t \\
 &= V_{j2}^3 \rightarrow 0 + \Gamma_{23}^3 V^3 = \cot \theta \sin \phi
 \end{aligned}$$

$$\begin{aligned}
 V_{j3}^3 &= V_{j3}^3 + \Gamma_{3t}^3 V^t = V_{j3}^3 + \Gamma_{31}^3 V^1 + \Gamma_{32}^3 V^2 \\
 &= (\sin \phi)_{,3} + \frac{1}{\lambda} (\lambda) + \cot \theta (\lambda^2 \cos \theta) \\
 &= \cos \phi + 1 + \frac{\lambda^2 \cos^2 \theta}{\sin \theta}
 \end{aligned}$$

Question: The covariant derivative of Kronecker delta is zero.

Solution: $\delta^i_{j;k} = ?$

$$\begin{aligned}\delta^i_{j;k} &= \delta^i_{j,k} + \Gamma^i_{lk} \delta^l_j - \Gamma^l_{jk} \delta^i_l \\ &= \Gamma^i_{jk} - \Gamma^i_{jk} \\ &= 0\end{aligned}$$

Question: Show that $g_{\alpha\beta;r} = 0$

Solution:

$$g_{\alpha\beta;r} = g_{\alpha\beta,r} - \Gamma_{\alpha r}^s g_{s\beta} - \Gamma_{\beta r}^s g_{\alpha s}$$

using the formulas:

$$\Gamma_{\alpha r}^s = \frac{1}{2} g^{st} (g_{tr,\alpha} + g_{t\alpha,r} - g_{\alpha r,t})$$

$$\Gamma_{\beta r}^s = \frac{1}{2} g^{st} (g_{tr,\beta} + g_{t\beta,r} - g_{\beta r,t})$$

Implies:

$$\begin{aligned} g_{\alpha\beta;\gamma} &= g_{\alpha\beta,\gamma} - g_{s\beta} \frac{g^{st}}{2} (g_{tr,\alpha} + g_{t\alpha,r} - g_{\alpha r,t}) \\ &\quad - g_{\alpha s} \frac{g^{st}}{2} (g_{tr,\beta} + g_{t\beta,r} - g_{\beta r,t}) \\ &= g_{\alpha\beta,\gamma} - \frac{1}{2} \delta_{\beta}^t (g_{tr,\alpha} + g_{t\alpha,r} - g_{\alpha r,t}) \\ &\quad - \frac{\delta_{\alpha}^t}{2} (g_{tr,\beta} + g_{t\beta,r} - g_{\beta r,t}) \\ \therefore g_{\alpha s} g^{st} &= \delta_{\alpha}^t, \quad g_{s\beta} g^{st} = \delta_{\beta}^t \end{aligned}$$

$$\Rightarrow g_{\alpha\beta;\gamma} = g_{\alpha\beta,\gamma} - \frac{1}{2} \cancel{g_{\beta\gamma,\alpha}} - \frac{1}{2} g_{\beta\alpha,\gamma} + \frac{1}{2} \cancel{g_{\alpha\gamma,\beta}}$$

$$- \frac{1}{2} \cancel{g_{\alpha\gamma,\beta}} - \frac{1}{2} g_{\alpha\beta,\gamma} + \frac{1}{2} \cancel{g_{\beta\gamma,\alpha}}$$

$$= g_{\alpha\beta,\gamma} - g_{\alpha\beta,\gamma}$$

$$= 0$$

$$\Rightarrow g_{\alpha\beta;\gamma} = 0$$

Question: If A_{ij} be a skew symmetric tensor of rank two
Then show that

$$A_{ij;k} + A_{jk;i} + A_{ki;j} = A_{ij,k} + A_{jk,i} + A_{ki,j}$$

Solution: consider the L.H.S and expand the formulas of covariant derivatives.

$$A_{ij;k} = A_{ij,k} - \Gamma_{ik}^l A_{lj} - \Gamma_{jk}^l A_{il}$$

$$A_{jk;i} = A_{jk,i} - \Gamma_{ji}^l A_{lk} - \Gamma_{ki}^l A_{jl}$$

$$A_{ki;j} = A_{ki,j} - \Gamma_{kj}^l A_{li} - \Gamma_{ij}^l A_{kl}$$

Adding all these we get:

$$A_{ij;k} + A_{jk;i} + A_{ki;j} = A_{ij,k} + A_{jk,i} + A_{ki,j}$$

$$- (A_{lj} + A_{jl}) \Gamma_{ik}^l$$

$$- (A_{il} + A_{li}) \Gamma_{jk}^l$$

$$- (A_{lk} + A_{kl}) \Gamma_{ij}^l$$

Since A_{ij} is skew symmetric, we have

$$A_{ij;k} + A_{jk;i} + A_{ki;j} = A_{ij,k} + A_{jk,i} + A_{ki,j}$$

Gradient, Divergence and curl:

1) Gradient:

If ϕ be a scalar function $\phi = \phi(x^i)$, then the gradient of ϕ is denoted by

$$\nabla \phi = \text{grad } \phi = \frac{\partial \phi}{\partial x^i} \hat{e}_i$$

$$\phi_{;i} = \phi_{,i}$$

Divergence:

The divergence of the contravariant vector A^i is defined by (in component form)

$$\begin{aligned}\text{div}(A^i) &= A^i_{;i} \\ &= A^i_{,i} + \Gamma^i_{ik} A^k\end{aligned}$$

and similarly:

$$\text{div}(A_i) = A_i_{;i} = A_{i,i} - \Gamma^k_{ii} A_k$$

Question: Prove that

$$\operatorname{div} \dot{A}^i = \frac{1}{\sqrt{g_{\alpha\alpha}}} \frac{\partial(\sqrt{g_{\alpha\alpha}} \dot{A}^k)}{\partial x^k}$$

Solution:

$$\operatorname{div} \dot{A}^i = \dot{A}^i_{;i} = \dot{A}^i_{,i} + \Gamma^i_{ki} \dot{A}^k$$

where

$$\Gamma^i_{ki} = \frac{1}{2} g^{is} [g_{si,k} + g_{sk,i} - g_{ki,s}]$$

we have proved that if g_{ij} is a diagonal metric then

$$\Gamma_{ki}^i = \frac{\partial}{\partial x^k} \left(\frac{1}{2} \ln(g_{ii}) \right) = \frac{\partial}{\partial x^k} (\ln \sqrt{g_{ii}}) = \frac{1}{\sqrt{g_{ii}}} \frac{\partial \sqrt{g_{ii}}}{\partial x^k}$$

so this implies

$$\begin{aligned} \Rightarrow \operatorname{div} A^i &= A^i_{,i} + \frac{1}{\sqrt{g_{ii}}} \frac{\partial \sqrt{g_{ii}}}{\partial x^k} A^k \\ &= \frac{1}{\sqrt{g_{ii}}} \left(\sqrt{g_{ii}} A^k_{,k} + A^k \frac{\partial \sqrt{g_{ii}}}{\partial x^k} \right) \end{aligned}$$

$$\Rightarrow \operatorname{div} A^i = \frac{1}{\sqrt{g_{ii}}} \frac{\partial}{\partial x^k} (\sqrt{g_{ii}} A^k)$$

Curl: The curl is defined as:

$$[\nabla \times \vec{A}]_i = \epsilon_{ijk} A_{k;j}$$

so in terms of covariant derivative we have

$$[\nabla \times \vec{A}]_i = \epsilon_{ijk} A_{k;j}$$

where

$$A_{k;j} = A_{k;j} - \Gamma_{kj}^s A_s$$

Laplacian:

The Laplacian operator ∇^2 is defined as:

$$\nabla^2 \phi = \nabla \cdot \nabla \phi \quad \text{i.e.} \quad \text{div}(\text{grad } \phi)$$

Since

$$\nabla \phi = \frac{\partial \phi}{\partial x^i} \hat{e}_i$$

Implies

$$\nabla \cdot \nabla \phi = \left(\frac{\partial \phi}{\partial x^i} \right)_{;i}$$

Absolute Differentiation Along a curve:

As T^i_{ij} is a tensor, the inner product of T^i_{ij} with another tensor is also a tensor.

Suppose that the other tensor is $\frac{dx^i}{dt}$, that is the tangent vector of the curve $C: x^i = x^i(t)$. Then the inner product $T^i_{ij} \frac{dx^j}{dt}$ is a tensor of the same type and order as the original tensor T^i_{ij} .

This tensor is known as the absolute derivative of T^i along the curve C , with components written as

$$\frac{\delta T^i}{\delta t} = \frac{dT^i}{dt} + \Gamma^i_{rs} T^s \frac{dx^r}{dt} ; \quad T^i = T^i(x^i(t))$$

It is clear that, in coordinate systems in which the g_{ij} are constant, absolute differentiation reduces to ordinary differentiation.

$$T^i_{;r} = \frac{\partial T^i}{\partial x^r} + \Gamma^i_{rs} T^s \Rightarrow T^i_{;r} \frac{dx^r}{dt} = \frac{\partial T^i}{\partial x^r} \frac{dx^r}{dt} + \Gamma^i_{rs} T^s \frac{dx^r}{dt}$$

Theorem: (Uniqueness of the absolute derivative)

The only tensor derivable from a given tensor T^i that coincides with the ordinary derivative $\frac{dT^i}{dt}$ along some curve in a rectangular coordinate system is the absolute derivative of T^i along that curve.

Rules for Tensor Differentiation:

The basic rules for differentiation from calculus carry over to covariant and absolute differentiation of tensors.

For covariant differentiation: For tensors T and S we have:

1) Sum : $(T+S)_{;k} = T_{;k} + S_{;k}$

2) outer product : $[TS]_{;k} = [T_{;k}S] + [TS_{;k}]$

3) Inner product : $(TS)_{;k} = T_{;k}S + TS_{;k}$

For Absolute differentiation:

$$1) \text{ Sum: } \frac{\delta}{\delta t}(T+S) = \frac{\delta T}{\delta t} + \frac{\delta S}{\delta t}$$

$$2) \text{ Outer product: } \frac{\delta}{\delta t}[TS] = \left[\frac{\delta T}{\delta t} S \right] + \left[T \frac{\delta S}{\delta t} \right]$$

$$3) \text{ Inner product: } \frac{\delta}{\delta t}(TS) = \frac{\delta T}{\delta t} S + T \frac{\delta S}{\delta t}$$

Question: Prove that absolute differentiation $\frac{\delta T^i}{\delta t}$ is the result of inner product of $T^i_{;j}$ with the tangent vector $\frac{dx^j}{dt}$ of the curve.

Proof: $T^i_{;j} = \frac{\partial T^i}{\partial x^j} + \Gamma^i_{kj} T^k$

Inner product:

$$\begin{aligned} T^i_{;j} \frac{dx^j}{dt} &= \left(\frac{\partial T^i}{\partial x^j} + \Gamma^i_{kj} T^k \right) \frac{dx^j}{dt} = \frac{\partial T^i}{\partial x^j} \frac{dx^j}{dt} + \Gamma^i_{kj} T^k \frac{dx^j}{dt} \\ &= \frac{dT^i}{dt} + \Gamma^i_{kj} T^k \frac{dx^j}{dt} = \frac{\delta T^i}{\delta t} \end{aligned}$$

The Torsion Tensor:

The torsion tensor is defined as:

$$T^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma} - \Gamma^{\alpha}_{\gamma\beta}$$

Even though Christoffel symbols are not tensors their difference is a tensor.

In torsion free spaces $\Gamma^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\gamma\beta}$

However, in some theories the torsion tensor does not vanish.

Acceleration in General Coordinates:

In rectangular coordinates, the acceleration vector of a particle is the time derivative of its velocity vector, or the second time derivative of its position function $\vec{r} = \vec{r}(x^i(t))$.

$$\vec{a} = (a^i) = \left(\frac{d}{dt} \cdot \frac{dx^i}{dt} \right) = \left(\frac{d^2 x^i}{dt^2} \right)$$

The length of this vector at time t is the instantaneous acceleration of the particle.

$$a = \sqrt{\delta_{ij} a^i a^j}$$

The generalization of $\frac{d}{dt} \left(\frac{dx^i}{dt} \right)$

$$\frac{\delta}{\delta t} \left(\frac{dx^i}{dt} \right) = \frac{d}{dt} \left(\frac{dx^i}{dt} \right) + \Gamma_{rs}^i \frac{dx^r}{dt} \frac{dx^s}{dt}$$

Hence,

$$\vec{a} = (a^i) = \left(\frac{d^2 x^i}{dt^2} + \Gamma_{rs}^i \frac{dx^r}{dt} \frac{dx^s}{dt} \right)$$

$$\text{and } a = \sqrt{|g_{ij} a^i a^j|}$$

Question: A particle is in motion along the circular arc given parametrically in Spherical coordinates by $x^1 = b$, $x^2 = \frac{\pi}{4}$, $x^3 = \omega t$. Find its acceleration using the formula

$$a = \sqrt{g_{ij} a^i a^j}$$

Solution: The Christoffel symbols for spherical coordinate system and parametric equations $x^1 = b$, $x^2 = \pi/4$, $x^3 = \omega t$ become:

$$\Gamma_{22}^1 = -x^1 = -b, \quad \Gamma_{33}^1 = -x^1 \sin^2 x^2 = -b \sin^2 \frac{\pi}{4} = -\frac{b}{2}$$

$$\Gamma_{12}^1 = \Gamma_{21}^1 = \frac{1}{x^1} = \frac{1}{b}, \quad \Gamma_{33}^2 = -\sin x^2 \cos x^2 = -\sin \frac{\pi}{4} \cos \frac{\pi}{4} = -\frac{1}{2}$$

$$\Gamma_{13}^3 = \Gamma_{31}^3 = \frac{1}{x^1} = \frac{1}{b}, \quad \Gamma_{23}^3 = \Gamma_{32}^3 = \cot x^2 = \cot \frac{\pi}{4} = 1$$

The components of acceleration are:

$$\begin{aligned} a^1 &= \frac{d^2 x^1}{dt^2} + \Gamma_{rs}^1 \frac{dx^r}{dt} \frac{dx^s}{dt} \\ &= 0 + \Gamma_{22}^1 \frac{dx^2}{dt} \frac{dx^2}{dt} + \Gamma_{33}^1 \frac{dx^3}{dt} \frac{dx^3}{dt} \end{aligned}$$

$$= -\frac{b}{2} (\omega)(\omega) = -\frac{b\omega^2}{2}$$

$$\begin{aligned} a^2 &= \frac{d^2 x^2}{dt^2} + \Gamma_{rs}^2 \frac{dx^r}{dt} \frac{dx^s}{dt} \\ &= \Gamma_{33}^2 \frac{dx^3}{dt} \frac{dx^3}{dt} = -\frac{\omega^2}{2} \end{aligned}$$

$$\begin{aligned}
 a^3 &= \frac{d^2 x^3}{dt^2} + \Gamma_{rs}^3 \frac{dx^r}{dt} \frac{dx^s}{dt} \\
 &= \Gamma_{13}^3 \frac{dx^1}{dt} \frac{dx^2}{dt} + \Gamma_{23}^3 \frac{dx^2}{dt} \frac{dx^3}{dt} \\
 &= 0
 \end{aligned}$$

Metric components are:

$$g_{11} = 1, \quad g_{22} = (x')^2 = b^2$$

$$g_{33} = (x')^2 \sin^2(x^2) = b^2 \sin^2 \frac{\pi}{4} = \frac{b^2}{2}$$

The acceleration is given by:

$$a = \sqrt{g_{ij} \dot{a}^i \dot{a}^j}$$

$$= \sqrt{g_{11} \dot{a}^1 \dot{a}^1 + g_{22} \dot{a}^2 \dot{a}^2 + g_{33} \dot{a}^3 \dot{a}^3}$$

$$= \left((1) \left(-\frac{b\omega^2}{2} \right)^2 + b^2 \left(-\frac{\omega^2}{2} \right)^2 + \frac{b^2}{2} (0) \right)^{1/2}$$

$$= \left(\frac{b^2 \omega^4}{4} + \frac{b^2 \omega^4}{4} \right)^{1/2} = \left(\frac{b^2 \omega^4}{2} \right)^{1/2}$$

$$= \frac{b\omega^2}{\sqrt{2}}$$

Curvature in General Coordinates:

In Euclidean geometry an important role is played by the curvature. The curvature of a curve $C: x^i = x^i(t)$ is defined as the norm of the second derivative of $(x^i(s))$:

$$K(s) = \sqrt{\delta_{ij} \frac{d^2 x^i}{ds^2} \frac{d^2 x^j}{ds^2}}$$

where

$$s(t) = \int_a^t \sqrt{\delta_{ij} \left(\frac{dx^i}{dt} \right) \left(\frac{dx^j}{dt} \right)} dt \text{ gives the arc length parameter.}$$

The generalized formula for curvature is:

$$K(s) = \sqrt{|g_{ij} b^i b^j|}$$

where

$$(b^i) = \left(\frac{\delta}{\delta s} \frac{dx^i}{ds} \right) = \left(\frac{d^2 x^i}{ds^2} + \Gamma_{pq}^i \frac{dx^p}{ds} \frac{dx^q}{ds} \right)$$

and

$$s(t) = \int_a^t \sqrt{g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt}} dt$$

Question: Using the formula

$$k(s) = (g_{ij} \cdot b^i b^j)^{1/2}$$

find the curvature of the circle
 $x' = b, x'' = t$ in polar coordinates.

Solution: For polar coordinates: $(g_{ij}) = \begin{bmatrix} 1 & 0 \\ 0 & (x')^2 \end{bmatrix}$

and christoffel symbols are:

$$\Gamma_{22}^1 = -x' \text{ and}$$

$$\Gamma_{12}^2 = \Gamma_{21}^2 = \frac{1}{x'} , \text{ remaining all are zero.}$$

now:

$$k(s) = (g_{ij} b^i b^j)^{1/2} = (g_{11} b^1 b^1 + g_{22} b^2 b^2)^{1/2}$$

$$b^i = \frac{dx^i}{ds} + \Gamma_{pq}^i \frac{dx^p}{ds} \frac{dx^q}{ds}$$

$$\text{where } \frac{ds}{dt} = \sqrt{g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt}}$$

$$\Rightarrow \frac{ds}{dt} = \left(g_{11} \overset{\nearrow 0}{\frac{dx^1}{dt}} \overset{\nearrow 0}{\frac{dx^1}{dt}} + g_{22} \frac{dx^2}{dt} \frac{dx^2}{dt} \right)^{1/2}$$

$$\frac{ds}{dt} = \left[(x')^2 (1)(1) \right]^{1/2} = x'$$

Now:

$$b' = \frac{d^2 x^1}{ds^2} + \Gamma_{pq}^1 \frac{dx^p}{ds} \frac{dx^q}{ds} = \frac{d^2 x^1}{ds^2} + \Gamma_{22}^1 \frac{dx^2}{ds} \frac{dx^2}{ds}$$

$$\frac{dx^2}{ds} = \frac{dx^2}{dt} \cdot \frac{dt}{ds} = \frac{(1)}{x^1} = \frac{1}{x^1}$$

$$\frac{dx^1}{ds} = \frac{dx^1}{dt} \cdot \frac{dt}{ds} = 0 \Rightarrow \frac{d^2 x^1}{ds^2} = 0$$

$$\Rightarrow b' = 0 + (-x^1) \left(\frac{1}{x^1}\right)^2 = -\frac{1}{x^1}$$

$$b^2 = \frac{dx^2}{ds^2} + \Gamma_{pq}^2 \frac{dx^p}{ds} \frac{dx^q}{ds} = \frac{dx^2}{ds^2} + 2\Gamma_{12}^2 \frac{dx^1}{ds} \frac{dx^2}{ds}$$

$$\text{as } \frac{dx^1}{ds} = 0$$

$$\frac{dx^2}{ds} = \frac{1}{x^1} \Rightarrow \frac{d}{ds} \left(\frac{dx^2}{ds} \right) = \frac{d}{dt} \left(\frac{1}{x^1} \right) \cdot \frac{dt}{ds}$$

$$= 0$$

$$\Rightarrow b^2 = 0$$

Hence:

$$K(s) = (g_{11} b' b' + g_{22} b^2 b^2)^{1/2}$$

$$= \left((1) \left(-\frac{1}{x'} \right)^2 \right)^{1/2}$$

$$= \frac{1}{x'}$$

where $x' = b$

$$\Rightarrow K(s) = \frac{1}{b}$$

Geodesics:

An important application of curvature $k(s)$ in curvilinear coordinates is the following.

Suppose that we seek those curves for which $k=0$, i.e., the straight lines or geodesics.

$$K=0 \Rightarrow b^i = 0 \Rightarrow \frac{d^2 x^i}{ds^2} + \Gamma_{pq}^i \frac{dx^p}{ds} \frac{dx^q}{ds} = 0$$

The solution of this system of second order differential equations will define the geodesics $x^i = x^i(s)$

Question:

Verify that $r' = a \sec r^2$ is a geodesic for the Euclidean metric in polar coordinates. (In rectangular coordinates the curve is $x = a$).

Solution: Given $r' = a \sec r^2$, let $r^2 = t$
then the parametric equations of curve

are: $r' = a \sec t$

$$r^2 = t \quad ; \quad (-\pi/2 < t < \pi/2)$$

we have the formula for geodesic:

$$\frac{d^2 x^i}{ds^2} + \Gamma_{pq}^i \frac{dx^p}{ds} \frac{dx^q}{ds} = 0$$

First we find how t is related to s :

$$\frac{ds}{dt} = \sqrt{g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt}}$$

In polar coordinates:

$$(g_{ij}) = \begin{bmatrix} 1 & 0 \\ 0 & (x')^2 \end{bmatrix}$$

$$\Rightarrow \frac{ds}{dt} = \left[\left(\frac{dx^1}{dt} \right)^2 + a^2 \sec^2 t \left(\frac{dx^2}{dt} \right)^2 \right]^{1/2}$$

$$\frac{dx^1}{dt} = a \sec t \tan t \quad \text{and} \quad \frac{dx^2}{dt} = 1$$

$$\frac{ds}{dt} = \left(a^2 \sec^2 t \tan^2 t + a^2 \sec^2 t \right)^{1/2}$$

$$= a \sec t (1 + \tan^2 t)^{1/2}$$

$$\frac{ds}{dt} = a \sec t (\sec^2 t)^{1/2} = a \sec^2 t$$

$$\text{and } \frac{dt}{ds} = \frac{\cos^2 t}{a} \quad \checkmark$$

$$\text{Now: } \frac{dx'}{ds} = \frac{dx'}{dt} \cdot \frac{dt}{ds} = \cancel{a} \sec t \tan t \cdot \frac{\cos^2 t}{\cancel{a}}$$

$$= \sin t$$

$$\text{and } \frac{d^2 x'}{ds^2} = \frac{d}{ds}(\sin t) = \frac{d}{dt}(\sin t) \frac{dt}{ds}$$

$$\Rightarrow \frac{d^2 x^1}{ds^2} = \cos t \frac{\cos^2 t}{a} = \frac{\cos^3 t}{a}$$

$$\text{and } \frac{dx^2}{ds} = \frac{d}{dt}(x^2) \frac{dt}{ds} = (1) \frac{\cos^2 t}{a} = \frac{\cos^2 t}{a}$$

$$\begin{aligned} \frac{d^2 x^2}{ds^2} &= \frac{d}{ds} \left(\frac{\cos^2 t}{a} \right) = \frac{d}{dt} \left(\frac{\cos^2 t}{a} \right) \cdot \frac{dt}{ds} \\ &= \frac{1}{a} (-2 \cos t \sin t) \frac{\cos^2 t}{a} \end{aligned}$$

$$= -\frac{2 \sin t \cos^3 t}{a^2}$$

$$\text{So, } \frac{d^2 x^1}{ds^2} + \Gamma_{pq}^1 \frac{dx^p}{ds} \frac{dx^q}{ds} = 0$$

The non-zero Christoffel symbols are:

$$\Gamma_{22}^1 = -x^1, \quad \Gamma_{12}^2 = \Gamma_{21}^2 = \frac{1}{x^1}$$

$$\Rightarrow \frac{\cos^2 t}{a} + \Gamma_{22}^1 \frac{dx^2}{ds} \frac{dx^2}{ds} = 0$$

$$\frac{\cos^2 t}{a} + (-a \sec t) \left(\frac{\cos^2 t}{a} \right)^2 = 0$$

$$\Rightarrow \frac{\cos^2 t}{a} \left[1 - \frac{a}{\cos t} \cdot \frac{\cos t}{a} \right] = 0 \Rightarrow 0 = 0$$

satisfied.

Now,

$$\frac{d^2 x^2}{ds^2} + \Gamma_{pq}^{12} \frac{dx^p}{ds} \frac{dx^q}{ds} = 0$$

$$\Rightarrow -\frac{2}{a^2} \sin t \cos^3 t + 2 \Gamma_{12}^{12} \frac{dx^1}{ds} \frac{dx^2}{ds} = 0$$

$$\Rightarrow \frac{-2}{a^2} \sin t \cos^3 t + \frac{2}{a \sec t} \left(\sin t \cdot \frac{\cos^2 t}{a} \right) = 0$$

$$\Rightarrow \frac{-2}{a^2} \sin t \cos^3 t + \frac{2}{a^2} \sin t \cos^3 t = 0$$

$\Rightarrow$ geodesic equation is satisfied

$\Rightarrow x' = a \sec x^2$ is a geodesic for the Euclidean metric.

Question: Write the geodesic equations on the surface of a sphere.

Solution: Here the non-zero Christoffel symbols are:

$$\Gamma_{12}^2 = \Gamma_{21}^2 = \frac{1}{x^1}$$
$$\Gamma_{31}^3 = \Gamma_{13}^3 = \frac{1}{x^1}, \quad \Gamma_{32}^3 = \Gamma_{23}^3 = \cot x^2$$

$$\Gamma_{22}^1 = -x^1, \quad \Gamma_{33}^1 = -x^1 \sin^2 x^2$$

$$\Gamma_{33}^2 = -\sin x^2 \cos x^2$$

using the formula:

$$\frac{d^2 x^i}{ds^2} + \Gamma_{pq}^i \frac{dx^p}{ds} \cdot \frac{dx^q}{ds} = 0$$

for $i=1$:

$$\frac{d^2 x^1}{ds^2} + \Gamma_{22}^1 \left(\frac{dx^2}{ds} \right)^2 + \Gamma_{33}^1 \left(\frac{dx^3}{ds} \right)^2 = 0$$

$$\Rightarrow \frac{d^2 x^1}{ds^2} - x^1 \left(\frac{dx^2}{ds} \right)^2 - x^1 \sin^2 x^2 \left(\frac{dx^3}{ds} \right)^2 = 0$$

for $i=2$:

$$\frac{d^2 x^2}{ds^2} + \Gamma_{pq}^2 \frac{dx^p}{ds} \frac{dx^q}{ds} = 0$$

$$\Rightarrow \frac{d^2 x^2}{ds^2} + 2\Gamma_{12}^2 \frac{dx^1}{ds} \frac{dx^2}{ds} + \Gamma_{33}^2 \left(\frac{dx^3}{ds} \right)^2 = 0$$

$$\Rightarrow \frac{d^2 x^2}{ds^2} + \frac{2}{x^1} \frac{dx^1}{ds} \frac{dx^2}{ds} - \sin x^2 \cos x^2 \left(\frac{dx^3}{ds} \right)^2 = 0$$

for $i = 3$:

$$\frac{d^2 x^3}{ds^2} + \Gamma_{pq}^3 \frac{dx^p}{ds} \frac{dx^q}{ds} = 0$$

$$\frac{d^2 x^3}{ds^2} + 2\Gamma_{13}^3 \frac{dx^1}{ds} \frac{dx^3}{ds} + 2\Gamma_{23}^3 \frac{dx^2}{ds} \frac{dx^3}{ds} = 0$$

$$\frac{d^2 x^3}{ds^2} + \frac{2}{x^1} \frac{dx^1}{ds} \frac{dx^3}{ds} + 2 \cot x^2 \frac{dx^2}{ds} \frac{dx^3}{ds} = 0$$

Remark: If the radius of the sphere is constant then

$$(x^1, x^2, x^3) \rightarrow (r, \theta, \phi)$$

$$x^1 = r = a \text{ (some constant)}$$

$\Rightarrow$ all derivatives of x^1 will vanish
and equations are reduced to:

$$\frac{d^2 x^2}{ds^2} - (\sin^2 \theta \cos^2 \theta) \left(\frac{dx^3}{ds} \right)^2 = 0$$

$$\frac{d^2 x^3}{ds^2} + 2 \cot \theta^2 \frac{dx^2}{ds} \frac{dx^3}{ds} = 0$$

Question: If the metric for (x^i) system is

$$(g_{ij}) = \begin{bmatrix} (x')^2 & 0 \\ 0 & 1 \end{bmatrix}$$

- a) write the differential equations of the geodesics in terms of the dependent variables $u = (x')^2$ and $v = x^2$
- b) Integrate these equations and eliminate the arc-length parameter from the solution.

Solution:^(a) Given $u = (x')^2$ and $v = x^2$

$$\text{and } (g_{ij}) = \begin{bmatrix} (x')^2 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow (g^{ij}) = \begin{bmatrix} \frac{1}{(x')^2} & 0 \\ 0 & 1 \end{bmatrix}$$

The geodesic equation is:

$$\frac{d^2 x^i}{ds^2} + \Gamma_{pq}^i \frac{dx^p}{ds} \frac{dx^q}{ds} = 0 \quad ; i = 1, 2$$

So, first we will find the Christoffel symbols using the formula:

$$\Gamma_{jk}^i = g^{in} \left[-g_{jk,n} + g_{kn,j} + g_{nj,k} \right]$$

$$\Gamma_{11}^1 = \frac{g^{11}}{2} \left[-\cancel{g_{11,1}} + \cancel{g_{11,1}} + g_{11,1} \right]$$

$$= \frac{1}{2(x')^2} (2x') = \frac{1}{x'}$$

$$\Gamma_{21}^1 = \frac{g^{11}}{2} \left[-g_{21,1}^{\rightarrow 0} + g_{11,2}^{\rightarrow 0} + g_{12,1}^{\rightarrow 0} \right] = 0 = \Gamma_{12}^1$$

$$\Gamma_{22}^1 = \frac{g^{11}}{2} \left[-g_{22,1}^{\rightarrow 0} + g_{21,2}^{\rightarrow 0} + g_{12,2}^{\rightarrow 0} \right] = 0$$

$$\Gamma_{11}^2 = \frac{g^{22}}{2} \left[-g_{11,2} + g_{12,2} + g_{21,1} \right] = 0$$

$$\Gamma_{21}^2 = \frac{g^{22}}{2} \left[-g_{21,2} + g_{22,1} + g_{21,2} \right] = 0 = \Gamma_{12}^2$$

$$\Gamma_{22}^2 = \frac{g^{22}}{2} \left[-g_{22,2} + g_{22,2} + g_{22,2} \right] = 0$$

$\Rightarrow$ The only non-zero Christoffel symbol
is $\Gamma_{11}^1 = \frac{1}{x^1}$.

Now for $i=1$, geodesic equation gives:

$$\frac{d^2 x^1}{ds^2} + \Gamma_{pq}^1 \frac{dx^p}{ds} \frac{dx^q}{ds} = 0$$

$$\Rightarrow \frac{d^2 x^1}{ds^2} + \Gamma_{11}^1 \frac{dx^1}{ds} \frac{dx^1}{ds} = 0$$

$$\Rightarrow \frac{d^2 x^1}{ds^2} + \frac{1}{x^1} \left(\frac{dx^1}{ds} \right)^2 = 0$$

$$\text{Given } u = (x^1)^2 \Rightarrow x^1 = \sqrt{u}$$

$$\Rightarrow \frac{dx^1}{ds} = \frac{1}{2\sqrt{u}} \frac{du}{ds}$$

$$\Rightarrow \frac{d^2 x'}{ds^2} = \frac{d}{ds} \left(\frac{1}{2\sqrt{u}} \frac{du}{ds} \right)$$

$$\Rightarrow \frac{d^2 x'}{ds^2} = \frac{1}{2\sqrt{u}} \frac{d^2 u}{ds^2} - \frac{1}{4u^{3/2}} \left(\frac{du}{ds} \right)^2$$

$$\text{So: } \frac{d^2 x'}{ds^2} + \frac{1}{x'} \left(\frac{dx'}{ds} \right)^2 = 0$$

$$\Rightarrow \frac{1}{2\sqrt{u}} \frac{d^2 u}{ds^2} - \frac{1}{4u\sqrt{u}} \left(\frac{du}{ds} \right)^2 + \frac{1}{\sqrt{u}} \frac{1}{4u} \left(\frac{du}{ds} \right)^2 = 0$$

$$\Rightarrow \frac{d^2 u}{ds^2} = 0$$

Similarly, for $i=2$:

$$\frac{d^2 x^2}{ds^2} + \Gamma_{\rho\gamma}^{2\alpha 0} \frac{dx^\rho}{ds} \frac{dx^\gamma}{ds} = 0$$

$$\Rightarrow \frac{d^2 x^2}{ds^2} = 0 \quad \text{given} \quad x^2 = v$$

$$\Rightarrow \frac{d^2 v}{ds^2} = 0$$

Implies Geodesic equations are:

$$\frac{d^2 u}{ds^2} = 0 \quad \text{and} \quad \frac{d^2 v}{ds^2} = 0$$

(b) Now we will integrate both equations and eliminate the arc length parameter s .

Consider, $\frac{d^2 u}{ds^2} = 0 \Rightarrow \frac{du}{ds} = c_1 \Rightarrow u = c_1 s + c_2$

and similarly: $\frac{d^2 v}{ds^2} = 0$

$$v = c_3 s + c_4$$

$$\Rightarrow S = \frac{1}{c_1} (u - c_2)$$

$$\text{and } S = \frac{1}{c_3} (v - c_4)$$

comparison yields:

$$\frac{1}{c_1} (u - c_2) = \frac{1}{c_3} (v - c_4)$$

$$\Rightarrow \frac{c_3}{c_1} u - \frac{c_2 c_3}{c_1} = v - c_4$$

$$\Rightarrow v = \frac{c_3}{c_1} u - \frac{c_2 c_3}{c_1} + c_4$$

$$\Rightarrow v = p u + q \quad \text{where } p = c_3/c_1$$

$$\text{and } q = c_4 - \frac{c_2 c_3}{c_1}$$

Question: Find the absolute differentiation of $T^i = (\cos \theta, \frac{\sin \theta}{r})$ along the curve C with parametric equations: $x^1 = b, x^2 = t$ in polar coordinates.

Solution: here $(x^1, x^2) \rightarrow (r, \theta)$

$$\Rightarrow x^1 = r = b \quad \text{and} \quad x^2 = \theta = t$$

$$\text{and } T^i = (\cos \theta, \frac{\sin \theta}{r}) = (\cos t, \frac{\sin t}{b})$$

The non-zero Christoffel Symbols for polar coordinates are:

$$\Gamma_{22}^1 = -x' \quad , \quad \Gamma_{12}^2 = \Gamma_{21}^2 = \frac{1}{x'}$$

$$\Rightarrow \Gamma_{22}^1 = -b \quad \text{and} \quad \Gamma_{12}^2 = \frac{1}{b}$$

Now using the formula for absolute derivative:

$$\frac{\delta T^{\dot{i}}}{\delta t} = \frac{d}{dt} (T^{\dot{i}}) + \Gamma_{\dot{j}k}^{\dot{i}} T^{\dot{j}} \frac{dx^k}{dt}$$

$$\Rightarrow \frac{\delta T^1}{\delta t} = \frac{dT^1}{dt} + \Gamma_{jk}^1 T^j \frac{dx^k}{dt}$$

$$= \frac{d}{dt} (\cos t) + \Gamma_{22}^1 T^2 \frac{dx^2}{dt}$$

$$= -\sin t + (-1) \left(\frac{\sin t}{1} \right) \frac{dt}{dt}$$

$$= -\sin t - \sin t (1)$$

$$\frac{\delta T^1}{\delta t} = -2 \sin t$$

Similarly:

$$\frac{\delta T^2}{\delta t} = \frac{dT^2}{dt} + \Gamma_{jk}^2 T^j \frac{dx^k}{dt}$$

$$= \frac{d}{dt} \left(\frac{\sin t}{b} \right) + \Gamma_{12}^2 T^1 \frac{dx^2}{dt} + \Gamma_{21}^2 T^2 \frac{dx^1}{dt}$$

$$= \frac{1}{b} \cos t + \left(\frac{1}{b} \right) \cos t \frac{dt}{dt} + \left(\frac{1}{b} \right) \frac{\sin t}{b} \frac{d(b)}{dt} \rightarrow 0$$

$$= \frac{\cos t}{b} + \frac{\cos t}{b}$$

$$\frac{\delta T^2}{\delta t} = \frac{2}{b} \cos t \Rightarrow \frac{\delta T^2}{\delta t} = \left(-2 \sin t, \frac{2}{b} \cos t \right)$$

Question: Find the divergence of

$$\vec{A} = \left(\frac{1}{x^1}, x^1 \cos x^2 \right)$$

in polar coordinates using christoffel symbols

$$\Gamma_{22}^1 = -x^1, \quad \Gamma_{12}^2 = \Gamma_{21}^2 = \frac{1}{x^1}$$

Solution: consider the formula:

$$\text{div} \vec{A} = A^{\hat{i}}_{;\hat{i}} = A^{\hat{i}}_{, \hat{i}} + \Gamma^{\hat{i}}_{\hat{j}\hat{i}} A^{\hat{j}}$$

$$\text{also } \text{div} \vec{A} = A^{\hat{i}}_{;\hat{i}} = A^1_{;1} + A^2_{;2}$$

$$\Rightarrow A^1_{;1} = A^1_{,1} + \Gamma^1_{j1} A^j$$

$$= \left(\frac{1}{r}\right)_{,r} + \Gamma^1_{j1} A^j = -\frac{1}{r^2}$$

$$A^2_{;2} = A^2_{,2} + \Gamma^2_{j2} A^j$$

$$= (r \cos \theta)_{,2} + \Gamma^2_{12} A^1$$

$$= -r \sin \theta + \frac{1}{r} \left(\frac{1}{r}\right) = -r \sin \theta + \frac{1}{r^2}$$

$$\Rightarrow \operatorname{div} \vec{A} = A^i_{;i} = -\frac{1}{r^2} - r \sin \theta + \frac{1}{r^2} = \underline{\underline{-r \sin \theta}}$$

Question: Find the curl of $\vec{A} = (A_i) = (1, \alpha', \frac{\cos \alpha'^2}{\alpha'})$ in spherical polar coordinates

Solution: Here we use the following formulas:

$$(\nabla \times \vec{A})_i = \varepsilon_{ijk} A_{kj}$$

$$(\nabla \times \vec{A})_j = \varepsilon_{jki} A_{i;k}$$

$$(\nabla \times \vec{A})_k = \varepsilon_{kij} A_{j;i}$$

$$(\nabla \times \vec{A})_i = \varepsilon_{ijk} A_{k;j}$$

$$(\nabla \times \vec{A})_1 = \varepsilon_{1jk} \left[A_{k;j} - \Gamma_{kj}^{k} A_k \right]$$

$$= \varepsilon_{123} \left[A_{3;2} - \Gamma_{23}^{k} A_k \right]$$

$$+ \varepsilon_{132} \left[A_{2;3} - \Gamma_{32}^{k} A_k \right]$$

$$= (-1) \left[x'_{,3} - \Gamma_{23}^{3} A_3 \right] + (1) \left[\left(\frac{\cos x^2}{x'} \right)_{,2} - \Gamma_{32}^{3} A_3 \right]$$

$$= 0 + (\cot x^2) \frac{\cos x^2}{x'} + \left[-\frac{\sin x^2}{x'} - \cot x^2 \frac{\cos x^2}{x'} \right]$$

$$= \frac{\cos^2 x^2}{x' \sin x^2} - \frac{\sin x^2}{x'} - \frac{\cos^2 x^2}{x' \sin x^2} = -\frac{\sin x^2}{x'}$$

$$(\nabla \times \vec{A})_j = \varepsilon_{j k i} A_{i;k}$$

$$(\nabla \times \vec{A})_2 = \varepsilon_{2 k i} A_{i;k}$$

$$= \varepsilon_{231} A_{1;3} + \varepsilon_{213} A_{3;1}$$

$$= (1) [A_{1,3} - \Gamma_{13}^1 A_1] + (-1) [A_{3,1} - \Gamma_{31}^1 A_1]$$

$$= 0 - \Gamma_{13}^3 A_3 - \left(\frac{\cos \kappa^2}{\kappa'} \right)_{,1} + \Gamma_{31}^3 A_3$$

$$= -\frac{1}{\kappa'} \left(\frac{\cos \kappa^2}{\kappa'} \right) + \frac{\cos \kappa^2}{(\kappa')^2} + \frac{1}{\kappa'} \left(\frac{\cos \kappa^2}{\kappa'} \right)$$

$$(\nabla \times \vec{A})_2 = \frac{\cos \kappa^2}{(\kappa')^2}$$

$$(\nabla \times \vec{A})_k = \varepsilon_{kij} A_{j,i}$$

$$\Rightarrow (\nabla \times \vec{A})_3 = \varepsilon_{3ij} A_{j,i}$$

$$= \varepsilon_{321} A_{1,2} + \varepsilon_{312} A_{2,1}$$

$$= (1) [A_{1,2} - \Gamma_{12}^2 A_2] + (-1) [A_{2,1} - \Gamma_{12}^2 A_2]$$

$$= 0 - \cancel{\Gamma_{12}^2} A_2 - (x')_{,1} + \cancel{\Gamma_{12}^2} A_2$$

$$= -1$$

$$\Rightarrow \nabla \times \vec{A} = \left(\frac{\sin x^2}{x'} \right) \hat{e}_r + \frac{\cos x^2}{(x')^2} \hat{e}_\theta + (-1) \hat{e}_\phi$$

$$\nabla \times \vec{A} = \frac{\sin \theta}{r} \hat{e}_r + \frac{\cos \theta}{r^2} \hat{e}_\theta - \hat{e}_\phi$$

Question: Find $\nabla^2 \phi$ for $\phi = \frac{x'x^2}{2}$ in polar coordinates.

Solution: First consider:

$$\nabla \phi = \frac{\partial \phi}{\partial x^i} \hat{e}_i = \frac{\partial \phi}{\partial x^1} \hat{e}_1 + \frac{\partial \phi}{\partial x^2} \hat{e}_2$$

$$\Rightarrow \nabla \phi = \frac{\partial}{\partial x^1} \left(\frac{x'x^2}{2} \right) \hat{e}_1 + \frac{\partial}{\partial x^2} \left(\frac{x'x^2}{2} \right) \hat{e}_2$$

$$\nabla \phi = \frac{x^2}{2} \hat{e}_1 + \frac{x'}{2} \hat{e}_2$$

$$\Rightarrow \nabla \phi = \left(\frac{x^2}{2}, \frac{x'}{2} \right) = \left(\frac{\theta}{2}, \frac{r}{2} \right)$$

now:

$$\nabla^2 \phi = \nabla \cdot \nabla \phi = \left(\frac{\partial \phi}{\partial x^i} \right)_{;i}$$

$$= \left(\frac{\partial \phi}{\partial x^1} \right)_{;1} + \left(\frac{\partial \phi}{\partial x^2} \right)_{;2} \quad (\nabla \phi \text{ is a covariant vector})$$

$$= \nabla \phi^1_{;1} + \nabla \phi^2_{;2}$$

$$= \nabla \phi^1_{;1} - \Gamma^1_{j1} \nabla \phi^j + \nabla \phi^2_{;2} - \Gamma^2_{j2} \nabla \phi^j$$

$$= \left(\frac{\partial}{\partial x^1} \right)_{;1} - \Gamma^1_{j1} \nabla \phi^j + \left(\frac{\partial}{\partial x^2} \right)_{;2} - \Gamma^2_{j2} \nabla \phi^j$$

$$= -\left(\frac{1}{x^1} \right) \left(x^2_{;2} \right) = -\frac{1}{2} \frac{x^2}{x^1} = -\frac{\partial}{2x}$$

Applications of Tensors in Special Relativity:

In 1905, Albert Einstein proposed the special theory of relativity based on three simple postulates:

Postulate 1: The principle of relativity.

The laws of physics are the same in all inertial reference frames.

Postulate 2: The speed of light is invariant.

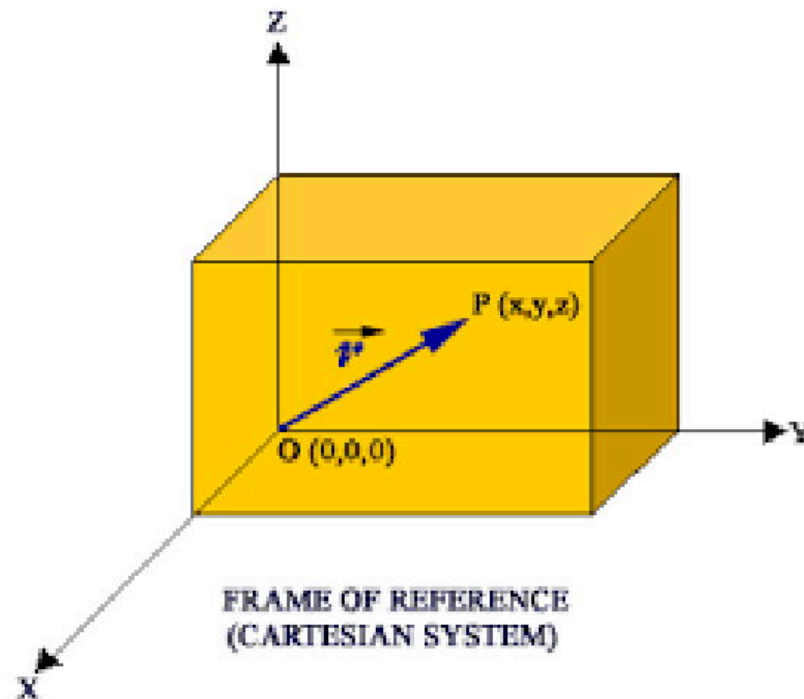
All observers in inertial frames will measure the same speed of light, regardless of their state of motion.

Postulate 3: Uniform motion is invariant.

A particle at rest or with constant velocity in one inertial frame will be at rest or have constant velocity in all inertial frames.

Frame of reference

The frame of reference is defined as any coordinate system with respect to which one can study the motion of another object. For example a bus, train, the Earth, the road etc.



Inertial frame of reference

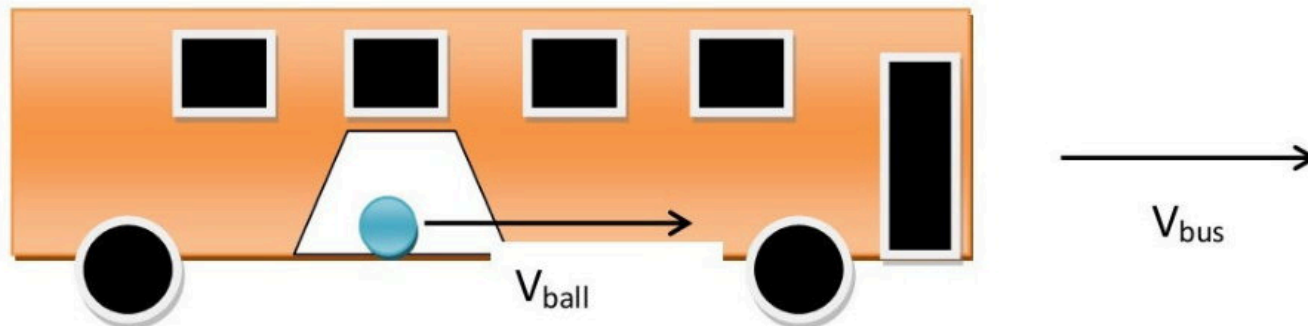
Inertia – resistance of an object to change its state of motion

Inertial Frame of Reference

- Non accelerating
- Newton's 1st law and other laws of physics are valid

For example:

Inside a bus moving at constant velocity with a ball inside



Here we will discuss some concepts of Special Relativity where tensors are being used.

Event Space:

Here the concept of space and time is combined. Thus each event is assigned four coordinates (t, x, y, z) where t is the time of the event and (x, y, z) is the location of the event in ordinary rectangular coordinates. Such coordinates

are called spacetime coordinates.

Definition: An event space is an $\mathbb{R}^4$ space whose points are events.

Event coordinates: $(x^i) = (x^0, x^1, x^2, x^3)$

where $x^0 = ct$ is the temporal coordinate, and $(x^1, x^2, x^3) = (x, y, z)$ the rectangular positional coordinates of an event.

let $E_1(x_1)$ and $E_2(x_2)$ are two events
then:

* They are identical if
 $x_1^i = x_2^i$ for all i

* They are simultaneous
if $x_1^0 = x_2^0$

* They are copositional
if $x_1^i = x_2^i$ for $i = 1, 2, 3$

* The spatial distance b/w E_1 and E_2 is
$$S_d = \sqrt{(dx^1)^2 + (dx^2)^2 + (dx^3)^2}$$

where $dx^i = x_2^i - x_1^i$ for $i = 1, 2, 3$.

Relativistic length:

For an arbitrary event $E(x^i)$, the relativistic distance from origin to $E(x^i)$ is the real number $s \geq 0$ such that

$$s^2 = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$$

More generally, the length of interval or relativistic distance between $E_1(x_1^i)$ and $E_2(x_2^i)$ is:

$$(ds)^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$$

Interval Types: $ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$

The interval between $E_1(x_1^i)$ and $E_2(x_2^i)$ is

(1) Space like if $(ds)^2 < 0$

$$\Rightarrow (dx^1)^2 + (dx^2)^2 + (dx^3)^2 > (dx^0)^2$$

(2) light like if $(ds)^2 = 0$

$$\Rightarrow (dx^0)^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2$$

(3) Time like if $(ds)^2 > 0$

$$\Rightarrow (dx^0)^2 > (dx^1)^2 + (dx^2)^2 + (dx^3)^2$$

Spacelike Interval:

If the interval between two events is spacelike then there is some frame of reference where the events occur at the same time, and there is no frame of reference where the events occur at the same place. The value of a spacelike interval is actually the proper distance between the two events—that is the distance as measured in a frame of reference where the two events occur at the same time.

Lightlike Interval:

We say that the interval is light-like or null if the interval is zero. If the interval is light-like (or null) then a ray of light can go from one event to the other. If the interval between two events is light-like then there is no frame of reference where the events occur at either the same time or at the same place.

Timelike Interval:

If the interval between two events is timelike then there is some frame of reference where the events occur at the same place, and there is no frame of reference where the events occur at the same time. And the event that occurs first in one frame of reference, occurs first in all frames of reference. The value of a timelike interval determines the proper time between the two events—that is the time between the events as measured in a frame of reference where they occur at the same position.

Exercises:

1) Calculate ds^2 for the event pairs:

a) $E_1(5, 1, -2, 0)$ and $E_2(0, 3, 1, -3)$

$$\begin{aligned} ds^2 &= (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \\ &= (0-5)^2 - (3-1)^2 - (1+2)^2 - (-3)^2 \\ &= 25 - 4 - 9 - 9 \end{aligned}$$

$$ds^2 = 3$$

$\Rightarrow$ Timelike interval

Also determine the interval type.

b) $E_1(7, 2, 4, 4)$ and $E_2(4, 1, 2, 6)$

$$ds^2 = (3)^2 - (1)^2 - (2)^2 - (2)^2$$

$$= 9 - 1 - 4 - 4 = 0$$

$\Rightarrow ds^2 = 0 \Rightarrow$ light like interval.

c) $E_1(5, 1, 3, 3)$ and $E_2(2, -1, 1, 1)$

$$ds^2 = (3)^2 - (2)^2 - (2)^2 - (2)^2$$

$$= 9 - 4 - 4 - 4 = -3$$

$\Rightarrow$ space like interval

Question: Show that

a) Simultaneous events have a spacelike interval.

Solution: Simultaneous events means

$$x_1^0 = x_2^0 \Rightarrow dx^0 = 0$$

for $E_1(x_1^i), E_2(x_2^i)$; $i = 0, 1, 2, 3$.

$$\Rightarrow ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$$

$$\Rightarrow ds^2 = 0 - [(dx^1)^2 + (dx^2)^2 + (dx^3)^2] < 0$$

$\Rightarrow$ Spacelike interval.

b) Copositional events have a timelike interval.

Solution: Copositional events means

$$x^i_1 = x^i_2 \text{ for } i = 1, 2, 3$$

$$\Rightarrow dx^i = 0 \text{ for } i = 1, 2, 3$$

So,

$$ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$$

$$\Rightarrow ds^2 = (dx^0)^2 > 0$$

$\Rightarrow$ timelike interval.

Minkowski Spacetime:

Minkowski space or Minkowski spacetime is named after the mathematician Hermann Minkowski.

In its setting the three ordinary dimensions of space are combined with a single dimension of time to form a four-dimensional spacetime.

Minkowski space is often contrasted with Euclidean space. While a Euclidean space has only spacelike dimensions, a Minkowski space also has one timelike dimension.

Therefore the symmetry group of a Euclidean space is the Euclidean group and for a Minkowski space it is the Poincaré group.

Locally flat spacetime:

The use of the Minkowski space to describe physical systems over finite distances applies only in the Newtonian limit of systems without significant gravitation.

In the case of significant gravitation, spacetime becomes curved and one must abandon special relativity in favor of the full theory of general relativity.

In the presence of gravity, spacetime is described by a curved 4-dimensional manifold for which the tangent space to any point is a 4-dimensional Minkowski space. Thus, the structure of Minkowski space is still essential in the description of general relativity.

Mathematical Definition

Minkowski spacetime is a 4-dimensional coordinate system in which the axes are $(ct, x, y, z) \rightarrow (x^0, x^1, x^2, x^3)$

The differential for arc length in Minkowski spacetime is given by

$$ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$$

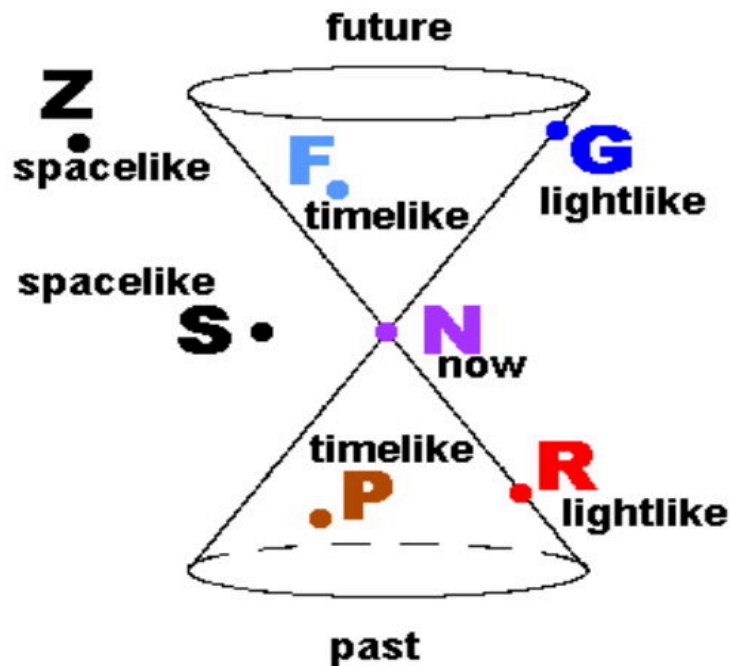
and in matrix notation the Minkowski spacetime is:

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

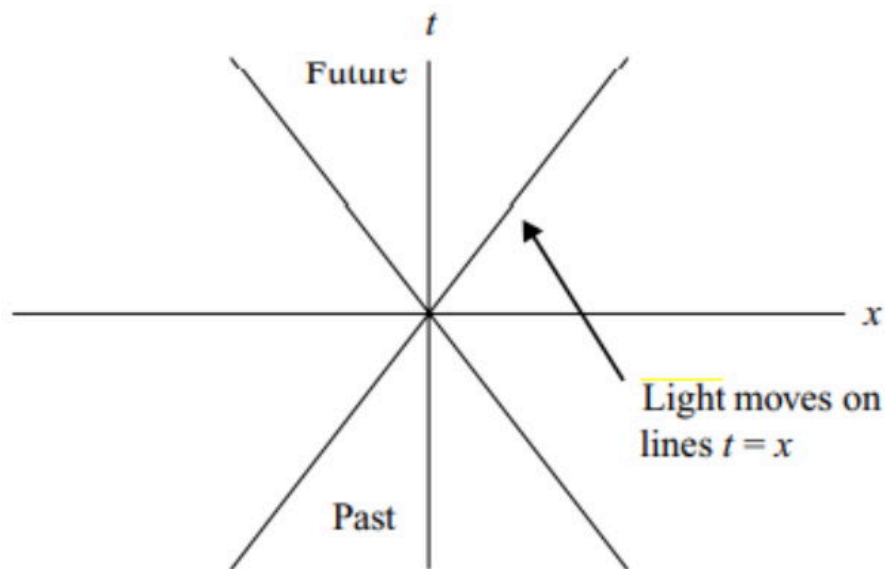
This metric is denoted by $\eta_{\mu\nu}$ and its inverse $\eta^{\mu\nu}$ is also same.

Light Cones and Spacetime Diagrams

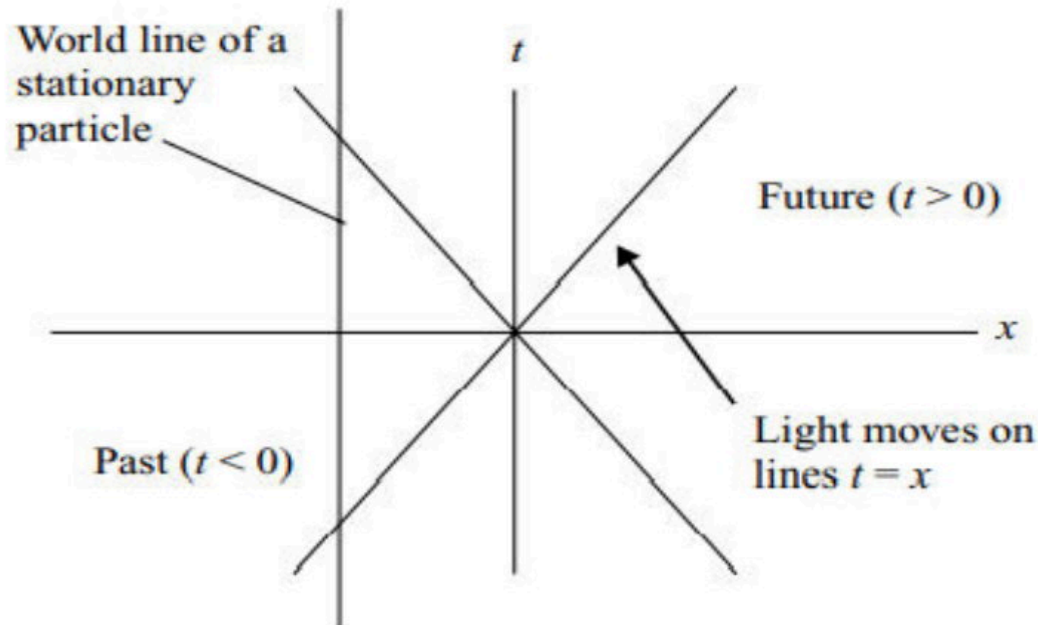
It is often helpful to visualize spacetime by considering a flash of light emitted at the origin. Such a flash of light is described by a spherical wavefront. However, our minds cannot visualize four dimensions and it's not possible to draw it on paper. So we do the next best thing and suppress one or more of the spatial dimensions.



Let's start with the simplest of all cases, where we suppress two spatial dimensions. Doing so gives us a simple spacetime diagram. In a spacetime diagram, the vertical axis represents time while one or two horizontal axes represent space. It is convenient to work in units where $c = 1$. The upper half plane where $t > 0$ represents events to the future of the origin. Past events are found in the lower half plane where $t < 0$.



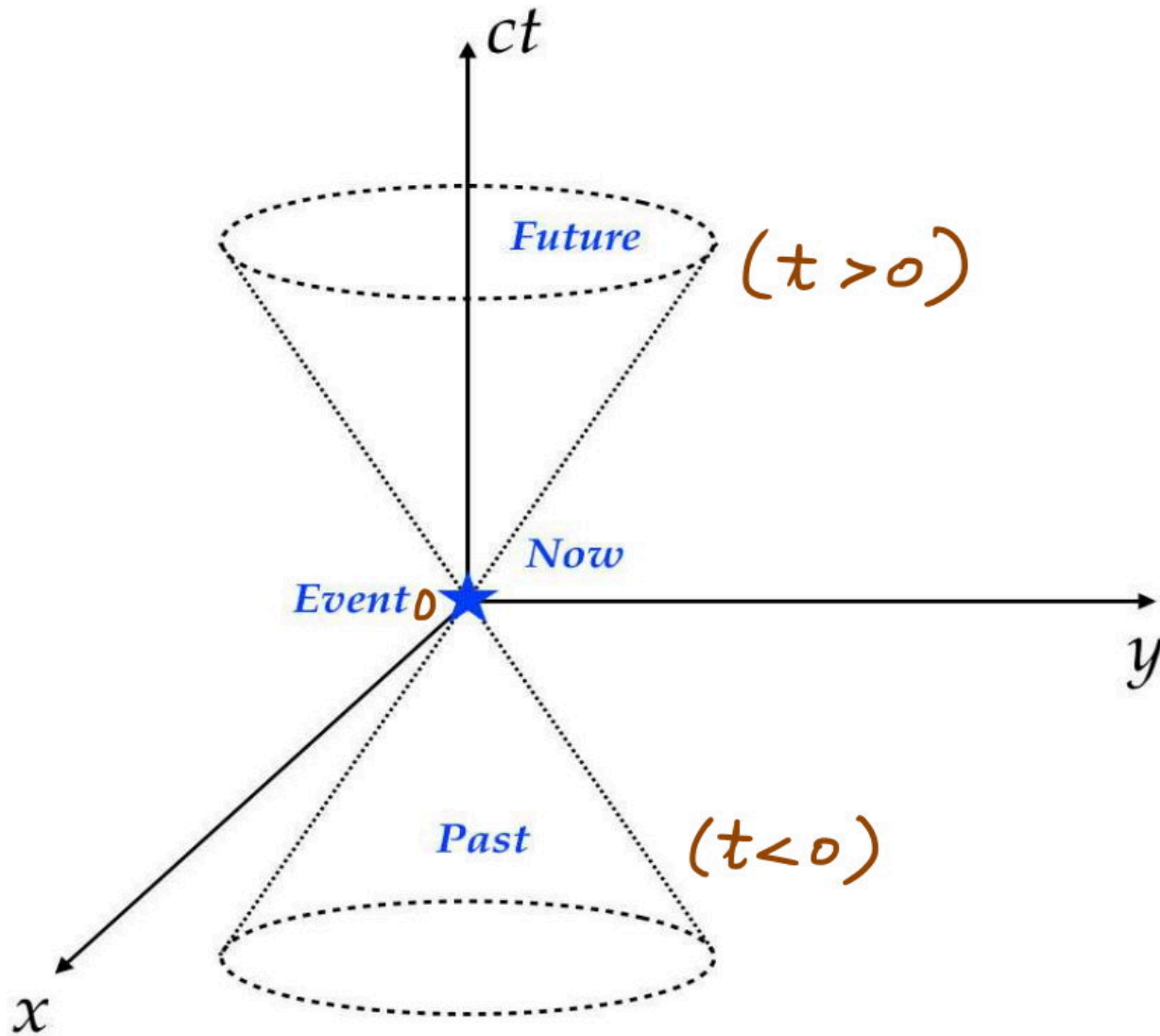
The motion of light in such a diagram is then described by lines that make a 45° angle with the x -axis, i.e., lines that satisfy $t^2 = x^2$. The motion of a particle through spacetime as depicted in a spacetime diagram is called a worldline.



The simplest of all particle motion is a particle just sitting somewhere. To indicate the worldline of a stationary particle on a spacetime diagram, we simply draw a straight vertical line that passes through the x-axis at the spatial location of the particle. This makes sense because the particle is located at some position x that does not change, but time keeps marching forward.

Special relativity tells us that a particle cannot move faster than the speed of light. On a spacetime diagram, this is indicated by the fact that particle motion is restricted to occur only inside the light cone. The region inside the light cone is called timelike. Regions outside the light cone, which are casually unrelated to the event E , are called spacelike.

The following figure shows the projection of the light cone onto the hyperplane $x^3 = 0$.



Light Cone: A flash of light at position $(0,0,0)$ and time $t=0$ sends out an expanding Spherical wave front, with equation

$$x^2 + y^2 + z^2 = c^2 t^2 \quad \text{or}$$

$$(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = 0$$

this is the equation of the light cone in event space, relative to the inertial frame (x^i) .

In any other inertial frame $(\tilde{x}^i)$ the equation of the light cone is exactly the same, (since all observers measure the velocity of light as c).

$$(\tilde{x}^0)^2 - (\tilde{x}^1)^2 - (\tilde{x}^2)^2 - (\tilde{x}^3)^2 = 0$$

Note : Relativistic distance is an invariant across all inertial frames. This is due to postulate of special relativity, i.e.,

$$ds^2 = d\tilde{s}^2.$$

The relativistic distance b/w two events is also called interval.

Lorentz Matrices and Transformations:

The invariance of the equation of the light cone can be expressed as:

$$ds^2 = 0 = d\tilde{s}^2$$

$$g_{ij} dx^i dx^j = 0 = \tilde{g}_{ij} d\tilde{x}^i d\tilde{x}^j$$

where g_{ij} is the Minkowski metric.

$$\Rightarrow g_{00} = 1, \quad g_{11} = g_{22} = g_{33} = -1 \text{ and } g_{ij} = 0 \text{ for } i \neq j.$$

$$\text{also } d\tilde{x}^i = a^i_j dx^j$$

$$\Rightarrow g_{ij} dx^i dx^j = g_{ij} a^i_r a^j_s dx^r dx^s$$

$$\Rightarrow g_{ij} \frac{dx^i}{dx^r} \frac{dx^j}{dx^s} = g_{ij} a^i_r a^j_s$$

$$\Rightarrow g_{ij} \delta^i_r \delta^j_s = g_{ij} a^i_r a^j_s$$

$$\Rightarrow g_{rs} = g_{ij} a^i_r a^j_s$$

or In matrix form:

$$A^t G A = G$$

$$\text{or: } (a_0^0)^2 - (a_0^1)^2 - (a_0^2)^2 - (a_0^3)^2 = 1$$

$$(a_j^0)^2 - (a_j^1)^2 - (a_j^2)^2 - (a_j^3)^2 = -1 \quad (j=1,2,3)$$

$$a_i^0 a_j^0 - a_i^1 a_j^1 - a_i^2 a_j^2 - a_i^3 a_j^3 = 0 \quad (i \neq j)$$

Definition: Any 4×4 matrix (or corresponding linear transformation) that preserves the quadratic form $A^T G A$ is called Lorentz.

★ The set of Lorentz matrices form a group under matrix multiplication called the Lorentz group.

Simple Lorentz Transformations:

The simple Lorentz transformations are

$$\tilde{x}^0 = \frac{x^0 - \beta x^1}{\sqrt{1 - \beta^2}}, \quad \tilde{x}^1 = -\frac{\beta x^0 + x^1}{\sqrt{1 - \beta^2}}, \quad \tilde{x}^2 = x^2, \quad \tilde{x}^3 = x^3$$

OR in matrix form:

$$A = \begin{bmatrix} \frac{1}{\sqrt{1 - \beta^2}} & -\frac{\beta}{\sqrt{1 - \beta^2}} & 0 & 0 \\ -\frac{\beta}{\sqrt{1 - \beta^2}} & \frac{1}{\sqrt{1 - \beta^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where $\beta = v/c$ is relative velocity.

Any 4×4 matrix of the form:

$$A = \begin{bmatrix} a & b & 0 & 0 \\ b & a & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ where } a^2 - b^2 = 1$$

is called Simple Lorentz and the relative velocity is recovered as:

$$\beta = \frac{-b}{a}$$

The inverse of a simple Lorentz matrix is:

$$A^{-1} = \begin{bmatrix} a & -b & 0 & 0 \\ -b & a & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

which is itself a simple Lorentz matrix, corresponding to $-\beta$.

i.e: If the velocity of $\tilde{O}$ (frame of reference) w.r.t O is v , then the velocity of O with respect to $\tilde{O}$ is $-v$.

Consequences of Simple Lorentz Transformation:

Length Contraction:

For any fixed x^0 , simple Lorentz transformation gives:

$$\Delta \tilde{x}^1 = \frac{1}{\sqrt{1-\beta^2}} \Delta x^1 \text{ or } \Delta x^1 = \sqrt{1-\beta^2} \Delta \tilde{x}^1 < \Delta x^1$$

If the frame $\tilde{O}$ is moving at a uniform velocity v relative to O , distance in $\tilde{O}$ appears to Observer O to run slow by the factor $\sqrt{1-\beta^2}$.

Two events: $(\tilde{x}_1^0, \tilde{x}_1^1, \tilde{x}_1^2, \tilde{x}_1^3) \rightarrow A$
 $(\tilde{x}_2^0, \tilde{x}_2^1, \tilde{x}_2^2, \tilde{x}_2^3) \rightarrow B$

x^0 is fixed, Implies:

$$\tilde{x}_1^1 = \frac{-\beta x_1^0 + x_1^1}{\sqrt{1-\beta^2}}, \quad \tilde{x}_2^1 = \frac{-\beta x_2^0 + x_2^1}{\sqrt{1-\beta^2}}$$

$$\Rightarrow \Delta \tilde{x}^1 = \frac{1}{\sqrt{1-\beta^2}} (-\beta \Delta x^0 + \Delta x^1)$$

$$\Rightarrow \Delta \tilde{x}^1 = \frac{1}{\sqrt{1-\beta^2}} \Delta x^1$$

Time Dilation:

For any fixed $\tilde{x}'$,

$$\Delta x^0 = \frac{1}{\sqrt{1-\beta^2}} \Delta \tilde{x}^0 \text{ or } \Delta t = \frac{1}{\sqrt{1-\beta^2}} \Delta \tilde{t} > \Delta \tilde{t}$$

If the frame $\tilde{O}$ is moving at a uniform velocity v relative to O , the clock of Observer $\tilde{O}$ appears to observer O to run slow by the factor $\sqrt{1-\beta^2}$.

consider inverse Lorentz transformation
as:

$$x_1^0 = \frac{\tilde{x}_1^0 + \beta \tilde{x}_1^1}{\sqrt{1-\beta^2}}, \quad x_2^0 = \frac{\tilde{x}_2^0 - \beta \tilde{x}_2^1}{\sqrt{1-\beta^2}}$$

$$\Rightarrow \Delta x^0 = \frac{\Delta \tilde{x}^0 + \beta \Delta \tilde{x}^1}{\sqrt{1-\beta^2}}$$

$$\Rightarrow \Delta x^0 = \frac{1}{\sqrt{1-\beta^2}} \Delta \tilde{x}^0$$

Question: Verify that the following matrix is Lorentz:

$$A = \begin{bmatrix} \sqrt{3}^{a_0^0} & \sqrt{2}^{a_1^0} & 0^{a_2^0} & 0^{a_3^0} \\ 1^{a_1^0} & \sqrt{6}/2^{a_1^1} & 1/2^{a_2^1} & 1/2^{a_3^1} \\ 1 & \sqrt{6}/2 & -1/2 & -1/2 \\ 0 & 0 & -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \quad g_{rs} = g_{ij} a_r^i a_s^j$$

Solution: To show that the above matrix is Lorentz, we have to show that:

$$(a_0^0)^2 - (a_1^0)^2 - (a_2^0)^2 - (a_3^0)^2 = 1 \rightarrow (A)$$

$$(a_j^0)^2 - (a_j^1)^2 - (a_j^2)^2 - (a_j^3)^2 = -1, \quad j=1,2,3 \rightarrow (B)$$

$$a_i^0 a_j^0 - a_i^1 a_j^1 - a_i^2 a_j^2 - a_i^3 a_j^3 = 0, \quad i \neq j \rightarrow (C)$$

For A:

$$(\sqrt{3})^2 - (1)^2 - (1)^2 - (0)^2 = 3 - 2 = 1$$

For B:

$$(\sqrt{2})^2 - \left(\frac{\sqrt{6}}{2}\right)^2 - \left(\frac{\sqrt{6}}{2}\right)^2 + (0)^2 = 2 - \frac{3}{2} - \frac{3}{2} = -1$$

$$(0)^2 - \left(\frac{1}{2}\right)^2 - \left(-\frac{1}{2}\right)^2 - \left(-\frac{\sqrt{2}}{2}\right)^2 = 0 - \frac{1}{4} - \frac{1}{4} - \frac{1}{2} = -1$$

$$(0)^2 - \left(\frac{1}{2}\right)^2 - \left(-\frac{1}{2}\right)^2 - \left(\frac{\sqrt{2}}{2}\right)^2 = 0 - \frac{1}{4} - \frac{1}{4} - \frac{1}{2} = -1$$

For C:

$$a_1^0 a_2^0 - a_1^1 a_2^1 - a_1^2 a_2^2 - a_1^3 a_2^3 = 0$$

$$(\sqrt{2})(0) - \left(\frac{\sqrt{6}}{2}\right)\left(\frac{1}{2}\right) - \left(\frac{\sqrt{6}}{2}\right)\left(-\frac{1}{2}\right) - (0)\left(-\frac{\sqrt{2}}{2}\right) = 0$$

$$a_0^0 a_1^0 - a_0^1 a_1^1 - a_0^2 a_1^2 - a_0^3 a_1^3 = 0$$

$$(\sqrt{3})(\sqrt{2}) - (1)\left(\frac{\sqrt{6}}{2}\right) - (1)\left(\frac{\sqrt{6}}{2}\right) - 0 = 0$$

and similarly:

for $i=0, j=2$:

$$(\sqrt{3})(0) - (1)\left(\frac{1}{2}\right) - (1)\left(-\frac{1}{2}\right) - (0)\left(-\frac{\sqrt{2}}{2}\right) = 0$$

for $i=0, j=3$:

$$(\sqrt{3})(0) - (1)\left(\frac{1}{2}\right) - (1)\left(-\frac{1}{2}\right) - (0)\left(\frac{\sqrt{2}}{2}\right) = 0$$

for $i=1, j=3$:

$$(\sqrt{2})(0) - \left(\frac{\sqrt{6}}{2}\right)\left(\frac{1}{2}\right) - \left(\frac{\sqrt{6}}{2}\right)\left(-\frac{1}{2}\right) - (0)\left(\frac{\sqrt{2}}{2}\right) = 0$$

for $i=2, j=3$:

$$0 - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = -\frac{1}{4} - \frac{1}{4} + \frac{1}{2} = 0$$

Question: Prove that for simple Lorentz transformations

$$ds^2 = d\tilde{s}^2$$

Solution: Consider:

$$\tilde{x}^0 = \frac{x^0 - \beta x^1}{\sqrt{1 - \beta^2}}, \quad \tilde{x}^1 = -\frac{\beta x^0 + x^1}{\sqrt{1 - \beta^2}}, \quad \tilde{x}^2 = x^2, \quad \tilde{x}^3 = x^3$$

also, $ds^2 = g_{ij} dx^i dx^j$

where g_{ij} is Minkowski spacetime.

$$\Rightarrow d\tilde{s}^2 = \tilde{g}_{ij} d\tilde{x}^i d\tilde{x}^j$$

here $g_{ij} = \tilde{g}_{ij}$

$$\Rightarrow d\tilde{s}^2 = g_{00}(d\tilde{x}^0)^2 + g_{11}(d\tilde{x}^1)^2 + g_{22}(d\tilde{x}^2)^2 + g_{33}(d\tilde{x}^3)^2$$

$$d\tilde{s}^2 = (1) \left[d\left(\frac{x^0 - \beta x^1}{\sqrt{1-\beta^2}}\right) \right]^2 - \left[d\left(\frac{-\beta x^0 + x^1}{\sqrt{1-\beta^2}}\right) \right]^2 - (dx^2)^2 - (dx^3)^2$$

$$= \frac{1}{1-\beta^2} \left[(dx^0)^2 + \beta^2 (dx^1)^2 - 2\beta dx^0 dx^1 - \beta^2 (dx^0)^2 - (dx^1)^2 + 2\beta dx^0 dx^1 \right] - (dx^2)^2 - (dx^3)^2$$

$$\Rightarrow (d\tilde{s})^2 = \frac{1}{1-\beta^2} (dx^0)^2 (1-\beta^2) - \frac{1}{1-\beta^2} (dx^1)^2 (1-\beta^2) \\ - (dx^2)^2 - (dx^3)^2$$

$$\Rightarrow (d\tilde{s})^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \\ = (ds)^2$$

Four vectors

In special relativity, we work with a unified entity called spacetime rather than viewing space as an arena with time flowing in the background.

As a result, a vector is going to have a time component in addition to the spatial components we are used to. This is called a four vector.

In general, (V^i) is a four vector if it transforms according to the law $\tilde{V}^i = a_j^i V^j$, where (a_j^i) is the Lorentz matrix.

We usually use the following notation:

$$(V^i) = (V^0, \vec{V}) \quad \text{where}$$

$V^0 = V_t$ is referred to as the time component of the vector and

$\vec{V} = (V^1, V^2, V^3) = (V_x, V_y, V_z)$ are the usual space components.

Four velocity and acceleration:

In the inertial frame $(x^i) = (ct, x, y, z)$,

let a particle describe the class c^2 -curve:

$$(x^i) = (ct, r(t)) = (ct, x(t), y(t), z(t))$$

$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$
 $x^0 \quad x^1 \quad x^2 \quad x^3$

Then we have the formulas:

$$v_i = \frac{dx^i}{dt} = (c, \vec{v})$$

where $\vec{v} = \frac{d\vec{r}}{dt}$ and $\|\vec{v}\| = \sqrt{v_x^2 + v_y^2 + v_z^2}$

$$(a_i) = \left(\frac{d^2 x^i}{dt^2} \right) = (0, \vec{a})$$

where $\vec{a} = \frac{d\vec{v}}{dt}$ and $\hat{a} = \|\vec{a}\| = \sqrt{a_x^2 + a_y^2 + a_z^2}$.

If $(\tilde{v}_i)$ and $(\tilde{a}_i)$ are the like quantities in $(\tilde{x}_i)$, then using the simple Lorentz transformations we have:

$$\begin{aligned} \tilde{v}_0 &= c = v_0, & \tilde{v}_y &= \frac{v_y \sqrt{1 - \beta^2}}{1 - v_x} \\ \tilde{v}_x &= \frac{v_x - v}{1 - v_x v / c^2}, & \tilde{v}_z &= \frac{v_z \sqrt{1 - \beta^2}}{1 - v_x v / c^2} \end{aligned}$$

Derivation of Lorentz transformation of Velocity:

consider the simple Lorentz transformation

$$\tilde{x}^0 = \frac{x^0 - \beta x^1}{\sqrt{1 - \beta^2}}, \quad \tilde{x}^1 = \frac{-\beta x^0 + x^1}{\sqrt{1 - \beta^2}}, \quad \tilde{x}^2 = x^2, \quad \tilde{x}^3 = x^3$$

$$\text{let } \gamma = (1 - \beta^2)^{-1/2} \quad \text{and} \quad x^0 = ct$$

$$\Rightarrow c\tilde{t} = \gamma(ct - \beta x)$$

$$\tilde{x} = \gamma(-\beta ct + x)$$

$$\tilde{y} = y$$

$$\tilde{z} = z$$

Now differentiate the first equation
w.r.t $\tilde{t}$ and use the chain rule:

$$c = \gamma (c - \beta v_x) \frac{dt}{d\tilde{t}} \quad \text{or} \quad \frac{d}{d\tilde{t}} = \frac{d}{dt} \frac{dt}{d\tilde{t}}$$

$$\frac{dt}{d\tilde{t}} = \frac{c}{\gamma (c - \beta v_x)} = \frac{c}{\gamma c (1 - \beta/c v_x)}$$

$$\text{Also } \beta = v/c$$

$$\Rightarrow \frac{dt}{d\tilde{t}} = \frac{1}{\gamma (1 - \frac{v v_x}{c^2})}$$

$$\tilde{x} = \gamma (-\beta c t + x)$$

$$\Rightarrow \frac{d\tilde{x}}{d\tilde{t}} = \frac{d}{dt} \gamma (-\beta c t + x) \frac{dt}{d\tilde{t}}$$

$$= \gamma (-\beta c + v_x) \frac{dt}{d\tilde{t}}$$

$$\Rightarrow \tilde{v}_x = \frac{\gamma (-\beta c + v_x)}{\gamma \left(1 - \frac{v v_x}{c^2}\right)}$$

$$= \frac{v_x - v}{1 - \frac{v v_x}{c^2}}$$

$$(\because \beta c = v)$$

$$\Rightarrow \tilde{V}_n = \frac{V_n - V}{1 - \frac{V V_n}{c^2}} \Rightarrow \tilde{V}_n = \frac{c^2(V_n - V)}{c^2 - V V_n}$$

$$\Rightarrow c^2 \tilde{V}_n - V V_n \tilde{V}_n = c^2 V_n - c^2 V$$

$$\Rightarrow c^2(\tilde{V}_n + V) = V_n(c^2 + V \tilde{V}_n)$$

$$\Rightarrow V_n = \frac{\tilde{V}_n + V}{1 + \frac{V \tilde{V}_n}{c^2}}$$

Now consider

$$\vec{y} = y$$

$$\Rightarrow \frac{d\vec{y}}{d\vec{t}} = \frac{dy}{dt} \cdot \frac{dt}{d\vec{t}} \Rightarrow \tilde{V}_y = v_y \frac{dt}{d\vec{t}}$$

$$\Rightarrow \tilde{V}_y = v_y \frac{1}{\gamma(1 - \frac{vv_x}{c^2})} ; \gamma = (1 - \beta^2)^{-1/2}$$

$$\Rightarrow \tilde{V}_y = \frac{v_y \sqrt{1 - \beta^2}}{1 - \frac{vv_x}{c^2}}$$

and similarly: $\tilde{V}_z = \frac{v_z \sqrt{1 - \beta^2}}{1 - \frac{vv_x}{c^2}}$

Lorentz Transformations of Acceleration:

By differentiating the velocity components we have:

$$\begin{aligned}\tilde{a}_x &= \frac{d\tilde{v}_x}{d\tilde{t}} = \frac{d\tilde{v}_x}{dt} \cdot \frac{dt}{d\tilde{t}} \\&= \frac{(a_x - 0)(1 - v_x v/c^2) - (v_x - v)(0 - \frac{a_x v}{c^2})}{(1 - \frac{v_x v}{c^2})^2} \cdot \frac{1}{\gamma(1 - \frac{v_x v}{c^2})} \\&= \frac{a_x - \cancel{a_x v_x v/c^2} + \cancel{v_x v a_x/c^2} - a_x v^2/c^2}{\gamma(1 - \frac{v_x v}{c^2})^3}\end{aligned}$$

$$\Rightarrow \tilde{a}_x = \frac{a_x (1 - \frac{v^2}{c^2})}{\gamma (1 - \frac{v v_x}{c^2})^3}$$

$$\text{also } \gamma = \frac{1}{\sqrt{1 - \beta^2}} \text{ or } \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\Rightarrow \tilde{a}_x = \frac{a_x (1 - \beta^2)^{3/2}}{(1 - \frac{v v_x}{c^2})^3}$$

Similarly:

$$\tilde{a}_y = \frac{a_y + (a_x v_y - v_x a_y) v/c^2}{(1 - v_x v/c^2)^3} (1 - \beta^2)$$

and

$$\tilde{a}_z = \frac{a_z + (a_x v_z - v_x a_z) v/c^2}{(1 - v_x v/c^2)^3} (1 - \beta^2)$$

Proper Time: $\sqrt{g_{00} \frac{dx^0}{du} \frac{dx^0}{du} + g_{11} \frac{dx^1}{du} \frac{dx^1}{du} + g_{22} \frac{dx^2}{du} \frac{dx^2}{du} + g_{33} \frac{dx^3}{du} \frac{dx^3}{du}}$

Let us reparameterize the curve, choosing the quantity:

$$\sqrt{C^2 - v_x^2 - v_y^2 - v_z^2} = c \sqrt{1 - \hat{v}^2/c^2}$$

$$\tau = \frac{s}{c} = \frac{1}{c} \int_{t_0}^t \sqrt{g_{ij} \frac{dx^i}{du} \frac{dx^j}{du}} du$$

$$\text{or } \frac{d\tau}{dt} = \sqrt{1 - \hat{v}^2/c^2}$$

where $\hat{v} < c$. The new parameter τ is known as the proper time for the particle.

consider:

$$\tau = \frac{1}{c} \int_{t_0}^t \sqrt{g_{ij} \frac{dx^i}{du} \frac{dx^j}{du}} du$$

$$\Rightarrow \frac{d\tau}{dt} = \frac{1}{c} \sqrt{g_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt}} \quad \text{where } g_{ij} \text{ is the Minkowski metric.}$$

$$\begin{aligned} \Rightarrow \frac{d\tau}{dt} &= \frac{1}{c} \left[\left(\frac{dx^0}{dt} \right)^2 - \left(\frac{dx^1}{dt} \right)^2 - \left(\frac{dx^2}{dt} \right)^2 - \left(\frac{dx^3}{dt} \right)^2 \right]^{1/2} \\ &= \frac{1}{c} \left[c^2 - \hat{v}^2 \right]^{1/2} = \left[1 - \frac{\hat{v}^2}{c^2} \right]^{1/2} \end{aligned}$$

Proper Velocity and Acceleration:

When τ -derivatives replace t -derivatives, the corresponding velocity and acceleration become proper velocity and acceleration.

That is we have:

$$u^i = \frac{dx^i}{d\tau}, \quad b^i = \frac{du^i}{d\tau} = \frac{d^2 x^i}{d\tau^2}$$

Using simple Lorentz transformations we have:

$$\tilde{u}^0 = \frac{u^0 - \beta u^1}{\sqrt{1 - \beta^2}}, \quad \tilde{u}^1 = \frac{-\beta u^0 + u^1}{\sqrt{1 - \beta^2}}, \quad \tilde{u}^2 = u^2, \quad \tilde{u}^3 = u^3$$

$$\tilde{x}^0 = \frac{x^0 - \beta x^1}{\sqrt{1 - \beta^2}}$$

and

$$\tilde{b}^0 = \frac{b^0 - \beta b^1}{\sqrt{1 - \beta^2}}, \quad \tilde{b}^1 = \frac{-\beta b^0 + b^1}{\sqrt{1 - \beta^2}}, \quad \tilde{b}^2 = b^2, \quad \tilde{b}^3 = b^3$$

$$\text{Also: } u_i u^i = c^2$$

Question: Derive $u_i u^i = c^2$.

Solution:

consider: $u_i u^i = g_{ij} u^i u^j$

$$= (u^0)^2 - (u^1)^2 - (u^2)^2 - (u^3)^2$$

$$= [(v_t)^2 - (v_x)^2 - (v_y)^2 - (v_z)^2] \left(\frac{dt}{d\tau} \right)^2$$

$$= [c^2 - \hat{v}^2] \frac{1}{1 - \hat{v}^2/c^2} = c^2$$

Question: Find the expressions

$$u^i = \frac{dx^i}{d\tau} \quad \text{and} \quad b^i = \frac{du^i}{d\tau}$$

Solution: $u^i = \frac{dx^i}{d\tau} = \frac{dx^i}{dt} \cdot \frac{dt}{d\tau}$

$$= \frac{v_i}{\sqrt{1 - \hat{v}^2/c^2}}$$

$$b^i = \frac{du^i}{d\tau} = \frac{d}{dt} \left(\frac{v_i}{\sqrt{1 - \hat{v}^2/c^2}} \right) \cdot \frac{dt}{d\tau}$$

$$= \frac{a_i}{1 - \hat{v}^2/c^2} + \frac{(\vec{v} \vec{a}) v_i}{c^2 (1 - \hat{v}^2/c^2)^2}$$

Instantaneous Rest Frame:

An instantaneous rest frame for the particle is an inertial frame that translates in such a manner that its origin coincides with P_1 at $t = t_1$, where $P_1 = \vec{r}(t_1)$ is instantaneous position of the particle at $t = t_1$.

* The particle's motion (with respect to frame O) is uniformly accelerated if its spatial acceleration relative to an instantaneous rest frame $\tilde{O}$ does not vary, i.e.,

$$\hat{a} = \sqrt{a_x^2 + a_y^2 + a_z^2} = \alpha$$

Relativistic Mass and Force:

1) Rest Mass and Relativistic Mass:

The rest mass of a particle is its mass as measured from Newtonian mechanics in any instantaneous rest frame for that particle.

The relativistic mass of a particle with spatial velocity $\vec{v}$ is:

$$\hat{m} = \frac{m}{\sqrt{1 - \hat{v}^2/c^2}}$$

where m is the rest mass of the particle and $\hat{m}$ is relativistic mass.

Relativistic Momentum and Force:

The 4-momentum of special Relativity is defined by:

$$(P^i) = (P^0, \vec{P}) = (\hat{m} v_i)$$

and the Lorentz force $(F_0, \vec{F})$ is defined as the time derivative of the 4-momentum:

$$F_0 = \frac{dP^0}{dt} = \frac{d}{dt} (\hat{m} v_0) = \frac{d}{dt} \left(\frac{m c}{\sqrt{1 - \hat{v}^2/c^2}} \right)$$

$$\vec{F} = \frac{d}{dt} \left(\frac{m \vec{v}}{\sqrt{1 - \hat{v}^2/c^2}} \right)$$

In terms of proper time τ . The 4-force also called Minkowski force is defined as:

$$(K^i) = \left(\frac{dP^i}{d\tau} \right)$$

$$\Rightarrow K^i = \frac{dP^i}{dt} \cdot \frac{dt}{d\tau} = F_i \frac{dt}{d\tau} = \frac{F_i}{\sqrt{1 - \hat{v}^2/c^2}}$$

The Riemann Tensor:

The Riemann tensor emerges from an analysis of a simple question. Starting with a covariant vector (V_i) and taking the covariant derivative with respect to x^j and then with respect to x^k produces the third order tensor $((V_i)_{;j})_{;k} = V_{i;jk}$

The Question is:

Does the order of differentiation matter?

OR does $V_{i;jk} = V_{i;kj}$ hold in general?

By definition of the covariant derivative:

$$V_{i;jk} = (V_{i;j})_{;k} = \frac{\partial}{\partial x^k} (V_{i;j}) - \Gamma_{ik}^{\lambda} (V_{\lambda;j}) - \Gamma_{jk}^{\lambda} (V_{i;\lambda})$$

$$\text{Also, } V_{i;j} = \frac{\partial V_i}{\partial x^j} - \Gamma_{ij}^s V_s$$

$$\begin{aligned} \Rightarrow V_{i;jk} &= \frac{\partial^2 V_i}{\partial x^k \partial x^j} - \frac{\partial \Gamma_{ij}^s}{\partial x^k} V_s - \Gamma_{ij}^s \frac{\partial V_s}{\partial x^k} - \Gamma_{ik}^{\lambda} \frac{\partial V_{\lambda}}{\partial x^j} \\ &\quad + \Gamma_{ik}^{\lambda} \Gamma_{\lambda j}^s V_s - \Gamma_{jk}^{\lambda} \frac{\partial V_i}{\partial x^{\lambda}} + \Gamma_{jk}^{\lambda} \Gamma_{i\lambda}^s V_s \end{aligned}$$

$\hookrightarrow (A)$

Interchanging j and k yields:

$$V_{i;jk} = \frac{\partial^2 V_i}{\partial x^j \partial x^k} - \frac{\partial \Gamma_{ik}^s}{\partial x^j} V_s - \Gamma_{ik}^s \frac{\partial V_s}{\partial x^k} - \Gamma_{ij}^r \frac{\partial V_r}{\partial x^k} \\ + \Gamma_{ij}^r \Gamma_{rk}^s V_s - \Gamma_{kj}^r \frac{\partial V_i}{\partial x^r} + \Gamma_{kj}^r \Gamma_{ir}^s V_s \rightarrow (B)$$

Subtracting (B) from (A), we have:

$$V_{i;jk} - V_{i;kj} = -\frac{\partial \Gamma_{ij}^s}{\partial x^k} V_s + \Gamma_{ik}^r \Gamma_{rj}^s V_s + \\ \frac{\partial \Gamma_{ik}^s}{\partial x^j} V_s - \Gamma_{ij}^r \Gamma_{rk}^s V_s$$

$$\Rightarrow V_{i;jk} - V_{i;kj} = \left[\frac{\partial \Gamma_{ik}^s}{\partial x^j} - \frac{\partial \Gamma_{ij}^s}{\partial x^k} + \Gamma_{ik}^l \Gamma_{lj}^s - \Gamma_{ij}^l \Gamma_{lk}^s \right] V_s$$

$$V_{i;jk} - V_{i;kj} = R_{ijk}^s V_s \longrightarrow (*)$$

$$\text{where } R_{ijk}^s = \frac{\partial \Gamma_{ik}^s}{\partial x^j} - \frac{\partial \Gamma_{ij}^s}{\partial x^k} + \Gamma_{ik}^l \Gamma_{lj}^s - \Gamma_{ij}^l \Gamma_{lk}^s$$

The Quotient theorem immediately implies:

"The n^4 components defined by Equation (*) are those of a fourth-order tensor, contravariant of order one and covariant of order three."

R^i_{jkl} is called the Riemann or

Riemann-Christoffel tensor of the second kind. Lowering the contravariant index gives

$$R_{ijkl} = g_{in} R^n_{jkl}$$

is the Riemann tensor of first kind.

Note: We may now say that covariant differentiation is order-dependent unless the metric is such as to make Riemann tensor vanish.

Question: Show that at any point where the Christoffel symbols vanish,

$$R^i_{jkl} + R^i_{klj} + R^i_{ljk} = 0$$

Solution: Given that the Christoffel symbols vanish, the expression for Riemann tensor becomes:

$$R^i_{jkl} = \frac{\partial \Gamma^i_{jl}}{\partial x^k} - \frac{\partial \Gamma^i_{jk}}{\partial x^l},$$

$$R^i_{klj} = \frac{\partial \Gamma^i_{kj}}{\partial x^l} - \frac{\partial \Gamma^i_{kl}}{\partial x^j}$$

$$R^i_{ljk} = \frac{\partial \Gamma^i_{lk}}{\partial x^j} - \frac{\partial \Gamma^i_{lj}}{\partial x^k}$$

and adding all these:

$$\begin{aligned} R^i_{jkl} + R^i_{klj} + R^i_{ljk} &= \frac{\partial \Gamma^i_{jl}}{\partial x^k} - \frac{\partial \Gamma^i_{jk}}{\partial x^l} + \frac{\partial \Gamma^i_{kj}}{\partial x^l} - \frac{\partial \Gamma^i_{kl}}{\partial x^j} \\ &\quad + \frac{\partial \Gamma^i_{lk}}{\partial x^j} - \frac{\partial \Gamma^i_{lj}}{\partial x^k} \\ &= 0 \end{aligned}$$

because christoffel symbols are symmetric
w.r.t lower indices.

Properties of the Riemann Tensor:

Interchange of k and l in the definition of R^i_{jkl} shows that

$$R^i_{jkl} = -R^i_{jlk}$$

where:

$$R^i_{jkl} = \frac{\partial \Gamma^i_{jl}}{\partial x^k} - \frac{\partial \Gamma^i_{jk}}{\partial x^l} + \Gamma^{\lambda}_{jl} \Gamma^i_{\lambda k} - \Gamma^{\lambda}_{jk} \Gamma^i_{\lambda l}$$

and

$$R^i_{jlk} = \frac{\partial \Gamma^i_{jk}}{\partial x^l} - \frac{\partial \Gamma^i_{jl}}{\partial x^k} + \Gamma^{\lambda}_{jk} \Gamma^i_{\lambda l} - \Gamma^{\lambda}_{jl} \Gamma^i_{\lambda k}$$

also $R_{ijkl} = -R_{ijlk}$

Below are the symmetry properties of Riemann tensor:

1) $R^i_{jkl} = -R^i_{jlk}$

2) $R_{ijkl} = -R_{jikl}$ (first skew symmetry)

3) $R_{ijkl} = -R_{ijlk}$ (second skew symmetry)

4) $R_{ijkl} = R_{klji}$

5) $R_{ijkl} + R_{iklj} + R_{iljk} = 0$ (Bianchi's Identity)

Number of Independent Components:

We count the separate types of potentially non-zero components, using the above symmetry properties.

The first two properties imply that R_{aacd} and R_{abcc} (not summed on a or c) are zero. In the following list, the repeated indices are not summed over.

$$R_{||cd} = -R_{||cd} \Rightarrow R_{||cd} = 0$$

(A) Type $R_{abab}, a < b:$ → $\begin{matrix} 1212 \\ 1313 \\ 1414 \end{matrix}$

$$n_A = n_{C_2} = \frac{n(n-1)}{2!}$$

(B) Type $R_{abac}, b < c:$ → $\begin{matrix} R_{1213} \\ 1314 \end{matrix}$

$$n_B = 3 n_{C_3} = \frac{n(n-1)(n-2)}{2}$$

(C) Type R_{abcd} or R_{acbd} , all different indices $a < b < c < d$

$$n_C = 2 n_{C_4} = \frac{n(n-1)(n-2)(n-3)}{12}$$

Theorem: There are a total of $\frac{n^2(n^2-1)}{12}$ components of the Riemann tensor R_{ijkl} that are not identically zero and that are independent.

Proof: The symmetries of the Riemann tensor imply that

$$n_A = \frac{n(n-1)}{2}, n_B = \frac{n(n-1)(n-2)}{2}$$

$$\text{and } n_C = \frac{n(n-1)(n-2)(n-3)}{12}$$

now adding all these

No of non-zero independent components of the

$$\text{Riemann tensor} = n_A + n_B + n_C$$

$$= \frac{n(n-1)}{2} \left[1 + (n-2) + \frac{(n-2)(n-3)}{6} \right]$$

$$= \frac{n(n-1)}{2} \left[\frac{6(n-1) + (n-2)(n-3)}{6} \right]$$

$$= \frac{n(n-1)}{12} [6n - \cancel{6} + n^2 - 5n + \cancel{6}]$$

$$= \frac{n(n-1)(n^2+n)}{12} = \frac{n^2(n^2-1)}{12}$$

Corollary:

In a two dimensional space, the only components of the Riemann tensor not identically zero are

$$R_{1212} = R_{2121} = -R_{1221} = -R_{2112}$$

Question: List the independent, potentially nonzero components of R_{ijkl} for $n=5$.

Solution:

Type A: R_{1212} , R_{1313} , R_{1414} , R_{1515} ,

R_{2323} , R_{2424} , R_{2525}

R_{3434} , R_{3535} ,

R_{4545}

Type B: $R_{1213}, R_{1214}, R_{1215}, R_{1314}, R_{1315}, R_{1415}$

$R_{2123}, R_{2124}, R_{2125}, R_{2324}, R_{2325}, R_{2425}$

$R_{3132}, R_{3134}, R_{3135}, R_{3234}, R_{3235}, R_{3435}$

$R_{4142}, R_{4143}, R_{4145}, R_{4243}, R_{4245}, R_{4345}$

$R_{5152}, R_{5153}, R_{5154}, R_{5253}, R_{5254}, R_{5354}$

Type C: $R_{1234}, R_{1235}, R_{1245}, R_{1345}, R_{2345}$

$R_{1324}, R_{1325}, R_{1425}, R_{1435}, R_{2435}$

Implies:

Type A: 10 components

Type B: 30 components

Type C: 10 components

$\Rightarrow$ Total 50 components

and also
$$\frac{n^2(n^2-1)}{12} = \frac{25(24)}{12} = 50$$

Question: Compute the components of the Riemann tensor for a unit 2-sphere, where

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2$$

and the non-zero christoffel symbols are:

$$\Gamma_{22}^1 = -\sin\theta \cos\theta \quad \left| \begin{array}{l} x^1, x^2, x^3 \rightarrow r, \theta, \phi \\ \text{but } r=1 \end{array} \right.$$

$$\Gamma_{21}^{12} = \Gamma_{12}^{21} = \cot\theta \quad \Rightarrow x^1, x^2 \rightarrow \theta, \phi$$

Solution:

Using the fact that the non-zero components of the Riemann tensor in two dimensions

$$\text{are given by } R_{1212} = R_{2121} = -R_{1221} = -R_{2112}$$

$$R'_{212} = \frac{\partial \Gamma'_{22}}{\partial x'} - \frac{\partial \Gamma'_{21}}{\partial x^2} + \Gamma'_{22} \Gamma'_{x1} - \Gamma'_{21} \Gamma'_{x2}$$

$$= \frac{\partial}{\partial \theta} (-\sin \theta \cos \theta) - 0 + \Gamma'_{22} \Gamma'_{11} + \Gamma'_{22} \Gamma'_{21}$$

$$- \Gamma'_{21} \Gamma'_{12} - \Gamma'_{21} \Gamma'_{22}$$

$$= -\cos^2 \theta + \sin^2 \theta - \cot \theta (-\sin \theta \cos \theta)$$

$$= \sin^2 \theta - \cos^2 \theta + \cos^2 \theta$$

$$= \sin^2 \theta$$

Question: For the metric of spherical coordinates,

$$ds^2 = (dx^1)^2 + (x^1 dx^2)^2 + (x^1 \sin x^2 dx^3)^2$$

list and calculate the non-zero components R_{ijkl} , if any.

Solution: There are $\frac{n^2(n^2-1)}{12} = \frac{9(8)}{12} = 6$

$\Rightarrow$ There are six non zero components:

A) R_{1212} , R_{1313} , R_{2323}

B) R_{1213} , R_{1232} , R_{1323}

As $R_{ijkl} = g_{ii} R^i_{jkl}$, So we find the mixed components.

The nonzero Christoffel symbols are:

$$\Gamma^1_{22} = -r, \quad \Gamma^1_{33} = -r \sin^2 \theta$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r}, \quad \Gamma^2_{33} = -\sin \theta \cos \theta$$

$$\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}, \quad \Gamma^3_{23} = \Gamma^3_{32} = \cot \theta$$

$$R'_{212} = \frac{\partial \Gamma'_{22}}{\partial x^1} - \frac{\partial \Gamma'_{21}}{\partial x^2} + \Gamma'_{22} \Gamma'_{21} - \Gamma'_{21} \Gamma'_{22}$$

$$= \frac{\partial}{\partial x} (-x) - \Gamma'_{21} \Gamma'_{22} = (-1) - \frac{1}{x} (-x)$$

$$R'_{212} = -1 + 1 = 0$$

$$R'_{313} = \frac{\partial \Gamma'_{33}}{\partial x^1} - \frac{\partial \Gamma'_{31}}{\partial x^1} + \Gamma'_{33} \Gamma'_{21} - \Gamma'_{31} \Gamma'_{23}$$

$$= -\sin^2 \theta - \Gamma'_{13} \Gamma'_{33}$$

$$= -\sin^2 \theta - \left(\frac{1}{x}\right) (-x \sin^2 \theta) = 0$$

Similarly:

$$R_{323}^2 = 0$$

$$R_{213}^1 = 0$$

$$R_{232}^1 = 0$$

$$R_{323}^1 = 0$$

Therefore $R_{ijkl} = 0$ for all i, j, k, l .

The Ricci Tensor and Ricci Scalar:

The Riemann tensor can be used to derive two more quantities that are used to define the Einstein tensor. The first of these is the Ricci tensor, which is calculated from the Riemann tensor by contraction on the first and third indices.

$$R_{ab} = R^c_{acb}$$

The Ricci tensor is symmetric, so

$$R_{ab} = R_{ba}$$

Using contraction on the Ricci tensor,
we obtain the Ricci scalar:

$$R = g^{ab} R_{ab} = R^a_a$$

Question: Show that $R=2$ for the unit 2-Sphere. Given
 $R'_{212} = \sin^2 \theta$, $R_{1212} = \sin^2 \theta$

Solution:

The Symmetry conditions of the Riemann tensor implies that:

$$R_{1212} = R_{2121} = -R_{1221} = -R_{2112}$$

The metric of 2-sphere is:

$$ds^2 = d\theta^2 + \sin^2 \theta d\phi^2$$

So,

$$R^2_{121} = g^{22} R_{2121} = \frac{1}{\sin^2 \theta} \sin^2 \theta = 1$$

$$R^1_{221} = g^{11} R_{1221} = -\sin^2 \theta$$

$$R^2_{112} = g^{22} R_{2112} = \left(\frac{1}{\sin^2 \theta} \right) (-\sin^2 \theta) = -1$$

The components of the Ricci tensor are given by:

$$R_{11} = R^{\alpha}_{1\alpha 1} = R^1_{111} + R^2_{121} = 0 + 1 = 1$$

$$R_{22} = R^{\alpha}_{2\alpha 2} = R^1_{212} + R^2_{222} = \sin^2 \theta$$

$$R_{12} = R^{\alpha}_{1\alpha 2} = R^1_{112} + R^2_{122} = 0$$

$$\Rightarrow R_{12} = R_{21} = 0$$

$$\text{So, } R = g^{\alpha\beta} R_{\alpha\beta} = g^{11} R_{11} + g^{22} R_{22}$$

$$R = 1 + \frac{1}{\sin^2 \theta} \sin^2 \theta = 1 + 1 = 2$$

Question: For the metric

$$ds^2 = (dx^1)^2 + 2x^1(dx^2)^2 + 2x^2(dx^3)^2$$

calculate

R_{ij} , R , Given:

$$R^1_{212} = \frac{1}{2x^1}, \quad R^2_{323} = \frac{1}{4x^1x^2}, \quad R^3_{132} = \frac{1}{4x^1x^2}$$

$$R_{1212} = \frac{1}{2x^1}, \quad R_{2323} = \frac{1}{2x^2}, \quad R_{3132} = \frac{1}{2x^1}$$

Solution: Using the formula:

$$R_{ij} = R^k_{ikj} = R^1_{i1j} + R^2_{i2j} + R^3_{i3j}$$

$$= g^{11} R_{1i1j} + g^{22} R_{2i2j} + g^{33} R_{3i3j}$$

Implies:

$$R_{11} = g^{11} R_{1111} + g^{22} R_{2121} + g^{33} R_{3131}$$

$$= g^{22} R_{2121}$$

$$\text{where } g^{22} = \frac{1}{2x'} \quad \text{and} \quad R_{2121} = \frac{1}{2x'}$$

$$\Rightarrow R_{11} = \frac{1}{2x'} \left(\frac{1}{2x'} \right) = \frac{1}{4(x')^2}$$

$$R_{22} = g^{11} R_{1212} + g^{33} R_{3232}$$

$$= (1) \left(\frac{1}{2x'} \right) + \frac{1}{2x^2} \left(\frac{1}{2x^2} \right)$$

$$= \frac{1}{2x'} + \frac{1}{4(x^2)^2}$$

$$R_{33} = g^{22} R_{2323} = \frac{1}{2x'} \left(\frac{1}{2x^2} \right)$$

$$= \frac{1}{4x'x^2}$$

$$R_{12} = g^{33} R_{3132} = \frac{1}{2x^2} \left(\frac{1}{2x'} \right)$$

$$R_{12} = \frac{1}{4x'x^2} = R_{21}$$

Hence

$$\begin{aligned} R &= g^{11} R_{11} + g^{22} R_{22} + g^{33} R_{33} \\ &= (1) \left(\frac{1}{4(x')^2} \right) + \frac{1}{2x'} \left(\frac{1}{2x'} + \frac{1}{4(x^2)^2} \right) \\ &\quad + \frac{1}{2x^2} \left(\frac{1}{4x'x^2} \right) \end{aligned}$$

$$\Rightarrow R = \frac{x' + 2(x^2)^2}{(2x'x^2)^2}$$

Physical Interpretation:

Curvature describes how a geometric object such as a curve deviates from a straight line or a surface from a flat plane.

Curvature can be expressed simply as a scalar that represents the magnitude of this deviation. Curvature can also be described as a vector that takes into account the direction of the curve along with the magnitude.

For more complex objects such as surfaces or n-dimensional spaces, a more complex object is needed to describe the curvature. **One such object, is the Riemann curvature tensor.**

Intrinsic objects are those which can be obtained by measuring distances and computing the derivatives of those distances in some space.

The **metric tensor** is an example of an intrinsic object. The metric tensor describes how to compute distances and lengths of curves in a given space.

Riemann generalized this idea and extended it to spaces of n -dimensions. Because the metric tensor is an intrinsic object, subsequent objects that can be described in terms of the metric tensor and its derivatives are also intrinsic.

One object that can be derived from the metric tensor is the Christoffel symbol. The Christoffel symbol describes the variation in basis vectors from one point to another in curvilinear coordinate systems. The Christoffel symbols measure the rate of change of the covariant basis with respect to the coordinate variables.

The Riemann tensor, Ricci tensor, and Ricci scalar are all derived from the metric tensor and are therefore intrinsic measures of curvature.

A **manifold** is a space that can be curved but is locally flat. That is, near each point the space resembles Euclidean space. One example of a curved manifold is the surface of a sphere.

The **Riemann tensor** is the most common tool used to describe the curvature of a Riemannian manifold.

The Riemann curvature tensor is a tool used to describe the curvature of n -dimensional spaces such as Riemannian manifolds in the field of differential geometry. The Riemann tensor plays an important role in the theories of general relativity and gravity as well as the curvature of spacetime.

Contraction, of the Riemann tensor produces the **Ricci tensor**. The Ricci tensor provides a way measure the degree to which a space differs from Euclidean space.

Contraction of the Ricci tensor produces the scalar curvature or **Ricci scalar**. The Ricci scalar is the simplest curvature invariant of a manifold.

Riemannian Curvature:

The Riemannian or sectional curvature relative to a given metric (g_{ij}) , a Riemannian manifold and two linearly independent tangent vectors at the same point, (U^i) and (V^i) is defined as:

$$K = K(x^i; u^i, v^i) = \frac{R_{ijkl} u^i v^j u^k v^l}{G_{pqrs} u^p v^q u^r v^s}$$

where,

$$G_{pqrs} = g_{pr} g_{qs} - g_{ps} g_{qr}$$

This curvature can be associated to Gaussian curvature of the surface.

This sort of curvature depends not only on position, but also on a pair of directions selected at each point.

By contrast, the curvature K of a curve depends only on the points along the curve.

In the calculations of K it is helpful that G_{ijkl} possess exactly the same symmetries as the R_{ijkl} .

$R_{ijkl} U^i V^j U^k V^l \rightarrow$ is an invariant

$G_{pqrs} U^p V^q U^r V^s \rightarrow$ is also an invariant

It follows that K is an invariant
and it serves to generalize the
Gaussian curvature of a surface
to higher dimensions.

Question: Using the fact that in a two-dimensional space the only non-zero components of Riemann tensor are

$$R_{1212} = R_{2121} = -R_{1221} = -R_{2112}$$

simplify the expression of Riemannian Curvature.

Solution:

$$K = \underline{R_{ijkl} U^i V^j U^k V^l} = R_{1212} U^1 V^2 U^1 V^2 + R_{2121} U^2 V^1 U^2 V^1 - \check{R}_{1221} \underline{U^1 V^2 U^2 V^1} - \check{R}_{2112} \underline{U^2 V^1 U^1 V^2}$$

$$G_{pqrs} U^p V^q U^r V^s$$

$$= R_{1212} [(U^1)^2 (V^2)^2 - \underbrace{2 U^1 V^2 U^2 V^1}_{\text{red bracket}} + (U^2)^2 (V^1)^2]$$

$$G_{1212} [(U^1)^2 (V^2)^2 - 2 U^1 V^2 U^2 V^1 + (U^2)^2 (V^1)^2]$$

$$= \frac{R_{1212}}{G_{1212}} = \frac{R_{1212}}{g_{11} g_{22} - g_{12}^2}$$

Example: Calculate K for the metric

$$ds^2 = (x^1)^{-2} (dx^1)^2 - (x^1)^{-2} (dx^2)^2, \text{ using}$$

the result

$$K = \frac{R_{1212}}{g_{11} g_{22} - g_{12}^2}$$

and $\Gamma_{11}^1 = -\frac{1}{x^1}$, $\Gamma_{22}^1 = -\frac{1}{x^1}$

$$\Gamma_{12}^2 = \Gamma_{21}^2 = -\frac{1}{x^1}$$

Solution:

$$R'_{212} = \frac{\partial \Gamma'_{22}}{\partial x'} - \frac{\partial \Gamma'_{21}}{\partial x^2} + \Gamma'_{22} \Gamma'_{21} - \Gamma'_{21} \Gamma'_{22}$$

$$= \frac{\partial}{\partial x'} \left(-\frac{1}{x'} \right) - \Gamma'_{21} \Gamma'_{22} + \Gamma'_{22} \Gamma'_{11}$$

$$= \frac{1}{(x')^2} - \left(-\frac{1}{x'} \right) \left(-\frac{1}{x'} \right) - \frac{1}{x'} \left(-\frac{1}{x'} \right)$$

$$= \frac{1}{(x')^2}$$

$$\Rightarrow R_{1212} = -g_{11} R'_{212} = (x')^{-2} (x')^{-2} = -(x')^{-4}$$

Now,

$$K = \frac{R_{1212}}{g_{11}g_{22} - g_{12}^2}$$
$$= \frac{-\cancel{\left(\frac{1}{x}\right)^2} \left(\cancel{\frac{1}{x}}\right)^2}{-\left(\cancel{\frac{1}{x}}\right)^2 \left(\cancel{\frac{1}{x}}\right)^2 - 0}$$

$$K = -1$$

Riemannian Space: A Riemannian space is the space R^n coordinatized by (x^i) , together with a fundamental form or Riemannian metric, $g_{ij} dx^i dx^j$, where $g = (g_{ij})$ obeys conditions for a metric.

Zero Curvature and the Euclidean metric:

$$ds^2 = dr^2 + r^2 d\theta^2, \quad ds^2 = dx^2 + dy^2 \quad \left| \begin{array}{l} x^1, x^2 \rightarrow \tilde{x}^1, \tilde{x}^2 \\ r, \theta \rightarrow x, y \end{array} \right.$$

A Riemannian metric $g = (g_{ij})$, specified in a coordinate system (x^i) , is the Euclidean metric if, under some permissible coordinate transformation, $(\tilde{g}_{ij}) = \delta_{ij}$.

Now a coordinate system $(\tilde{x}^i)$ in which $(\tilde{g}_{ij}) = \delta_{ij}$ is a rectangular coordinate system. Then $\tilde{K} = 0$. Since all Christoffel

Symbols vanish in $(\tilde{x}^i)$.

But Riemannian curvature is an invariant, so that $K=0$ in the original coordinates (x^i) as well. Hence

Theorem:

"A Riemannian metric (g_{ij}) is the Euclidean metric if and only if the Riemannian Curvature K is zero at all points."

Proof: Here the
necessity part of the statement is:
If the metric is Euclidean then this implies

$$K = 0, \text{ and}$$

the sufficiency portion is if

$K = 0$ then the metric is Euclidean metric.

To prove the first part consider that the
metric is Euclidean then there exist
a transformation b/w (x^i) system and

the rectangular coordinate system $(\tilde{x}^i)$.

Also, all christoffel symbols vanish in rectangular coordinates implying $\tilde{K}=0$, but $\tilde{K}$ is an invariant $\Rightarrow K=0$.

To prove the Sufficiency part consider $K=0$. Also to prove the corresponding metric is Euclidean we establish a system of first order PDEs b/w (x^i) and $(\tilde{x}^i)$ system.

Here we set up a system of first order partial differential equations for $(\tilde{x}^i)$ as a function of (x^j) .

So consider, $G = J^T J$

$$\text{or } \frac{\partial \tilde{x}^k}{\partial x^i} \frac{\partial x^k}{\partial \tilde{x}^j} = g_{ij}(x^i)$$

also the transformation law of Γ_{jk}^i

$$\tilde{\Gamma}_{jk}^i = \Gamma_{st}^i \frac{\partial \tilde{x}^i}{\partial x^s} \frac{\partial x^s}{\partial \tilde{x}^j} \frac{\partial x^t}{\partial \tilde{x}^k} + \frac{\partial^2 x^i}{\partial \tilde{x}^j \partial \tilde{x}^k} \frac{\partial \tilde{x}^i}{\partial x^i}$$

Interchanging tilde and non-tilde
Coordinates and replacing $\tilde{\Gamma}_{jk}^i = 0$

$$\Rightarrow \Gamma_{jk}^i = \frac{\partial^2 x^i}{\partial \tilde{x}^j \partial \tilde{x}^k} \frac{\partial \tilde{x}^i}{\partial x^i}$$

$$\Rightarrow \frac{\partial^2 x^i}{\partial \tilde{x}^j \partial \tilde{x}^k} \frac{\partial \tilde{x}^i}{\partial x^i} = \Gamma_{jk}^i$$

$$\Rightarrow \frac{\partial^2 x^i}{\partial \tilde{x}^j \partial \tilde{x}^k} = \Gamma_{jk}^i \frac{\partial \tilde{x}^i}{\partial x^i}$$

$$\text{let } w = \tilde{x}^i \text{ and } u_i = \frac{\partial \tilde{x}^i}{\partial x^i}$$

$$\Rightarrow \frac{\partial w}{\partial x^i} = u_i$$

$$\frac{\partial u_i}{\partial x^j} = \Gamma_{ij}^k u_k$$

Flat Riemannian Spaces:

A Riemannian Space is flat if and only if $K=0$ at all points.

Theorem: If $K=0$, then $R=0$.

Proof: If $K=0$, then the space is flat and hence the $\tilde{g}_{ij}$ are constant for some coordinate system $(\tilde{x}^i)$. It follows that all $\tilde{R}^i_{jk}$, $\tilde{R}^i_{jkl}$, $\tilde{R}_{ij}$ vanish and hence $\tilde{R} = \tilde{R}^i_i = 0$.

and since the Ricci curvature is invariant,
 $R = 0$.

★ The converse of this theorem does not hold i.e., if $R = 0$ then it does not implies $K = 0$.

Question: Show that $\mathbb{R}^3$ under the metric

$$ds^2 = [(x^1)^2 + (x^2)^2] (dx^1)^2 + [(x^1)^2 + (x^2)^2] (dx^2)^2 + (dx^3)^2$$

is Euclidean. Given that $\Gamma_{jk}^i = 0$

whenever i, j or k equals to 3, and

$$\Gamma_{11}^1 = \frac{x^1}{w}, \quad \Gamma_{12}^1 = \Gamma_{21}^1 = \frac{x^2}{w}, \quad \Gamma_{22}^1 = -\frac{x^1}{w}$$

$$\Gamma_{11}^2 = -\frac{x^2}{w}, \quad \Gamma_{21}^2 = \Gamma_{12}^2 = \frac{x^1}{w}, \quad \Gamma_{22}^2 = \frac{x^2}{w}$$

$$\text{where } w = (x^1)^2 + (x^2)^2$$

Solution: The six independent components of the Riemann tensor are:

$$R_{1212}, R_{2323}, R_{1313}, R_{1213}, R_{2123}, R_{3132}$$

All components of the Riemann tensor involving index 3 goes to zero. So the only possibly non zero component is

$$R_{1212}.$$

So,

$$R'_{212} = \frac{\partial \Gamma'_{22}}{\partial x^1} - \frac{\partial \Gamma'_{21}}{\partial x^2} + \Gamma'_{22} \Gamma'_{11} + \Gamma'^2_{22} \Gamma'_{21}$$

$$- \Gamma'_{21} \Gamma'_{12} - \Gamma'^2_{21} \Gamma'_{22}$$

$$\Rightarrow R'_{212} = -\frac{\omega + x^1(2x^1)}{\omega^2} - \frac{\omega - x^2(2x^2)}{\omega^2} + \left(-\frac{x^1}{\omega}\right)\left(\frac{x^1}{\omega}\right)$$

$$+ \frac{x^2}{\omega} \left(\frac{x^2}{\omega}\right) - \frac{x^2}{\omega} \left(\frac{x^2}{\omega}\right) - \frac{x^1}{\omega} \left(-\frac{x^1}{\omega}\right)$$

$$= 0$$

As all the components of Riemann tensor are zero, we have

$$K = 0$$

$\Rightarrow$ The metric is Euclidean.

Question: Verify that

$$ds^2 = (dx^1)^2 + (x^1)^2 (dx^2)^2$$

represent the Euclidean metric.

Solution: This is actually polar metric:

$$ds^2 = dr^2 + r^2 d\theta^2 \quad (x^1, x^2 \rightarrow r, \theta)$$

with christoffel symbols:

$$\Gamma_{22}^1 = -x^1, \quad \Gamma_{12}^2 = \Gamma_{21}^2 = \frac{1}{x^1} \quad (\text{remaining all zero})$$

The independent components of Riemann tensor are: $\frac{n^2(n^2-1)}{12} = \frac{4(3)}{12} = 1$

i.e., R_{1212}

$$R'_{212} = \frac{\partial \Gamma'_{22}}{\partial x^1} - \frac{\partial \Gamma'_{21}}{\partial x^2} + \Gamma'_{22} \Gamma'_{11} - \Gamma'_{21} \Gamma'_{12}$$

$$= \frac{\partial}{\partial x^1}(-x^1) - \frac{\partial}{\partial x^2}(0) + 0 - \Gamma'_{21} \Gamma'_{22}$$

$$= -1 - (-x^1) \left(\frac{1}{x^1} \right) = -1 + 1$$

$$R'_{212} = 0$$

$$\Rightarrow K = 0$$

$\Rightarrow$ metric is Euclidean.

Question: Determine whether the following metric is Euclidean or not.

$$ds^2 = (dx^1)^2 + (x^3)^2 (dx^2)^2 + (dx^3)^2$$

Solution: First we will find the nonzero Christoffel symbols which are:

$$\Gamma_{23}^{12} = \Gamma_{32}^{12} = \frac{1}{x^3}, \quad \Gamma_{22}^{13} = -x^3$$

The independent components of the Riemann tensor are:

$$R_{1212}, R_{1313}, R_{2323}, R_{1312}, R_{1323}, R_{2321}$$

the only possibly non zero component is R_{2323} .

$$R^2_{323} = \frac{\partial \Gamma^1_{33}}{\partial x^2} - \frac{\partial \Gamma^1_{32}}{\partial x^3} + \Gamma^1_{33} \Gamma^2_{32} - \Gamma^1_{32} \Gamma^2_{33}$$

$$= -\left(-\frac{1}{(x^3)^2}\right) - \left(\frac{1}{x^3}\right)\left(\frac{1}{x^3}\right) = 0$$

$\Rightarrow K=0 \Rightarrow$ metric is Euclidean.

Question: Consider the Riemannian metric

$$ds^2 = (dx^1)^2 + 4(x^2)^2(dx^2)^2 + 4(x^3)^2(dx^3)^2 - 4(x^4)^2(dx^4)^2$$

a) Calculate the Riemannian Curvature

b) Find a solution of $\frac{\partial u_i}{\partial x^j} = \Gamma_{ij}^k u_k$, $\frac{\partial w}{\partial x^i} = u_i$

where,

$$\Gamma_{22}^2 = \frac{1}{x^2}, \quad \Gamma_{33}^3 = \frac{1}{x^3}, \quad \Gamma_{44}^4 = \frac{1}{x^4}$$

Solution: $K = \frac{R_{ijkl} U^i V^j U^k V^l}{G_{pqrs} U^p V^q U^r V^s}$

$$R^i_{jkl} = \frac{\partial \Gamma^i_{jl}}{\partial x^k} - \frac{\partial \Gamma^i_{jk}}{\partial x^l} + \Gamma^r_{jl} \Gamma^i_{rk} - \Gamma^r_{jk} \Gamma^i_{rl}$$

here $\Gamma^i_{jk} = 0$ unless $i=j=k$, Implies

$$R^i_{jkl} = 0 \Rightarrow R_{ijkl} = 0$$

and hence $K=0$

b) Consider, $\frac{\partial u_i}{\partial x^j} = \Gamma_{ij}^k u_k$

$$\Rightarrow \frac{\partial u_1}{\partial x^1} = 0, \quad \frac{\partial u_2}{\partial x^2} = \frac{u_2}{x^2}$$

$$\frac{\partial u_3}{\partial x^3} = \frac{u_3}{x^3}, \quad \frac{\partial u_4}{\partial x^4} = \frac{u_4}{x^4}$$

and $\frac{\partial u_i}{\partial x^j} = 0$ for $i \neq j$

$$\frac{\partial u_2}{u_2} = \frac{\partial x^2}{x^2}$$

$$\Rightarrow \ln u_2 = \ln x^2 + \ln f_2$$

$$\Rightarrow e^{\ln u_2} = e^{\ln x^2 + \ln f_2}$$

$$\Rightarrow u_2 = x^2 f_2(x^1, x^3, x^4)$$

Integration yields:

$$u_1 = f_1(x^2, x^3, x^4)$$

$$u_2 = x^2 f_2(x^1, x^3, x^4)$$

$$u_3 = x^3 f_3(x^1, x^2, x^4)$$

and $u_4 = x^4 f_4(x^1, x^2, x^3)$

where f_i , $i=1,2,3,4$ are arbitrary functions.

$$\text{and } \frac{\partial \omega}{\partial x^i} = u_i$$

$$\Rightarrow \int \partial \omega = \int u_i \partial x^i$$

$$\Rightarrow \omega = \int u_1 \partial x^1 + u_2 \partial x^2 + u_3 \partial x^3 + u_4 \partial x^4$$

consider $f_i = c_i = \text{constant}$, we have

$$\omega = \int c_1 \partial x^1 + c_2 x^2 \partial x^2 + c_3 x^3 \partial x^3 + c_4 x^4 \partial x^4$$

$$\omega = c_1 x^1 + c_2 \frac{(x^2)^2}{2} + c_3 \frac{(x^3)^2}{2} + c_4 \frac{(x^4)^2}{2} + c_5$$

$$\Rightarrow \omega = a_1 x^1 + a_2 (x^2)^2 + a_3 (x^3)^2 + a_4 (x^4)^2 + a_5$$

and as $\omega = \tilde{x}^k$

$$\Rightarrow \tilde{x}^k = a_1^k x^1 + a_2^k (x^2)^2 + a_3^k (x^3)^2 + a_4^k (x^4)^2 + a_5^k$$

where a_i^k are constants.

Normal Coordinates:

It is possible to introduce local, quasi rectangular coordinates in Riemannian space the use of which greatly simplify the proofs of certain complicated tensor identities.

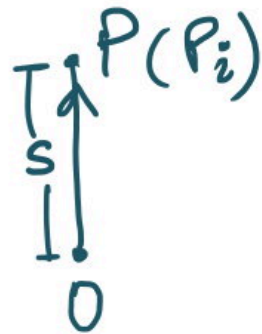
Let '0' denote an arbitrary point of R^n , and (P^i) an arbitrary unit vector at 0.

consider the differential Equations for geodesics,

$$\frac{d^2 x^i}{ds^2} + \Gamma_{jk}^i \frac{dx^j}{ds} \frac{dx^k}{ds} = 0$$

along with the initial conditions:

$$\left. \frac{dx^i}{ds} \right|_{s=0} = p^i$$



For a given (P^i) the previous system has a unique solution. Moreover, for each point P in some neighbourhood N of O , there is a unique choice of direction (P^i) at O such that the solution curve $x^i = x^i(s)$ (i.e., a geodesic) passes through P .

Accordingly, for each P in N , define the coordinates of P as:

$$y^i = s p^i$$

where s is the distance along the geodesic from O to P . The numbers (y^i) are called the normal coordinates }
or the geodesic coordinates }
or the Riemannian coordinates }
of point P .

Theorem: At the origin of a Riemannian coordinate system (y^i) , all $\frac{\partial g_{ij}}{\partial y^k}$, $\frac{\partial g^{ij}}{\partial y^k}$,

Γ_{ijk} and Γ^i_{jk} are zero.

Proof: If (y^i) are normal coordinates, then the geodesic through 0 and any point P in some neighbourhood N of 0 has

the parametric form:

$$y^1 = sp^1 \quad \dots \quad y^n = sp^n$$

$$y^i = sp^i \quad (p^i \rightarrow \text{constant})$$

$$p^i = (p^1, p^2, \dots, p^n)$$

$$y^i = (y^1, y^2, \dots, y^n)$$

$$\Rightarrow \frac{dy^i}{ds} = p^i \text{ and } \frac{d^2 y^i}{ds^2} = 0$$

Also (y^i) coordinates must satisfy the geodesic equation:

$$\frac{d^2 y^i}{ds^2} + \Gamma_{jk}^i \frac{dy^j}{ds} \frac{dy^k}{ds} = 0$$

$\rightarrow 0$

$$\Rightarrow \Gamma_{jk}^i p^j p^k = 0 \text{ (for all } (p^i))$$

now p^i is constant.

$$\Rightarrow \Gamma_{jk}^i = 0$$

$$\text{also } \Gamma_{ijk} = g_{kl} \Gamma_{ij}^l$$

$$\Rightarrow \Gamma_{ijk} = 0$$

now consider:

$$\frac{\partial g_{ik}}{\partial y^j} = \Gamma_{ijk} + \Gamma_{jki}$$

$$\Rightarrow \frac{\partial g_{ik}}{\partial y^j} = 0$$

and

$$g^{ij} g_{jr} = \delta^i_r$$

$$\Rightarrow g^{ij} \frac{\partial g_{jr}}{\partial y^k} + \frac{\partial g^{ij}}{\partial y^k} g_{jr} = 0$$

$$\Rightarrow 0 + \frac{\partial g^{ij}}{\partial y^k} g_{jr} = 0$$

$$g_{jr} \neq 0$$

$$\Rightarrow \frac{\partial g^{ij}}{\partial y^k} = 0$$

Question: Prove Bianchi's first identity.

$$R_{ijkl} + R_{iklj} + R_{iljk} = 0 \quad (\text{using normal coordinates})$$

Solution: consider the Riemannian coordinates.

Then

$$R_{ijkl} = \frac{\partial \Gamma_{jli}}{\partial x^k} - \frac{\partial \Gamma_{jki}}{\partial x^l} + \Gamma_{ilk} \Gamma_{jk}^l - \Gamma_{ikl} \Gamma_{jl}^k$$

becomes:

$$R_{ijkl} = \frac{\partial \Gamma_{jli}}{\partial x^k} - \frac{\partial \Gamma_{jki}}{\partial x^l}$$

at origin of Riemannian coordinates.

$$\Rightarrow R_{ijkl} = \Gamma_{jli,k} - \Gamma_{jki,l}$$

and similarly:

$$R_{iklj} = \Gamma_{kji,l} - \Gamma_{kli,j}$$

$$R_{iljk} = \Gamma_{lki,j} - \Gamma_{lj i,k}$$

$\therefore \Gamma_{ijk}$ is symmetric in first two indices

$$\Rightarrow R_{ijkl} + R_{iklj} + R_{iljk} = 0$$

Question: Prove Bianchi's second identity.

$$R_{ijkl,u} + R_{ijlu,k} + R_{ijuk,l} = 0$$

Solution: Consider,

$$R^i_{jkl} = \frac{\partial \Gamma^i_{jl}}{\partial y^k} - \frac{\partial \Gamma^i_{jk}}{\partial y^l} + \Gamma^i_{jl} \Gamma^i_{lk} - \Gamma^i_{jk} \Gamma^i_{kl}$$

$$R^i_{jkl,u} = \frac{\partial}{\partial y^u} \left[\frac{\partial \Gamma^i_{jl}}{\partial y^k} - \frac{\partial \Gamma^i_{jk}}{\partial y^l} + \Gamma^i_{jl} \Gamma^i_{lk} - \Gamma^i_{jk} \Gamma^i_{kl} \right]$$

$$R^i_{jkl,u} = \Gamma^i_{jl,ku} - \Gamma^i_{jk,lu} \quad \text{at '0' of}$$

Riemannian coordinates.

$$\Rightarrow R_{jlu,k}^i = \Gamma_{ju,kl}^i - \Gamma_{jl,lu}^i$$

and

$$R_{juk,l}^i = \Gamma_{jk,lu}^i - \Gamma_{ju,lk}^i$$

Implies:

$$R_{jkl,u}^i + R_{jlu,k}^i + R_{juk,l}^i = 0$$

$$\Rightarrow g^{hi} [R_{ijkl,u} + R_{ijlu,k} + R_{ijuk,l}] = 0$$

$$\Rightarrow R_{ijkl,u} + R_{ijlu,k} + R_{ijuk,l} = 0$$

The Einstein Tensor:

The Einstein tensor is defined in terms of the Ricci tensor R_{ij} and the curvature invariant R as:

$$G^i_j = R^i_j - \frac{1}{2} \delta^i_j R$$

This is a mixed tensor of order 2.

Theorem: For any Riemannian metric, the divergence of the Einstein tensor is zero at all points.

Proof: We have to show that

$$G^{\mu}_{\nu;\mu} = 0$$

$$A^i_{;i} = 0$$

consider,

$$R_{ijkl;\mu} + R_{ij\mu k;l} + R_{ij\mu l;k} = 0$$

Multiply both sides by $g^{il} g^{jk}$

$$\Rightarrow g^{il} g^{jk} R_{ijkl;u} + g^{il} g^{jk} R_{ijlu;k} + g^{il} g^{jk} R_{ijuk;l} = 0$$

$$\Rightarrow g^{il} g^{jk} R_{ijkl;u} - g^{il} g^{jk} R_{ijul;k} - g^{il} g^{jk} R_{jiuk;l} = 0$$

$$\Rightarrow g^{jk} R^l_{jkl;u} - g^{jk} R^l_{jul;k} - g^{il} R^k_{iuk;l} = 0$$

$$\Rightarrow R^l_{l;u} - R^k_{u;k} - R^l_{u;l} = 0$$

$$R^k_{k;u} - R^k_{u;k} - R^k_{u;k} = 0$$

$$2 \left(\frac{1}{2} R_{;u} - R^k_{u;k} \right) = 0$$

$$\left(R^k_u - \frac{1}{2} \delta^k_u R \right)_{;k} = 0$$

$$\Rightarrow G^k_u{}_{;k} = 0$$

Question: Show that the Einstein tensor G_{ij} obtained by lowering the index i in G^i_j is symmetric.

Solution: $G_{ij} = g_{ik} G^k_j$

$$G_{ij} = g_{ik} \left(R^k_j - \frac{1}{2} \delta^k_j R \right)$$

$$G_{ij} = \underset{\substack{\downarrow \\ \text{Symmetric}}}{R_{ij}} - \frac{1}{2} g_{ij} \underset{\substack{\downarrow \\ \text{Symmetric}}}{R}$$

$\Rightarrow G_{ij}$ is symmetric.

Question: Show that Einstein invariant $G = G^i_i$, vanishes if the space is flat.

Solution:

$$G^i_j = R^i_j - \frac{1}{2} \delta^i_j R$$

$$\begin{aligned} \Rightarrow G = G^i_i &= R^i_i - \frac{1}{2} R \\ &= R - \frac{1}{2} R = \frac{1}{2} R \end{aligned}$$

$$\Rightarrow G = \frac{1}{2} R$$

using the result that if space is flat $K=0$ and this implies $R=0$.

$\Rightarrow G=0$ if space is flat.

Energy Momentum Tensor:

- The quantity T^{ij} ; $i, j = 0, 1, 2, 3$ is the so-called energy-momentum-stress tensor or in a shorter version the energy-momentum tensor or the stress-energy tensor.
- It is a rank-2 symmetric tensor encoding all the information about energy density, momentum density, stress, pressure etc.
- In general relativity, the stress-energy or energy-momentum tensor T^{ij} acts as the source of the gravitational field.

$$T^{\alpha\beta} = \begin{pmatrix} \overset{T^{00}}{\text{energy density}} & \overset{T^{0i}}{\text{energy flux}} \\ \overset{T^{i0}}{\text{mom. density}} & \overset{T^{ij} \rightarrow i,j=1,2,3}{\text{stress tensor}} \end{pmatrix}.$$

The ten components of this tensor have the following interpretation:

- T^{00} is the local energy density, including any potential contribution from forces between particles and their kinetic energy.

The T^{00} component represents energy density. To see why, consider the momentum four vector $p^\mu = (E, p^i)$. Using this definition, we see that in this case T^{00} is the p^0 component of the momentum four vector. This is energy density.

* Energy density is the amount of energy in a system per unit volume.

- T^{0i} is the energy flux in the i th direction. This includes not only the bulk motion but also any other processes giving rise to transfers of energy, as for instance heat conduction.
- T^{i0} is the density of the momentum component in the i th direction, i.e. the 3-momentum density. As the previous case, it also takes into account the changes in momentum associated to heat conduction.

Momentum density is momentum per unit volume. This is the flow of momentum crossing a surface of constant time. As energy-momentum tensor is symmetric $T^{0i} = T^{i0}$ represents the energy flow across the surface.

Energy flux is the rate of transfer of energy through a surface.

the rate of transfer of momentum across a unit area.

- T^{ij} is the 3-momentum flux or stress tensor, i.e the rate of flow of the i th momentum component per unit area in the plane orthogonal to the j th -direction. The component T^{ii} encodes the isotropic pressure in the i th direction while the components T^{ij} with $i \neq j$ refer to the viscous stresses of the fluid.

$$T^{ij} = \begin{bmatrix} T^{11} & T^{12} & T^{13} \\ T^{21} & T^{22} & T^{23} \\ T^{31} & T^{32} & T^{33} \end{bmatrix}$$

viscous stresses.

isotropic stresses $\rightarrow$ pressure

Covariant Derivative of Energy Momentum Tensor:

The covariant derivative of energy momentum tensor is given by:

$$\nabla_j T^{ij} = 0$$

The covariant derivative of energy momentum tensor is actually the generalization of the conservation of energy and momentum in flat space.

This relation is called the local conservation of energy-momentum because it reduces to the conservation law

$\frac{\partial T^{ij}}{\partial x^j} = 0$ (in flat spacetime) in a local inertial frame. However, it is not a conservation law like in flat space, as the energy of matter is not conserved in the presence of dynamic spacetime curvature but changes in response to it.

Energy Momentum Tensor of Dust:

One of the simplest energy-momentum tensors is the dust energy momentum tensor. This type of matter field consists of noninteracting matter.

However, consider that the dust particles carry energy and momentum. The energy and momentum of the moving dust particles give rise to a gravitational field.

In this case, there are only two quantities that can be used to describe the matter field in the problem, the energy density and how fast (and in what direction) the dust is moving.

The simplest way to obtain the first quantity, the energy density, is to jump over to the co-moving frame. If you're in the comoving frame, then you're moving along with the dust particles. In that case there is a number of dust particles per unit volume n , and each dust particle has energy m . So the energy density is given by $\rho = mn$.

The second item of interest is none other than the velocity four vector u . This of course will give us the momentum carried by the dust. Generally speaking, to get the stress-energy tensor for dust, we put this together with the energy density. So for dust, the stress-energy tensor is given by

$$T^{ij} = \rho u^i u^j$$

Where u^i is the proper velocity.

The Components of Energy Momentum Tensor of Dust in Minkowski spacetime (Flat space)

Here we discuss two cases: when frame is comoving and non comoving.

Case I:

A frame moving with the flow of the dust fluid is called comoving frame.

For a co-moving observer, the four velocity reduces to

$$u^i = (c, 0, 0, 0)$$

In this case, the stress-energy tensor takes on the remarkably simple form

$$T^{ij} = \rho u^i u^j \rightarrow T^{00} = \rho u^0 u^0 = \rho c^2$$

$$T^{ij} = \begin{bmatrix} c^2 \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Here u^i is the proper velocity.

So Recall

$$v^i = \frac{dx^i}{dt} = \frac{d}{dt} (ct, x, y, z) = (c, v^x, v^y, v^z)$$

$$\text{and } \frac{d\tau}{dt} = \sqrt{1 - \frac{v^2}{c^2}} \Rightarrow \frac{dt}{d\tau} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma$$

Implies proper velocity is:

$$u^i = \frac{d}{d\tau} (ct, x, y, z) = \frac{dt}{d\tau} \cdot \frac{d}{dt} (ct, x, y, z)$$

$$= \frac{dt}{d\tau} (c, v^x, v^y, v^z)$$

$$u^i = \gamma (c, v^x, v^y, v^z)$$

Now in a comoving frame: $V=0$

$$\Rightarrow \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = 1$$

$$\Rightarrow u^\mu = (c, 0, 0, 0)$$

Case II:

Now consider the case of a stationary observer seeing the dust particles go by with four velocity u . In that case, we have

$$u = (\gamma c, \gamma v^x, \gamma v^y, \gamma v^z)$$

where u^i are the ordinary components of three velocity and $\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$. In this case the stress-energy tensor is

$$T^{ij} = \rho \gamma^2 \begin{bmatrix} c & cv^x & cv^y & cv^z \\ cv^x & (v^x)^2 & v^x v^y & v^x v^z \\ cv^y & v^y v^x & (v^y)^2 & v^y v^z \\ cv^z & v^z v^x & v^z v^y & (v^z)^2 \end{bmatrix}$$

Question: Show that the conservation equations for the energy-momentum tensor in case of dust lead to the equation of continuity of a fluid. (in flat space)

Solution: In a flat space the conservation equation is

$$\frac{\partial T^{ij}}{\partial x^j} = 0$$

for $i=0$ or $i=t$

$$\frac{\partial T^{tj}}{\partial x^j} = \frac{\partial T^{tt}}{c \partial t} + \frac{\partial T^{tx}}{\partial x} + \frac{\partial T^{ty}}{\partial y} + \frac{\partial T^{tz}}{\partial z}$$

$$= \frac{\partial}{c \partial t} (\gamma^2 c^2) + \frac{\partial}{\partial x} (\gamma^2 c v^x) + \frac{\partial}{\partial y} (c \gamma^2 v^y) + \frac{\partial}{\partial z} (c \gamma^2 v^z)$$

$$= c \left[\frac{\partial}{\partial t} (\gamma^2) + \frac{\partial}{\partial x} (\gamma^2 v^x) + \frac{\partial}{\partial y} (\gamma^2 v^y) + \frac{\partial}{\partial z} (\gamma^2 v^z) \right]$$

$\Rightarrow \frac{\partial}{\partial t} (\gamma^2) + \nabla \cdot (\gamma^2 \vec{u}) = 0$, This is the
equation of continuity.

$$\gamma^2 \vec{u} = \gamma^2 (v^x, v^y, v^z) \rightarrow \nabla \cdot (\gamma^2 \vec{u})$$

Question: Find Trace of energy momentum tensor defined by

$$x^0 = ct$$

$$T = T^i_i = T^{ij} g_{ij}$$

when $g_{ij} = \eta_{ij} = (cdt)^2 - dx^2 - dy^2 - dz^2$

and $T^{ij} = \rho u^i u^j$ in a comoving frame.

Solution: here only

$$T^{00} = T^{tt} = c^2 \rho \text{ is non zero}$$

in a comoving frame.

$$T = T^{ij} g_{ij} = T^{ij} \eta_{ij}$$

$$= T^{00} \eta_{00} + T^{11} \eta_{11} + T^{22} \eta_{22} + T^{33} \eta_{33}$$

$$= c^2 \rho (1) + 0 + 0 + 0$$

$$T = c^2 \rho$$

Question: Find the trace of $T^{ij} = \rho u^i u^j$ with $g_{ij} = \eta_{ij}$ for a non comoving frame.

Solution:

$$\begin{aligned} T &= T^{ij} \eta_{ij} \\ &= T^{00} \eta_{00} + T^{11} \eta_{11} + T^{22} \eta_{22} + T^{33} \eta_{33} \\ &= \rho \gamma^2 \left[c^2 + (v^x)^2 (-1) + (v^y)^2 (-1) + (v^z)^2 (-1) \right] \\ \Rightarrow T &= \rho \gamma^2 \left[c^2 - (v^x)^2 - (v^y)^2 - (v^z)^2 \right] \end{aligned}$$

Question: Find the components of $T^{ij} = \rho u^i u^j$

when $x^0 = ct$ $(dx^0)^2 = c^2 dt^2$

$$g_{ij} = c^2 dt^2 - a^2(t)(dx^2 + dy^2 + dz^2)$$

considering a comoving frame and

$$u^i = \frac{1}{\sqrt{g_{00}}}(c, 0, 0, 0)$$

Solution: $u^i = \frac{1}{\sqrt{g_{00}}}(c, 0, 0, 0)$ here $(g_{00} = 1)$

$$\Rightarrow u^i = \frac{1}{\sqrt{1}}(c, 0, 0, 0) = (c, 0, 0, 0)$$

$$\Rightarrow T^{00} = \rho u^0 u^0$$

$$\Rightarrow T^{00} = \rho(c)(c) = \rho c^2$$

remaining all components are zero.

$$\text{Also } T = T^{ij} g_{ij} = T^{00} g_{00} = \rho c^2$$

Energy-Momentum Tensor of Perfect Fluid

A perfect fluid is a fluid that has no heat conduction or viscosity. Such fluid is characterized by its mass density ρ and the pressure P . The stress-energy tensor that describes a perfect fluid is

$$T^{ij} = \left(\rho + \frac{P}{c^2} \right) u^i u^j - P g^{ij}$$

The Components of Energy Momentum Tensor of perfect fluid in Minkowski spacetime (Flat space)

Here we discuss two cases: when frame is comoving and non comoving.

Case I:

A frame moving with the flow of the perfect fluid is called comoving frame.

For a co-moving observer, the four velocity reduces to

$$u^i = (c, 0, 0, 0)$$

In this case, the stress-energy tensor takes on the remarkably simple form

$$T^{ij} = \begin{bmatrix} c^2 \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{bmatrix}$$

Consider,

$$T^{ij} = \left(\rho + \frac{p}{c^2}\right) u^i u^j - p g^{ij}$$

$$u^i = (c, 0, 0, 0)$$

$$\Rightarrow T^{00} = \left(\rho + \frac{p}{c^2}\right) u^0 u^0 - p g^{00}$$

$$\Rightarrow T^{00} = \left(\rho + \frac{p}{c^2}\right) c^2 - p = \rho c^2$$

$$T^{11} = 0 - p(-1) = p$$

$$T^{22} = 0 - p(-1) = p$$

$$T^{33} = 0 - p(-1) = p$$

Case II:

Now consider the case of a stationary observer seeing the perfect fluid particles go by with four velocity u^i . In that case, we have

$$u^i = (\gamma c, \gamma v^x, \gamma v^y, \gamma v^z)$$

where v^i are the ordinary components of three velocity and $\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$. In this case the stress-energy tensor is

$$T^{ij} = \left(\rho + \frac{p}{c^2} \right) u^i u^j - p g^{ij}$$

$$\Rightarrow T^{00} = \left(\rho + \frac{p}{c^2} \right) \gamma^2 c^2 - p(1)$$

$$= \rho c^2 \gamma^2 + p \gamma^2 - p$$

$$T^{00} = \rho c^2 \gamma^2 + (\gamma^2 - 1) p$$

$$\begin{aligned}
 T^{01} &= (\rho + p/c^2) u^0 u^1 \\
 &= (\rho + p/c^2) (\gamma c) (\gamma v^x) \\
 &= \gamma^2 c v^x (\rho + p/c^2)
 \end{aligned}$$

$$T^{02} = \gamma^2 c v^y (\rho + p/c^2)$$

$$T^{03} = \gamma^2 c v^z (\rho + p/c^2)$$

$$T^{11} = (\rho + p/c^2) \gamma^2 (v^x)^2 + p$$

$$T^{12} = (S + P/c^2) u^1 u^2$$

$$= \gamma^2 v^x v^y (S + P/c^2)$$

$$T^{13} = (S + P/c^2) \gamma^2 v^x v^z$$

$$T^{22} = (S + P/c^2) \gamma^2 (v^y)^2 + P$$

$$T^{23} = (S + P/c^2) \gamma^2 v^y v^z$$

$$T^{33} = (S + P/c^2) \gamma^2 (v^z)^2 + P$$

Question: Find the trace of Energy-momentum tensor of perfect fluid in Minkowski space and comoving frame.

Solution: $T = T^{ij} g_{ij}$ (where $g_{ij} = \eta_{ij}$)

$$g_{00} = 1, \quad g_{11} = -1 = g_{22} = g_{33}$$

$$\begin{aligned}
\Rightarrow T &= T^{00} g_{00} + T^{11} g_{11} + T^{22} g_{22} + T^{33} g_{33} \\
&= 3c^2 (1) + P(-1) + P(-1) + P(-1) \\
&= 3c^2 - 3P
\end{aligned}$$

Question: Find the trace of Perfect fluid energy-momentum tensor in flat space and non-comoving frame.

Solution: $T = T^{ij} g_{ij}$

$$\Rightarrow T = T^{00} g_{00} + T^{11} g_{11} + T^{22} g_{22} + T^{33} g_{33}$$

$$\text{here } g_{00} = 1, g_{11} = -1 = g_{22} = g_{33}$$

$$\text{and } T^{00} = (\rho + p/c^2) \gamma^2 c^2 - p$$

$$T^{11} = (\rho + p/c^2) \gamma^2 (v^x)^2 + p$$

$$T^{22} = (\rho + p/c^2) \gamma^2 (v^y)^2 + p$$

$$T^{33} = (\rho + p/c^2) \gamma^2 (v^z)^2 + p$$

$$\Rightarrow T = (\rho + p/c^2) \gamma^2 \{ c^2 - (v^x)^2 - (v^y)^2 - (v^z)^2 \}$$

-4p

Question: Find the components of

$$T^{ij} = (\rho + P/c^2) u^i u^j - P g^{ij}$$

where

$$g_{ij} = c^2 dt^2 - a^2(t) (dx^2 + dy^2 + dz^2)$$

and

$$u^i = \frac{1}{\sqrt{g_{00}}} (c, 0, 0, 0)$$

Solution:

$$u^i = \frac{1}{\sqrt{1}} (c, 0, 0, 0) = (c, 0, 0, 0)$$

$$T^{00} = (\rho + p/c^2) u^0 u^0 - p g^{00}$$

$$= (\rho + p/c^2) c^2 - p (1)$$

$$= \rho c^2 + p - p$$

$$\Rightarrow T^{00} = \rho c^2$$

$$T^{11} = (\rho + p/c^2) \dot{u}^1 \dot{u}^1 - p g^{11} \quad g^{11} = 1/g_{11}$$

$$= 0 - p (-1/a^2)$$

$$T^{11} = p/a^2$$

Similarly $T^{22} = p/a^2$ and $T^{33} = p/a^2$

and $T = \rho c^2 (1) + p/a^2 (-a^2) + p/a^2 (-a^2)$
 $+ p/a^2 (-a^2)$

$$T = \rho c^2 - 3p$$

Question: Discuss the conservation equation for perfect fluid in flat space and comoving frame.

Solution: The conservation equation in flat space is:

$$\frac{\partial T^{ij}}{\partial x^j} = 0$$

for $\vec{v}=0$, we have:

$$\frac{\partial T^{0j}}{\partial x^j} = \frac{\partial T^{00}}{\partial x^0} + \frac{\partial T^{01}}{\partial x^1} + \frac{\partial T^{02}}{\partial x^2} + \frac{\partial T^{03}}{\partial x^3}$$

$$= \frac{\partial (\rho c^2)}{\partial t} + 0 + 0 + 0 = 0$$

$$\Rightarrow c \frac{\partial \rho}{\partial t} = 0$$

$$\Rightarrow \frac{\partial \rho}{\partial t} = 0 \Rightarrow \rho = \text{constant}$$

Question: Find the conservation equation for perfect fluid in flat space and non-moving frame, for $i=0$.

Solution: Using the conservation equation $\frac{\partial T^{ij}}{\partial x^j} = 0$

for $i=0$:

$$\frac{\partial T^{00}}{\partial x^0} + \frac{\partial T^{01}}{\partial x^1} + \frac{\partial T^{02}}{\partial x^2} + \frac{\partial T^{03}}{\partial x^3} = 0 \rightarrow (*)$$

where

$$T^{00} = (\rho + p/c^2) \gamma^2 c^2 - p$$

$$T^{01} = \gamma^2 c v^x (\rho + p/c^2)$$

$$T^{02} = \gamma^2 c v^y (\rho + p/c^2)$$

$$T^{03} = \gamma^2 c v^z (\rho + p/c^2)$$

$$x^0 = ct, \quad x^1 = x, \quad x^2 = y, \quad x^3 = z$$

Eq. (★). Yields:

$$\frac{1}{c} \frac{\partial}{\partial t} \left[\left(\rho + \frac{p}{c^2} \right) \gamma^2 c^2 - p \right] + \frac{\partial}{\partial x} \left[\left(\rho + \frac{p}{c^2} \right) \gamma^2 c v^x \right]$$

$$+ \frac{\partial}{\partial y} \left[\left(\rho + \frac{p}{c^2} \right) \gamma^2 c v^y \right] + \frac{\partial}{\partial z} \left[\left(\rho + \frac{p}{c^2} \right) \gamma^2 c v^z \right] = 0$$

$$\Rightarrow \frac{\partial}{\partial t} \left[\left(\rho + \frac{p}{c^2} \right) \gamma^2 - \frac{p}{c^2} \right] + \nabla \cdot \left[\left(\rho + \frac{p}{c^2} \right) \gamma^2 \vec{v} \right] = 0$$

THE EINSTEIN FIELD EQUATIONS

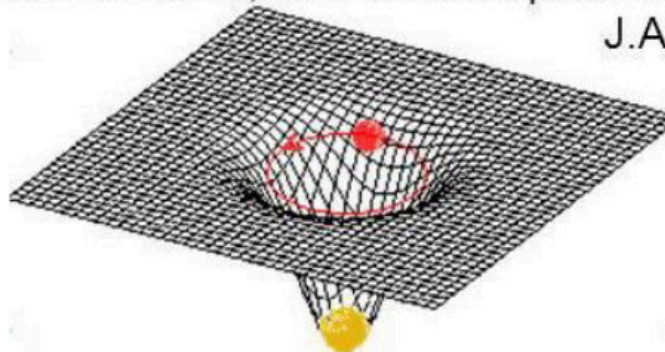
- To take into account spacetime curvature, tensor calculus was necessary in Einstein's General Relativity.

THE EINSTEIN FIELD EQUATION

$$G_{\mu\nu} = 8\pi T_{\mu\nu}$$

"matter tells spacetime how to curve, and curved space tells matter how to move"

J.A. Wheeler



Einstein's Formulation of the Field Equation:

Einstein did not derive the field equation, i.e., the law of gravitation, but formulated it from a number of considerations which we will discuss here.

1) At first, the field equation must be a tensor equation.

2) Secondly, under appropriate conditions the new law of gravitation must yield an equation of the form

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = 4\pi G \rho \quad \left. \vphantom{\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2}} \right\} \text{Poisson law}$$

So, the field equations must not contain higher derivatives of $g_{\mu\nu}$ than

the Second. So that the ten metric functions $g_{\mu\nu}$ of Einstein theory can correspond to a single function ψ of Newton's theory.

3) Thirdly, the right hand side of Poisson law should be related to $T_{\mu\nu}$.

Obeying the conservation law

$$T^{\alpha\beta}_{;\beta} = 0 \quad \text{or} \quad \frac{\partial T^{\alpha\beta}}{\partial x^\beta} = 0 \quad (\text{in flat space})$$

4) Finally,

$$(R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R)_{;\beta} = 0$$

Inspired by these consideration,
Einstein write down the general
relativistic analogue of the Poisson
equation as:

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} = -\kappa T_{\alpha\beta}$$

where $\kappa = 8\pi G/c^4$

Question: Write down the field equations for the cosmological model $ds^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2)$, in case of dust fluid, using

$$R_{00} = 3 \frac{\ddot{a}}{a}$$
$$R_{11} = -(a\ddot{a} + 2\dot{a}^2) = R_{22} = R_{33}$$

$$\Rightarrow R = R_{ij} g^{ij}$$

$$= R_{00} g^{00} + R_{11} g^{11} + R_{22} g^{22} + R_{33} g^{33}$$

$$= \frac{3\ddot{a}}{a} (1) + \frac{3}{a^2} (a\dot{a} + 2\dot{a}^2)$$

$$= \frac{6\ddot{a}}{a} + \frac{6\dot{a}^2}{a^2}$$

$$= \frac{6[\ddot{a}^2 + a\ddot{a}]}{a^2}$$

$$\Rightarrow G_{00} = R_{00} - \frac{1}{2} R g_{00}$$

$$= \frac{3\ddot{a}}{a} - \frac{1}{2}(1) \left[3\frac{\ddot{a}}{a} + 3\frac{\ddot{a}}{a^3} + \frac{6\dot{a}^2}{a^4} \right]$$

$$= \frac{3}{2} \frac{\ddot{a}}{a} + 3 \frac{\ddot{a}}{a^3} + 6 \frac{\dot{a}^2}{a^4}$$

Similarly

$$G_{11} = R_{11} - \frac{1}{2} R g_{11}$$

Now $T^{\hat{i}\hat{j}} = \rho u^{\hat{i}} u^{\hat{j}}$

$$u^{\hat{i}} = (1, 0, 0, 0)$$

$$\Rightarrow T^{00} = \rho, \quad T^{11} = T^{22} = T^{33} = 0$$

$$\Rightarrow T_{00} = g_{00} g_{00} T^{00}$$

$$T_{00} = (-1)(-1)\rho = \rho$$

So the field equations:

$$G_{\mu\nu} = \kappa T_{\mu\nu} \quad \text{Yield:}$$

$$G_{00} = \kappa T_{00}$$

$$T_{00} = g_{00} g_{00} T^{00}$$

$$= (-1)(-1) \rho = \rho$$

$$T'' = 0 \Rightarrow T_{11} = 0$$

$$\Rightarrow -\frac{3\dot{a}^2}{a^2} = \kappa \rho$$

and $G_{11} = \kappa T_{11}$

$$\Rightarrow 2a\ddot{a} + \dot{a}^2 = \kappa(0)$$

$$\Rightarrow 2a\ddot{a} + \dot{a}^2 = 0$$

Electrodynamics

It is the branch of Physics which deals with rapidly changing electric and magnetic fields.

Here we will discuss the applications of tensors or use of tensors in the well known Maxwell field equations of electrodynamics.

Maxwell Field Equations;

Maxwell's equations are four equations that together form a complete description of the production and interrelation of electric and magnetic fields.

In the 19th century, the physicist James Clerk Maxwell based his description of electromagnetic fields on these four equations, which express experimental laws.

Maxwell Field Equations

electric field $\uparrow$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

$\rho \rightarrow$ charge density
 $\epsilon_0 \rightarrow$ permittivity of Vacuum

$$\nabla \cdot \mathbf{B} = 0$$

Magnetic $\downarrow$ flux

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}$$

electric current $\swarrow$
permeability of vacuum $\downarrow$

$\frac{1}{\sqrt{\mu_0 \epsilon_0}} = c$
 $\downarrow$
speed of light

Maxwell Field Equations

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \rightarrow \text{Gauss' law of electricity}$$

$$\nabla \cdot \mathbf{B} = 0 \rightarrow \text{Gauss' law of Magnetism}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \rightarrow \text{Faraday's law of Induction}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} \rightarrow \text{Ampere's law}$$

These are differential equations for the electric field $\vec{E}$ and magnetic field $\vec{B}$ in the presence of a charge function ρ and an electrical current $\vec{j}$. The quantities ϵ_0 and μ_0 are physical constants called the permittivity and permeability of vacuum, respectively.

Where electric field is the physical field that surrounds electrically charged particles and exerts force to all other charged particles in the field.

A magnetic field describes the magnetic influence on moving electric charges, electric currents and magnetic materials.

Here we will discuss how different electromagnetic quantities are transformed in relativity under Lorentz transformations.

Charge Density:

Charge density is defined as the measure of electric charge per unit volume.

Consider two frames of reference S and S' such that frame S' is moving with velocity $\vec{v}$ away from frame S . Now if charge density is defined as

$$\rho_0 = \frac{Ne}{l_0^3}$$

in the frame S' .

$\left\{ \begin{array}{l} N \rightarrow \text{No of electrons} \\ e \rightarrow \text{charge} \end{array} \right.$

$l_0 \rightarrow \text{length of one side of cube}$

Then in frame S the charge density appears to be

$$\rho = \frac{Ne}{l_0^3 \sqrt{1 - v^2/c^2}} \rightarrow \text{this factor appears due to length contraction.}$$

$$\Rightarrow \rho = \frac{\rho_0}{\sqrt{1 - v^2/c^2}} \quad \left(\because \rho_0 = \frac{Ne}{l^3} \right)$$

Current Density: It is the amount of current flowing across the given region.

There are two frames of reference here, S and S' . Where S' frame is moving away from S with a velocity $\vec{v}$. Then there is no current for frame S' .

However, for frame S we have:

$$\vec{j} = \beta \vec{V}$$

$$\Rightarrow \vec{j} = \frac{\beta_0 \vec{V}}{\sqrt{1 - v^2/c^2}}$$

or in component form:

$$j_x = \frac{\beta_0 v_x}{\sqrt{1 - v^2/c^2}}, \quad j_y = \frac{\beta_0 v_y}{\sqrt{1 - v^2/c^2}}, \quad j_z = \frac{\beta_0 v_z}{\sqrt{1 - v^2/c^2}}$$

Mass:

If m_0 is the mass of the charge density ρ_0 according to frame S' , then mass according to S is:

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

Consider, $m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$ and $\mathcal{E} = \frac{\mathcal{E}_0}{\sqrt{1 - v^2/c^2}}$

$$\frac{m}{m_0} = \frac{\mathcal{E}}{\mathcal{E}_0} \Rightarrow \frac{\mathcal{E}_0}{m_0} = \frac{\mathcal{E}}{m}$$

$$\Rightarrow \frac{\mathcal{E}}{m} = \frac{\mathcal{E}_0}{m_0} \Rightarrow \mathcal{E} = \left(\frac{\mathcal{E}_0}{m_0} \right) m$$

and $\vec{j} = \mathcal{E} \vec{u}$

$$\mathcal{E} = \left(\frac{\mathcal{E}_0}{m_0} \right) m$$

$$\Rightarrow \vec{j} = \left(\frac{\mathcal{E}_0}{m_0} \right) m \vec{u} \Rightarrow \vec{j} = \left(\frac{\mathcal{E}_0}{m_0} \right) \vec{p}$$

Four current:

It is a four vector whose three space components are those of the ordinary current density and its time component is the charge density.

Mathematically,

$$\begin{aligned} J_\mu &= g_{\mu\nu} J^\nu \\ &= g_{00} J^0 + g_{11} J^1 + g_{22} J^2 + g_{33} J^3 \\ &= \rho c - J^1 - J^2 - J^3 \\ &= (\rho c, -\vec{J}) \end{aligned}$$

$$J^\mu = (\rho c, \vec{J})$$

where $\vec{J} = J_x \hat{i} + J_y \hat{j} + J_z \hat{k}$

$\rho \rightarrow$ charge density

or $J_\mu = (\rho c, -\vec{J})$ [in Minkowski space]

Conservation equation for four current density is:

$$\nabla_{\alpha} J^{\alpha} = 0 \text{ or } J^{\alpha}_{;\alpha} = 0$$

In special relativity:

$$J^{\alpha}_{;\alpha} = 0 \Rightarrow J^0_{;0} + J^1_{;1} + J^2_{;2} + J^3_{;3} = 0$$

$$\Rightarrow \frac{1}{q} \frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0 \quad \checkmark$$

Electromagnetic four potential:

The four potential has time and space components defined as

$$A^\mu = (\phi/c, \vec{A})$$

$\vec{A} \rightarrow$ vector potential

$\phi \rightarrow$ scalar potential

The scalar and vector potential are related to the potential energy U and potential momentum $\vec{Q}$ of a particle having charge q by:

$$U = q\phi$$

$$\vec{Q} = q\vec{A}$$

The magnetic field $\vec{B}$ is defined in terms of vector potential as:

$$\vec{B} = \nabla \times \vec{A}$$

and electric field as:

$$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t}$$

Four Potential Conservation:

Gives: $\nabla_\alpha A^\alpha = 0$ or $A^\alpha_{;\alpha} = 0$

In flat Space:

$$A^\alpha_{,\alpha} = 0 \Rightarrow A^0_{,0} + A^1_{,1} + A^2_{,2} + A^3_{,3} = 0$$

$$\Rightarrow \frac{\partial(\Phi/c)}{c \partial t} + \nabla \cdot \vec{A} = 0$$

$$\Rightarrow \frac{1}{c^2} \frac{\partial \Phi}{\partial t} + \nabla \cdot \vec{A} = 0$$

Electromagnetic field:

The electromagnetic field contains the combine effects of electric and magnetic fields produced by charged objects.

The area in the surrounding of a charged particle within which it can exert the electrostatic force

force on other particles is the electric field.

Magnetic field is similar to electric field and describes the effects of nearby magnetized material and electric currents at a point.

When charged particles are in motion then magnetic field is also produced such that both fields are orthogonal to each other as well as to the direction of motion.

Maxwell field equations describe the evolution of electromagnetic field.

Electromagnetic field tensor:

The electromagnetic field tensor also called Faraday's tensor is defined as

$$F^{\alpha\beta} = \nabla^{\alpha} A^{\beta} - \nabla^{\beta} A^{\alpha}$$

where A^{α} is the four vector potential.

Also, $A^\alpha = (\phi/c, \vec{A})$

In flat space:

$$F^{\alpha\beta} = \begin{bmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{bmatrix}$$

$F^{\alpha\beta}$ is an antisymmetric tensor.

and

$$F_{\alpha\beta} = g_{\alpha\gamma} F^{\gamma\delta} g_{\delta\beta}$$

$$= \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{bmatrix}$$

Components of $F^{\alpha\beta}$ in Flat space

$$F^{\alpha\beta} = \nabla^\alpha A^\beta - \nabla^\beta A^\alpha \rightarrow A^{\beta;\alpha}_{\downarrow} - A^{\alpha;\beta}_{\downarrow}$$

$$F^{\alpha\beta} = g^{\alpha\gamma} \nabla_\gamma A^\beta - g^{\beta\gamma} \nabla_\gamma A^\alpha$$

where $\alpha, \beta = 0, 1, 2, 3$

$$\text{and } F^{00} = 0 = F^{11} = F^{22} = F^{33}$$

$$F^{01} = g^{0\gamma} \nabla_\gamma A^1 - g^{1\gamma} \nabla_\gamma A^0$$

$$= g^{00} A'_{,0} - g^{11} A^0_{,1}$$

$$x^0 = ct \quad (\partial x^0 = c \partial t)$$

$$F^{01} = (1) \frac{1}{c} \frac{\partial A^1}{\partial t} - (-1) \frac{1}{c} \frac{\partial \phi}{\partial x}$$

Recall that,

$$\vec{B} = \nabla \times \vec{A} \text{ and } \vec{E} = -\nabla \phi - \frac{\partial \vec{A}}{\partial t}$$

$$\Rightarrow F^{01} = \frac{1}{c} \left(\frac{\partial \phi}{\partial x} + \frac{\partial A^1}{\partial t} \right) = -\frac{1}{c} E_x$$

$$\Rightarrow F^{10} = E_x / c$$

$$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t}$$

$$= -\left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}\right) - \left(\frac{\partial A^1}{\partial t}\hat{i} + \frac{\partial A^2}{\partial t}\hat{j} + \frac{\partial A^3}{\partial t}\hat{k}\right)$$

$$= -\left(\frac{\partial\phi}{\partial x} + \frac{\partial A^1}{\partial t}\right)\hat{i} \rightarrow E_x = -\left(\frac{\partial\phi}{\partial x} + \frac{\partial A^1}{\partial t}\right)$$

$$- \left(\frac{\partial\phi}{\partial y} + \frac{\partial A^2}{\partial t}\right)\hat{j} \rightarrow E_y = -\left(\frac{\partial\phi}{\partial y} + \frac{\partial A^2}{\partial t}\right)$$

$$- \left(\frac{\partial\phi}{\partial z} + \frac{\partial A^3}{\partial t}\right)\hat{k} \rightarrow E_z = -\left(\frac{\partial\phi}{\partial z} + \frac{\partial A^3}{\partial t}\right)$$

$$\vec{B} = \nabla \times \vec{A}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A^1 & A^2 & A^3 \end{vmatrix}$$

$$= \hat{i} \left(\frac{\partial A^3}{\partial y} - \frac{\partial A^2}{\partial z} \right) - \hat{j} \left(\frac{\partial A^3}{\partial x} - \frac{\partial A^1}{\partial z} \right)$$

$$+ \hat{k} \left(\frac{\partial A^2}{\partial x} - \frac{\partial A^1}{\partial y} \right)$$

Similarly,

$$F^{02} = -\frac{1}{c} E_y \text{ and } F^{03} = \frac{1}{c} E_z$$

and

$$\begin{aligned} F^{12} &= g^{1r} \nabla_r A^2 - g^{2r} \nabla_r A^1 \\ &= g^{11} A^2_{,1} - g^{22} A^1_{,2} \\ &= -\frac{\partial A^2}{\partial x} + \frac{\partial A^1}{\partial y} = \frac{\partial A^1}{\partial y} - \frac{\partial A^2}{\partial x} \end{aligned}$$

$$F^{12} = -B_z \Rightarrow F^{21} = B_z$$

$$\begin{aligned}
 F^{13} &= g^{1\gamma} \nabla_\gamma A^3 - g^{3\gamma} \nabla_\gamma A^1 \\
 &= g^{11} A^3_{,1} - g^{33} A^1_{,3}
 \end{aligned}$$

$$F^{13} = B_y \Rightarrow F^{31} = -B_y.$$

$$F^{23} = -B_x$$

In Short, we can write

$$F_{ij} = \epsilon_{ijk} B^k ; i, j, k = 1, 2, 3$$

$$\text{and } F_{i0} = -\frac{E_i}{c} ; i = 1, 2, 3$$

$$\star F^{i0} = \frac{E_i}{c}$$

$$\text{and } F^{ij} = \epsilon^{ijk} B_k$$

Using:

$\left\{ \begin{array}{l} \text{cyclic } 123 \\ \text{permutations } 231 \\ 312 \end{array} \right\}$

$$F_{ij} = \epsilon_{ijk} B^k$$

$$F_{12} = \epsilon_{12k} B^k = \epsilon_{123} B^3$$

$$= (1) B^3 = B^3$$

$$F_{12} = g^{3\mu} B_\mu = g^{33} B_3 = (-1) B_3$$

$$= -B_z \Rightarrow F_{21} = B_z$$

$$F_{13} = B_y \text{ and } F_{31} = -B_y$$

$$\text{and } F_{23} = -B_x \text{ and } F_{32} = B_x$$

$$F_{i0} = -\frac{\vec{E}_i}{c}$$

$$\Rightarrow F_{10} = -\frac{E_1}{c} = -\frac{E_x}{c}$$

Similarly

$$F_{20} = -\frac{\vec{E}_y}{c} \text{ and } F_{30} = -\frac{\vec{E}_z}{c}$$

$$\Rightarrow F_{01} = \frac{E_x}{c}, F_{02} = \frac{E_y}{c}, F_{03} = \frac{E_z}{c}$$

Now using:

$$F^{ij} = \epsilon^{ijk} B_k$$

$$\Rightarrow F^{12} = \epsilon^{123} B_3$$

$$\Rightarrow F^{12} = g^{11} g^{22} g^{33} \epsilon_{123} B_3$$

$$= (-1)(-1)(-1)(1) B_z$$

$$= -B_z$$

$$\text{and } F^{21} = B_z$$

Maxwell field equations in Tensor form.

These equations in tensor form are given below:

$$F_{\alpha\beta;\gamma} + F_{\beta\gamma;\alpha} + F_{\gamma\alpha;\beta} = 0$$

$$F^{\alpha\beta}{}_{;\beta} = \mu_0 J^{\alpha}$$

and in flat space covariant derivatives are replaced by partial derivatives.

$$F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = 0$$

$$F^{\alpha\beta}_{,\beta} = \mu_0 J^\alpha$$

Now we will derive the four Maxwell equations from tensorial form in flat space.

1) consider,

$$F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = 0$$

for $\alpha=1$, $\beta=2$ and $\gamma=3$

we have

$$\Rightarrow F_{12,3} + F_{23,1} + F_{31,2} = 0$$

$$(-B_z)_{,z} + (-B_x)_{,x} + (-B_y)_{,y} = 0$$

$$\Rightarrow \frac{\partial B_z}{\partial z} + \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = 0$$

$$\Rightarrow \nabla \cdot \vec{B} = 0$$

$$\text{or } \operatorname{div} \vec{B} = 0$$

In Short: $F_{jk} = \epsilon_{jki}^{\uparrow} \underline{B^i}$, $F_{ki} = \epsilon_{kij}^{\uparrow} \underline{B^j}$

$$F_{ij} = \epsilon_{ijk} B^k = B^k \left\{ \begin{array}{l} \text{when } i \neq j \neq k \\ \text{and 'ijk' is} \\ \text{even permutation} \end{array} \right.$$

so

$$F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = 0$$

for $\alpha=i, \beta=j, \gamma=k$

$$\Rightarrow F_{ij,k} + F_{jk,i} + F_{ki,j} = 0$$

$$\Rightarrow B^k_{,k} + B^i_{,i} + B^j_{,j} = 0 \Rightarrow \operatorname{div} B = 0$$

2) Now again:

$$F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = 0$$

for $\alpha = i, \beta = j$ and $\gamma = 0$

$$\Rightarrow F_{ij,0} + F_{j0,i} + F_{0i,j} = 0$$

$$\text{now } F_{ij} = \varepsilon_{ijk} B^k$$

$$F_{j0} = -E_j/c \quad \text{and} \quad F_{0i} = E_i/c$$

$$\Rightarrow \epsilon_{ijk} B^k_{,0} + \left(-\frac{E_j}{c}\right)_{,i} + \left(\frac{E_i}{c}\right)_{,j} = 0$$

$$\Rightarrow \left[\frac{1}{c} \frac{\partial B^k}{\partial t} + \frac{1}{c} (E_{i,j} - E_{j,i}) = 0 \right]^{k^{\text{th}} \text{ component of}}$$

$$\Rightarrow \frac{\partial \vec{B}}{\partial t} + \nabla \times \vec{E} = 0$$

$$\Rightarrow \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

3) Now consider

$$F^{\alpha\beta}{}_{,\beta} = \mu_0 J^\alpha$$

For $\alpha=0$ and $\beta=i$, we have

$$F^{0i}{}_{,i} = \mu_0 J^0 \quad (i=1,2,3)$$

$$\Rightarrow \left(-\frac{E_i}{c} \right)_{,i} = \mu_0 J^0$$

$$\Rightarrow (-E_{i,i}) = \mu_0 c J^0$$

also here $g_{ii} = -1$ and $g^{ii} = -1$

$$\text{so } (-E_i)_{,i} = (g^{ii} E_i)_{,i} = E^i_{,i}$$

$$\Rightarrow E^i_{,i} = \mu_0 c J^0$$

$$\text{where } J^0 = \rho c \text{ and } c^2 = \frac{1}{\mu_0 \epsilon_0}$$

$$\Rightarrow E^i_{,i} = \mu_0 \frac{1}{\mu_0 \epsilon_0} \rho = \rho / \epsilon_0$$

$$\Rightarrow \operatorname{div} E = \rho / \epsilon_0$$

4) Consider,

$$F^{\alpha\beta}_{,\beta} = \mu_0 J^\alpha$$

for $\alpha = i$

$$\Rightarrow F^{i0}_{,0} + F^{ij}_{,j} = \mu_0 J^i$$

$$\text{As } F^{i0} = E^i/c$$

$$(E_i = -E^i)$$

$$\Rightarrow F^{i0} = -E^i/c$$

$$\Rightarrow F^{i0}_{,0} = \frac{1}{c} \frac{\partial}{\partial t} \left(-\frac{E^i}{c} \right)$$

$$F^{i0}_{,0} = -\frac{1}{c^2} \frac{\partial E^i}{\partial t}$$

$$\begin{aligned} x^0 &= ct \\ \partial x^0 &= c \partial t \end{aligned}$$

$$\text{and } F^{ij} = \epsilon^{ijk} B_k$$

$$\Rightarrow F^{ij}_{,j} = \epsilon^{ijk} B_{k,j}$$

Implies:

$$-\frac{1}{c^2} \frac{\partial E^i}{\partial t} + \varepsilon^{ijk} B_{k,j} = \mu_0 J^i$$

$$\Rightarrow \varepsilon^{ijk} B_{k,j} = \mu_0 J^i + \frac{1}{c^2} \frac{\partial E^i}{\partial t}$$

$$\Rightarrow (\nabla \times \vec{B})^i = \mu_0 J^i + \frac{1}{c^2} \frac{\partial E^i}{\partial t}$$

$$\Rightarrow \nabla \times \vec{B} = \mu_0 \vec{J} + \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}$$

Electromagnetic Energy Momentum Tensor:

It is defined by:

$$T^{\alpha}_{\beta} = \frac{1}{4} \delta^{\alpha}_{\beta} (F_{\mu\gamma} F^{\mu\gamma}) - F_{\mu\beta} F^{\mu\alpha}$$

or

$$T^{\alpha\beta} = \frac{1}{4} g^{\alpha\beta} (F_{\mu\nu} F^{\mu\nu}) - F^{\alpha\mu} F^{\beta}_{\mu}$$

Trace:

$$T^\alpha_\alpha = T^0_0 + T^1_1 + T^2_2 + T^3_3$$

$$T^0_0 = \frac{1}{4} F_{\mu\gamma} F^{\mu\gamma} - F_{\mu 0} F^{\mu 0}$$

$$T^1_1 = \frac{1}{4} F_{\mu\gamma} F^{\mu\gamma} - F_{\mu 1} F^{\mu 1}$$

$$T^2_2 = \frac{1}{4} F_{\mu\gamma} F^{\mu\gamma} - F_{\mu 2} F^{\mu 2}$$

$$T^3_3 = \frac{1}{4} F_{\mu\sigma} F^{\mu\sigma} - F_{\mu 3} F^{\mu 3}$$

$$\Rightarrow T^\alpha_\alpha = F_{\mu\tau} F^{\mu\tau} - \left[F_{\mu 0} F^{\mu 0} + F_{\mu 1} F^{\mu 1} + F_{\mu 2} F^{\mu 2} + F_{\mu 3} F^{\mu 3} \right]$$

$$= F_{\mu\tau} F^{\mu\tau} - F_{\mu\tau} F^{\mu\tau}$$

$$= 0$$

Components of electromagnetic Energy

Momentum tensor in Flat Space:

consider

$$T^{\alpha}_{\beta} = \frac{1}{4} \delta^{\alpha}_{\beta} (F_{\mu\nu} F^{\mu\nu}) - F_{\mu\beta} F^{\mu\alpha}$$

consider

$$F_{\mu\nu} F^{\mu\nu} = F_{0\nu} F^{0\nu} + F_{1\nu} F^{1\nu} + F_{2\nu} F^{2\nu} + F_{3\nu} F^{3\nu}$$

where $\nu = 0, 1, 2, 3$ in each term

$$\Rightarrow F_{\mu\nu} F^{\mu\nu} = 2B^2 - \frac{2}{c^2} E^2$$

Now

$$T^0_0 = \frac{1}{4} \left(2B^2 - \frac{2}{c^2} E^2 \right) - F_{\mu 0} F^{\mu 0}$$

$$= \frac{1}{4} \left(2B^2 - \frac{2}{c^2} E^2 \right) + \frac{E_x^2}{c^2} + \frac{E_y^2}{c^2} + \frac{E_z^2}{c^2}$$

$$= \frac{1}{2} B^2 - \frac{1}{2c^2} E^2 + \frac{1}{c^2} E^2 = \frac{1}{2} \left(B^2 + \frac{E^2}{c^2} \right)$$

Similarly:

$$\begin{aligned} T^0_1 &= \frac{1}{4} \delta^0_1 F_{\mu\nu} F^{\mu\nu} - F_{\mu 1} F^{\mu 0} \\ &= -F_{01} F^{00} - F_{11} F^{10} - F_{21} F^{20} - F_{31} F^{30} \\ &= -(B_z) \left(\frac{E_y}{c} \right) - (-B_y) \left(+\frac{E_z}{c} \right) \\ &= \frac{1}{c} [B_y E_z - B_z E_y] \end{aligned}$$

Properties of $F^{\mu\nu}$:

1) Antisymmetry $F^{\mu\nu} = -F^{\nu\mu}$

2) It has six independent components.

3) $F_{\mu\nu} F^{\mu\nu} = 2 (B^2 - E^2/c^2)$

4) Trace:

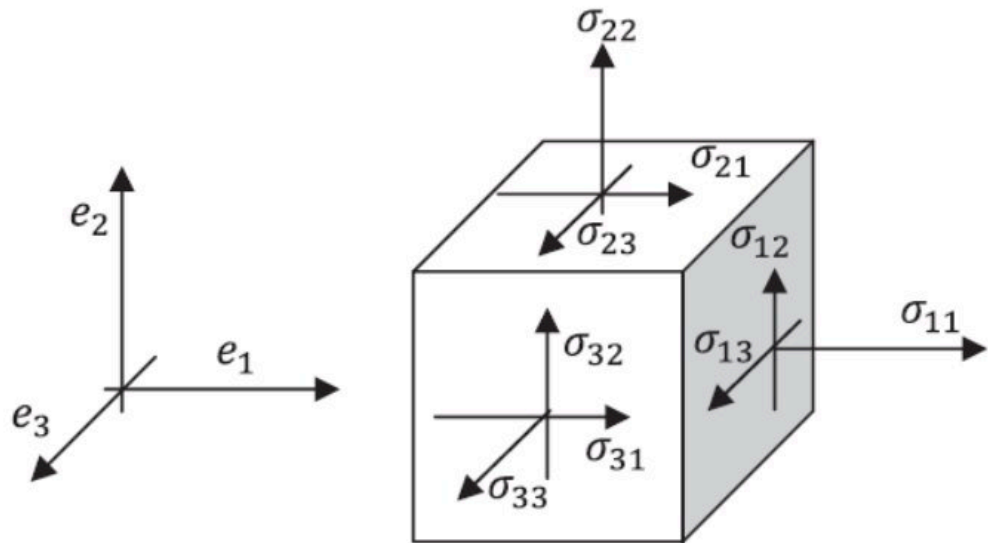
$$F = F^{\mu}_{\mu} = 0$$

The Stress Tensor

Stress is defined as force per unit area. If we take a cube of material and subject it to an arbitrary load we can measure the stress on it in various directions. These measurements will form a second rank tensor; called the stress tensor. Whose each component σ_{ij} is characterized by its sub-indices:

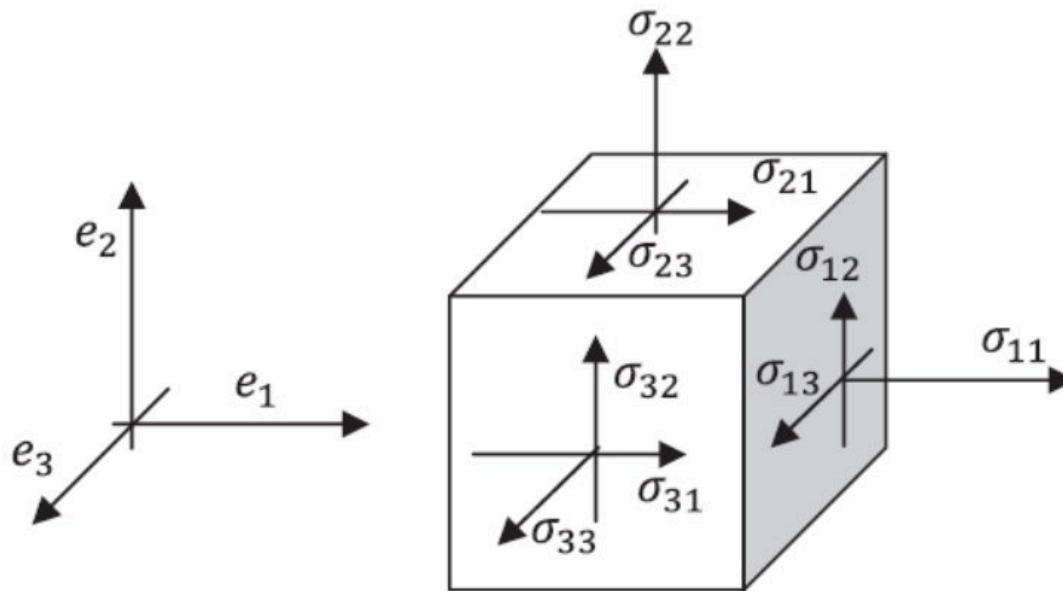
- Index i designates the coordinate plane on which the component acts.
- Index j identifies the coordinate direction in which the component acts.

$$\boldsymbol{\sigma} \equiv \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$



Normal and Shear Stresses:

- σ_{ii} are called the normal stresses acting on plane i .
- Whereas $\sigma_{ij}; i \neq j$ are called the tangential or shear stresses, acting on the plane perpendicular to the i -axis in the direction of j -axis.



The sign criterion for the stress components:

$$1) \sigma_{ii} = \begin{cases} \text{positive} \Rightarrow \text{tensile stress} \\ \text{negative} \Rightarrow \text{compressive stress} \end{cases}$$

$$2) \sigma_{ij}, i \neq j \begin{cases} \text{positive} \Rightarrow \text{positive direction of} \\ \text{the } j\text{-axis.} \\ \text{negative} \Rightarrow \text{negative direction of} \\ \text{ } j\text{-axis.} \end{cases}$$

★ tensile stress means material stretches.

★ compressive stress means material compresses.

Cauchy's Equations of Motion:

Cauchy's first law of motion

According to the principle of conservation of linear momentum, if the body is in static equilibrium it can be demonstrated that the components of the Cauchy stress tensor in every material point in the body satisfy the equilibrium equations.

$$\sigma_{ji,j} + F_i = 0$$

here F_i represents body force component.

★ A body force is a force that acts throughout the volume of the body for example:
gravity, electric fields and magnetic fields.

So the Cauchy's equilibrium equations
are: (in 3-dimensions)

$$\frac{\partial \sigma_{11}}{\partial x^1} + \frac{\partial \sigma_{21}}{\partial x^2} + \frac{\partial \sigma_{31}}{\partial x^3} + \underline{\underline{F_1}} = 0$$

$$\frac{\partial \sigma_{12}}{\partial x^1} + \frac{\partial \sigma_{22}}{\partial x^2} + \frac{\partial \sigma_{32}}{\partial x^3} + \underline{\underline{F_2}} = 0$$

$$\frac{\partial \sigma_{13}}{\partial x^1} + \frac{\partial \sigma_{23}}{\partial x^2} + \frac{\partial \sigma_{33}}{\partial x^3} + \underline{\underline{F_3}} = 0$$

Cauchy's second law of motion

According to the principle of conservation of angular momentum, equilibrium requires that the summation of moments with respect to an arbitrary point is zero, which leads to the conclusion that the stress tensor is symmetric, thus having only six independent stress components, instead of the original nine in three dimensions.

$$\sigma_{ij} = \sigma_{ji}$$

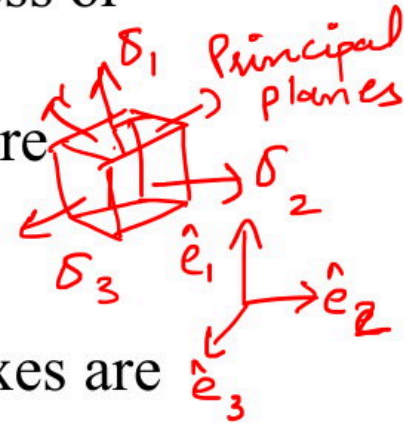


Moment: The moment of a force is a measure of its tendency to cause a body to rotate about a specific point or axis.

Angular momentum is the rotational analog of linear momentum. $\vec{L} = \vec{r} \times \vec{p}$

Principle Stresses: $\sigma = (\sigma_{ij}) = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$

- Regardless of the state of stress, it is always possible to choose a special set of axes (principal axes of stress or principal stress directions) so that the shear stress components vanish when the stress components are referred to this system.
- The three planes perpendicular to the principal axes are the principal planes.
- The normal stress components in the principal planes are the principal stresses.



Principle Stresses:

- The Eigen values of σ_{ij} tensor; represented as $\sigma_1, \sigma_2, \sigma_3$ are referred to as the principle stresses. The Eigen vectors are the principle stress directions.

$$\begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix} \rightarrow \text{principal stress matrix}$$

- The stress tensor is a symmetric 2nd order tensor so its eigenvalues are real numbers.

For the eigenvalue λ and its corresponding eigenvector v :

$$\sigma_{ij} \cdot v = \lambda v$$

$$\Rightarrow (\sigma_{ij} - \lambda I) \cdot v = 0$$

and $|\sigma_{ij} - \lambda I| = 0 \rightarrow$ characteristic equation.

$$\Rightarrow \lambda^3 - I_1 \lambda^2 - I_2 \lambda - I_3 = 0$$



Invariants

The three eigen values

$\lambda_1, \lambda_2, \lambda_3 \rightarrow \sigma_1, \sigma_2, \sigma_3$ are
principal stresses.

→ Recall that the invariants
associated with a tensor are values
which do not change with the
change of coordinate system.

Stress Invariants

- Principal stresses are invariants of the stress state:

Invariant w.r.t. rotation of the coordinate axes to which the stresses are referred.

The principal stresses are combined to form the stress invariants $I_i; i=1,2,3$ as:

$$I_1 = \text{Tr}(\sigma) = \sigma_{ii} = \sigma_1 + \sigma_2 + \sigma_3$$

$$I_2 = -(\sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3)$$

$$I_3 = \det(\sigma)$$

$$\text{where } \sigma = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix} = (\sigma_{ij})$$

1) Mean Stress:

Given the Cauchy stress tensor and its principal stresses, mean stress is defined as

$$\sigma_m = \frac{1}{3} \text{Tr}(\sigma_{ij}) = \frac{1}{3} \sigma_{ii} = \frac{1}{3} (\sigma_1 + \sigma_2 + \sigma_3)$$

2) Mean Pressure:

$$\bar{P} = -\sigma_m = -\frac{1}{3} (\sigma_1 + \sigma_2 + \sigma_3)$$

3) A spherical or hydrostatic state of stress:

$$\sigma_1 = \sigma_2 = \sigma_3 \Rightarrow \sigma_{ij} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma \end{bmatrix}$$

In a hydrostatic state of stress, the stress tensor is isotropic and, thus, its components are the same in any Cartesian coordinate system. As a consequence, any direction is a principal direction and the stress state (traction vector) is the same in any plane.

Question: A body in equilibrium
subject to the following stress state

$$\sigma = (\sigma_{ij}) = \begin{bmatrix} xy & 5y \\ 5y & 4x \end{bmatrix} \quad \sigma_{ji,j} + F_i = 0$$

determine body forces.

Solution: here we use the following eqs:

$$\frac{\partial \sigma_{11}}{\partial x^1} + \frac{\partial \sigma_{21}}{\partial x^2} = -F_1 \quad \text{and} \quad \frac{\partial \sigma_{12}}{\partial x^1} + \frac{\partial \sigma_{22}}{\partial x^2} = -F_2$$

$$\Rightarrow \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(5y) = -F_1$$

$$\Rightarrow F_1 = -y + 5 \Rightarrow F_1 = 5 - y$$

and

$$\frac{\partial}{\partial x}(5y) + \frac{\partial}{\partial y}(4x) = -F_2$$

$$\Rightarrow F_2 = -(0 + 0) = 0$$

$$\vec{F} = (5 - y, 0) \checkmark$$

Question: For the stress matrix

$$(\sigma_{ij}) = \begin{bmatrix} xy & 5y \\ 5y & 4x \end{bmatrix} \text{ write normal}$$

and shear stresses.

Solution: Normal stresses:

$$\sigma_{11} = xy, \sigma_{22} = 4x$$

Shear Stresses:

$$\sigma_{12} = \sigma_{21} = 5y$$

Question: Find the constraint for the stress matrix

$$(\sigma_{ij}) = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix}$$

for tensile and compressive stress.

Solution:

For tensile stress: $\sigma_{ii} > 0$

$$\Rightarrow \sigma_{11} > 0 \text{ and } \sigma_{22} > 0$$

$$\Rightarrow \sigma_x > 0 \text{ and } \sigma_y > 0$$

$$\Rightarrow x > 0 \text{ and } y > 0$$

For compressive stress:

$$\sigma_{ii} < 0$$

$$\Rightarrow \sigma_{11} < 0 \text{ and } \sigma_{22} < 0$$

$$\Rightarrow \kappa \gamma < 0 \text{ and } 4\kappa < 0$$

$$\Rightarrow \kappa < 0 \text{ and } \gamma > 0$$

$$\kappa < 0 \text{ and } \gamma > 0.$$

Question: Find the principal stresses for the stress state:

$$(\sigma_{ij}) = \begin{bmatrix} x_y & \sigma_y \\ \sigma_y & 4x \end{bmatrix}$$

Solution: consider:

$$|\sigma_{ij} - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} x_y - \lambda & \sigma_y \\ \sigma_y & 4x - \lambda \end{vmatrix} = 0$$

$$\Rightarrow (x_y - \lambda)(4x - \lambda) - 2\sigma_y^2 = 0$$

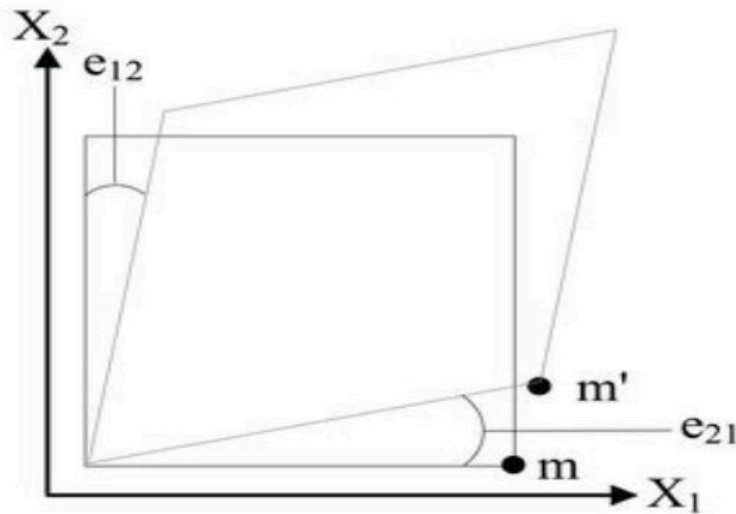
$$\Rightarrow 4x^2y - 4\lambda x - \lambda xy + \lambda^2 - 25y^2 = 0$$

$$\Rightarrow \lambda^2 - (4x + xy)\lambda + 4x^2y - 25y^2 = 0$$

$$\Rightarrow \lambda = \frac{4x + xy \pm \sqrt{(4x + xy)^2 - 4(4x^2y - 25y^2)}}{2}$$

The Strain Tensor

Strain is defined as the relative change in the position of points within a body that has undergone deformation. The classic example in two dimensions is of the square which has been deformed to a parallelepiped.



where $\varepsilon_{ij} = e_{ij} - \tilde{\omega}_{ij}$

is defined as the strain

is defined as rigid body rotation.

and

$$\varepsilon_{ij} = \frac{1}{2} (e_{ij} + e_{ji}) \rightarrow \varepsilon_{ji} = \frac{1}{2} (e_{ji} + e_{ij}) = \varepsilon_{ij}$$

$$\tilde{\omega}_{ij} = \frac{1}{2} (e_{ij} - e_{ji}) \rightarrow \tilde{\omega}_{ji} = \frac{1}{2} (e_{ji} - e_{ij}) = -\tilde{\omega}_{ij}$$

For the two-dimensional case:

$$\begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix} = \begin{pmatrix} e_{11} & \frac{1}{2}(e_{12} + e_{21}) \\ \frac{1}{2}(e_{21} + e_{12}) & e_{22} \end{pmatrix} + \begin{pmatrix} 0 & \frac{1}{2}(e_{12} - e_{21}) \\ \frac{1}{2}(e_{21} - e_{12}) & 0 \end{pmatrix}$$

$\rightarrow \epsilon_{ij}$
 $\rightarrow \tilde{\omega}_{ij}$

* ϵ_{ij} is a symmetric tensor and $\tilde{\omega}_{ij}$ is an antisymmetric tensor; the leading diagonal of $\tilde{\omega}_{ij}$ is always zero.

* The tensor ϵ_{ij} has Eigen values which are called the principal strains $\epsilon_1, \epsilon_2, \epsilon_3$.

Question: Given the matrix:

$$e_{ij} = \begin{bmatrix} a & c \\ c & a \end{bmatrix} \text{ find } \varepsilon_{ij} \text{ and } \tilde{w}_{ij}.$$

Solution:

$$\varepsilon_{ij} = \frac{1}{2} (e_{ij} + e_{ji})$$

$$\Rightarrow \varepsilon_{11} = \frac{1}{2} (e_{11} + e_{11}) = a$$

$$\varepsilon_{22} = a$$

$$\varepsilon_{12} = \frac{1}{2} (e_{12} + e_{21}) = \frac{1}{2} (c + c) = c$$

$$\varepsilon_{21} = \frac{1}{2} (c + c) = c$$

$$\Rightarrow \epsilon_{ij} = \begin{bmatrix} a & c \\ c & a \end{bmatrix} = e_{ij}$$

Similarly: $\tilde{\omega}_{ij} = \frac{1}{2} (e_{ij} - e_{ji})$

$$\Rightarrow \tilde{\omega}_{11} = \tilde{\omega}_{22} = 0$$

$$\tilde{\omega}_{12} = \frac{1}{2} (e_{12} - e_{21}) = \frac{1}{2} (c - c) = 0 = -\tilde{\omega}_{21}$$

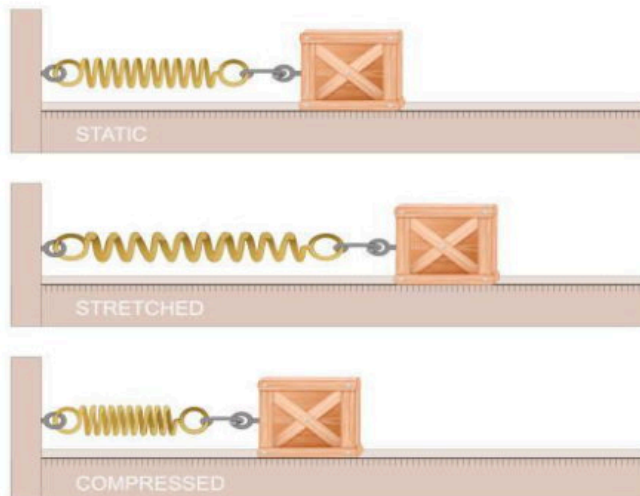
$$\Rightarrow \tilde{\omega}_{ij} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Elasticity

Unlike stress and strain, elasticity is an intrinsic property of a material. The elastic properties of Earth materials affects everything from the variation of density with depth in the planet to the speed at which seismic waves pass through the interior.

If the deformation in a body under stress does not exceed a certain limit, called the elastic limit, the body will return to its initial shape when the stress is removed.

Elasticity



Relation of Elasticity with Stress and Strain:

If the amount of stress (σ) is infinitesimally small then the amount of strain (ϵ), which is also infinitesimal, is linearly proportional to the strain and may be written as:

$$\epsilon = s \sigma$$

$$\text{or } \sigma = c \epsilon$$

Where s is the elastic compliance and c is the elastic stiffness. In order to relate two second rank tensors, a fourth rank tensor is necessary.


stress and strain

Elasticity Tensor:

$$\text{So, } \epsilon_{ij} = S_{ijkl} \sigma_{kl}$$

$$\text{or } \sigma_{ij} = C_{ijkl} \epsilon_{kl}$$

* $S_{ijkl} \rightarrow$ elasticity tensor $\rightarrow$ related to change in volume when subjected to an applied force. It is reciprocal of stiffness tensor.

* $C_{ijkl} \rightarrow$ stiffness tensor

Components of S_{ijkl} :

$$\text{In } 2D \rightarrow 2^4 = 16$$

$$\rightarrow 3^4 = 81$$

For the three-dimensional case there are 81 terms in a fourth rank tensor. However, both stress and strain are symmetric tensors;

$$\sigma_{ij} = \sigma_{ji} \quad \text{and} \quad \epsilon_{ij} = \epsilon_{ji} \quad \text{each only has 6 independent terms.}$$

There are only 6 equations needed to calculate ϵ_{ij} from σ_{ij} and in each equation there will only be 6 independent terms. Therefore, there can be no more than 36 independent values in S_{ijkl} .

Components of ϵ_{ij} using: $\epsilon_{ij} = S_{ijkl} \sigma_{kl}$

To calculate ϵ_{ij} for the three-dimensional case we would begin like so:

$$\epsilon_{11} = S_{1111} \sigma_{11} + S_{1112} \sigma_{12} + S_{1113} \sigma_{13} + S_{1121} \sigma_{21} + S_{1122} \sigma_{22} \\ + S_{1123} \sigma_{23} + S_{1131} \sigma_{31} + S_{1132} \sigma_{32} + S_{1133} \sigma_{33}$$

$$\epsilon_{12} = S_{1211} \sigma_{11} + S_{1212} \sigma_{12} + S_{1213} \sigma_{13} + S_{1221} \sigma_{21} + \\ S_{1222} \sigma_{22} + S_{1223} \sigma_{23} + S_{1231} \sigma_{31} + \\ S_{1232} \sigma_{32} + S_{1233} \sigma_{33}$$